nを自然数とし、$I_n=\int_0^1 \frac{dx}{\left(x^3+1\right)^n}$で定義する。

（１）I_1を求めよ。

（２）I_2を求めよ。

（３）I_3を求めよ。

（解答）

（１）$I_1=\int_0^1 \frac{dx}{x^3+1}$

$\frac{1}{x^3+1}=\frac{a}{x+1}+\frac{bx+c}{x^2-x+1}$が実数$x$について成り立つように$a,b,c$を求めると、

$\frac{1}{x^3+1}=\frac{a\left(x^2-x+1\right)+(bx+c)(x+1)}{x^3+1}=\frac{(a+b)x^2+(-a+b+c)x+a+c}{x^3+1}$より、

$a+b=0,-a+b+c=0,a+c=1$より、$a=\frac{1}{3},b=-\frac{1}{3},c=\frac{2}{3}$

よって、$I_1=\frac{1}{3}\left(\int_0^1 \frac{dx}{x+1}+\int_0^1 \frac{-x+2}{x^2-x+1}dx\right)$

$J_1=\int_0^1 \frac{dx}{x+1},J_2=\int_0^1 \frac{-x+2}{x^2-x+1}dx$とおくと、

$J_1=\left[\log|x+1|\right]_0^1=\log 2$

$J_2=-\frac{1}{2}\int_0^1 \frac{2x-1}{x^2-x+1}dx+\frac{3}{2}\int_0^1 \frac{dx}{x^2-x+1}$

$J_{2,1}=\int_0^1 \frac{2x-1}{x^2-x+1}dx,J_{2,2}=\int_0^1 \frac{dx}{x^2-x+1}$とおくと、

$J_{2,1}=\left[\log\left|x^2-x+1\right|\right]_0^1=0$

$J_{2,2}=\int_0^1 \frac{dx}{x^2-x+1}=\int_0^1 \frac{dx}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$

$x-\frac{1}{2}=\frac{\sqrt{3}}{2}\tan\theta\left(-\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2}\right)$とおくと、$dx=\frac{\sqrt{3}}{2}\cdot\frac{d\theta}{\cos^2\theta}$,

$x=0$のとき$-\frac{1}{2}=\frac{\sqrt{3}}{2}\tan\theta$より、$\theta=-\frac{\pi}{6}$

$x=1$のとき$\frac{1}{2}=\frac{\sqrt{3}}{2}\tan\theta$より、$\theta=\frac{\pi}{6}$

よって、

$$J_{2,2}=\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\frac{\frac{\sqrt{3}}{2}\frac{1}{\cos^2\theta}}{\left(\frac{\sqrt{3}}{2}\right)^2\tan^2\theta+\left(\frac{\sqrt{3}}{2}\right)^2}d\theta=\frac{2\sqrt{3}}{3}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}d\theta=\frac{2\sqrt{3}}{3}\left(\frac{\pi}{6}+\frac{\pi}{6}\right)=\frac{2\sqrt{3}}{9}\pi$$

$$J_2=-\frac{1}{2}J_{2,1}+\frac{3}{2}J_{2,2}=-\frac{1}{2}\cdot 0+\frac{3}{2}\cdot\frac{2\sqrt{3}}{9}\pi=\frac{\sqrt{3}}{3}\pi$$

$$I_1=\frac{1}{3}(J_1+J_2)=\frac{1}{3}\left(\log 2+\frac{\sqrt{3}}{3}\pi\right)=\frac{1}{3}\log 2+\frac{\sqrt{3}}{9}\pi$$

（2） $I_2=\int_0^1\frac{dx}{\left(x^3+1\right)^2}$

$$I_1=\int_0^1\frac{dx}{x^3+1}=\left[\frac{x}{x^3+1}\right]_0^1-\int_0^1\frac{x\cdot\left(-3x^2\right)}{\left(x^3+1\right)^2}dx=\frac{1}{2}+3\int_0^1\frac{x^3}{\left(x^3+1\right)^2}dx$$

$$\int_0^1\frac{x^3}{\left(x^3+1\right)^2}dx=\int_0^1\frac{x^3+1-1}{\left(x^3+1\right)^2}dx=\int_0^1\frac{1}{x^3+1}dx-\int_0^1\frac{1}{\left(x^3+1\right)^2}dx=I_1-I_2$$

より

$$I_1=\frac{1}{2}+3(I_1-I_2)\Leftrightarrow I_2=\frac{1}{6}+\frac{2}{3}I_1=\frac{1}{6}+\frac{2}{9}\log 2+\frac{2\sqrt{3}}{27}\pi$$

（3） $I_2=\int_0^1\frac{dx}{\left(x^3+1\right)^2}=\left[\frac{x}{\left(x^3+1\right)^2}\right]_0^1-\int_0^1\frac{x\cdot\left(-2\cdot 3x^2\right)}{\left(x^3+1\right)^3}dx=\frac{1}{4}+6\int_0^1\frac{x^3}{\left(x^3+1\right)^3}dx$

$$\int_0^1\frac{x^3}{\left(x^3+1\right)^3}dx=\int_0^1\frac{x^3+1-1}{\left(x^3+1\right)^3}dx=\int_0^1\frac{1}{\left(x^3+1\right)^2}dx-\int_0^1\frac{1}{\left(x^3+1\right)^3}dx=I_2-I_3$$

より

$$I_2=\frac{1}{4}+6(I_2-I_3)\Leftrightarrow I_3=\frac{1}{24}+\frac{5}{6}I_2=\frac{13}{72}+\frac{5}{27}\log 2+\frac{5\sqrt{3}}{81}\pi$$

(参考)

$$I_n=\int_0^1\frac{dx}{\left(x^3+1\right)^n}=\left[\frac{x}{\left(x^3+1\right)^n}\right]_0^1-\int_0^1\frac{x\cdot\left(-n\cdot 3x^2\right)}{\left(x^3+1\right)^{n+1}}dx=\frac{1}{2^n}+3n\int_0^1\frac{x^3}{\left(x^3+1\right)^{n+1}}dx$$

$$\int_0^1 \frac{x^3}{\left(x^3+1\right)^{n+1}}dx = \int_0^1 \frac{x^3+1-1}{\left(x^3+1\right)^{n+1}}dx = \int_0^1 \frac{1}{\left(x^3+1\right)^{n}}dx - \int_0^1 \frac{1}{\left(x^3+1\right)^{n+1}}dx = I_n - I_{n+1}$$

より

$$I_n = \frac{1}{2^n} + 3n\left(I_n - I_{n+1}\right) \Leftrightarrow I_{n+1} = \frac{1}{3n\cdot 2^n} + \frac{3n-1}{3n} I_n$$

（参考 2）

$$J_n = \int_0^1 \frac{x^q}{\left(x^p+a\right)^n}dx = \left[\frac{x^{q+1}}{q+1}\frac{1}{\left(x^p+a\right)^n}\right]_0^1 - \frac{1}{q+1}\int_0^1 \frac{x^{q+1}\cdot\left(-npx^{p-1}\right)}{\left(x^p+a\right)^{n+1}}dx$$

$$= \frac{1}{(q+1)(1+a)^n} + \frac{pn}{q+1}\int_0^1 \frac{x^{p+q}}{\left(x^p+a\right)^{n+1}}dx$$

$$\int_0^1 \frac{x^{p+q}}{\left(x^p+a\right)^{n+1}}dx = \int_0^1 \frac{x^{p+q}+ax^q-ax^q}{\left(x^p+a\right)^{n+1}}dx = \int_0^1 \frac{x^q}{\left(x^p+a\right)^{n}}dx - a\int_0^1 \frac{x^q}{\left(x^p+a\right)^{n+1}}dx = J_n - aJ_{n+1}$$

より

$$J_n = \frac{1}{(q+1)(1+a)^n} + \frac{pn}{q+1}\left(J_n - aJ_{n+1}\right) \Leftrightarrow J_{n+1} = \frac{1}{apn(1+a)^n} + \frac{pn-q-1}{apn} J_n$$