$x^{16}+x^8+1$ を実数係数の範囲で因数分解せよ。

（解答）

$$x^{16}+x^8+1=x^{16}+2x^8+1-x^8=\left(x^8+1\right)^2-\left(x^4\right)^2=\left(x^8+x^4+1\right)\left(x^8-x^4+1\right)$$

$$x^8+x^4+1=x^8+2x^4+1-x^4=\left(x^4+1\right)^2-\left(x^2\right)^2=\left(x^4+x^2+1\right)\left(x^4-x^2+1\right)$$

$$x^8-x^4+1=x^8+2x^4+1-3x^4=\left(x^4+1\right)^2-\left(\sqrt{3}x^2\right)^2=\left(x^4+\sqrt{3}x^2+1\right)\left(x^4-\sqrt{3}x^2+1\right)$$

$$x^4+x^2+1=x^4+2x^2+1-x^2=\left(x^2+1\right)^2-x^2=\left(x^2+x+1\right)\left(x^2-x+1\right)$$

$$x^4-x^2+1=x^4+2x^2+1-3x^2=\left(x^2+1\right)^2-\left(\sqrt{3}x\right)^2=\left(x^2+\sqrt{3}x+1\right)\left(x^2-\sqrt{3}x+1\right)$$

$$x^4+\sqrt{3}x^2+1=x^4+2x^2+1-\left(2-\sqrt{3}\right)x^2=\left(x^2+1\right)^2-\left(\sqrt{2-\sqrt{3}}x\right)^2$$

$$=\left(x^2+\sqrt{2-\sqrt{3}}x+1\right)\left(x^2-\sqrt{2-\sqrt{3}}x+1\right)$$

$$x^4-\sqrt{3}x^2+1=x^4+2x^2+1-\left(2+\sqrt{3}\right)x^2=\left(x^2+1\right)^2-\left(\sqrt{2+\sqrt{3}}x\right)^2$$

$$=\left(x^2+\sqrt{2+\sqrt{3}}x+1\right)\left(x^2-\sqrt{2+\sqrt{3}}x+1\right)$$

ここで、

$$x^2\pm x+1=\left(x\pm\frac{1}{2}\right)^2+\frac{3}{4}>0$$

$$x^2\pm\sqrt{3}x+1=\left(x\pm\frac{\sqrt{3}}{2}\right)^2+\frac{1}{4}>0$$

$$x^2\pm\sqrt{2\pm\sqrt{3}}x+1=\left(x\pm\frac{\sqrt{2\pm\sqrt{3}}}{2}\right)^2+\frac{2\mp\sqrt{3}}{4}>0$$

より実数係数の範囲でこれ以上因数分解することができないので、

$x^{16}+x^8+1$

$=\left(x^2+x+1\right)\left(x^2-x+1\right)\left(x^2+\sqrt{3}x+1\right)\left(x^2-\sqrt{3}x+1\right)$

$\times\left(x^2+\sqrt{2-\sqrt{3}}x+1\right)\left(x^2-\sqrt{2-\sqrt{3}}x+1\right)\left(x^2+\sqrt{2+\sqrt{3}}x+1\right)\left(x^2-\sqrt{2+\sqrt{3}}x+1\right)$