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# KOMATU-LOEWNER DIFFERENTIAL EQUATIONS

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ABSTRACT. The classical Loewner differential equation for simply connected planar domains is generalized to the Komatu-Loewner differential equation (K-L equation) for three types of canonical multiply connected domains, namely, (1) standard slit domain, (2) annulus, and (3) circularly slit annulus. First, the K-L equation in the left-derivative sense is derived for (1), (2), (3) in a unified manner in terms of the Brownian motion with darning (BMD). This K-L left-differential equation for (1) (resp., for (2)) is then converted into a genuine ODE by using BMD and a method of interior variations in PDE (resp., by using an annulus version of the Carathéodory kernel convergence theorem). Further, based on the K-L equation for (1), a K-L evolution determined by a pair $(\xi(t), \mathbf{s}(t))$ will be formulated, where $\xi(t)$ is a given real-valued continuous function and $\mathbf{s}(t)$ is an induced motion of slits.

## 1. INTRODUCTION

Let $\mathbb{H}$ be the upper half-plane and let $\gamma = \{\gamma(t) : 0 \le t \le t_\gamma\}$ be a Jordan arc satisfying $\gamma(0) \in \partial\mathbb{H}$, $\gamma(0, t_\gamma] \subset \mathbb{H}$. For each $t \in (0, t_\gamma]$, there exists a unique Riemann map $g_t$ from $\mathbb{H}\setminus\gamma(0,t]$ onto $\mathbb{H}$ satisfying $\lim_{z\to\infty}(g_t(z)-z) = 0$ and, under a suitable continuous reparametrization of $t$, it fulfills a simple ordinary differential equation

$$\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - \xi(t)}, \quad z \in \mathbb{H} \setminus \gamma(0,t], \quad g_0(z) = z, \tag{1.1}$$

called a *Loewner equation.* Here $\xi(t) = g_t(\gamma(t))$, which is a continuous function taking value in the boundary $\partial\mathbb{H}$. The Loewner equation had been formulated for canonical simply connected domains such as the unit disk $\mathbb{D}$ ([25, 27, 35, 40]) and effectively utilized in solving the Bieberbach conjecture, a long-standing problem in complex analysis. It was also used when L. de Branges gave a final solution to the conjecture in 1985 ([11]).

Conversely, given a $\partial\mathbb{H}$-valued continuous function $\xi(t)$, $t \ge 0$, let $\{g_t(z),\ 0 \le t < t_z\}$ be the unique solution of (1.1) with the right maximal interval $[0, t_z)$ of existence ([22]) and define the set $K_t = \{z \in \overline{\mathbb{H}} : t_z \le t\}$ for each $t \ge 0$. Then $K_t$ is a compact subset of $\overline{\mathbb{H}}$, $\mathbb{H} \setminus K_t$ is simply connected, and $g_t$ becomes a conformal map from $\mathbb{H} \setminus K_t$ onto $\mathbb{H}$. Such an increasing family $\{K_t\}$ of sets is called *a family of growing hulls driven by* $\xi(t)$.

In 2000, Oded Schramm [38] made a deep observation on possible scaling limits of lattice models in statistical physics, found it natural to adopt $\xi(t) = B(\kappa t)$ for the one-dimensional Brownian motion $B(t)$ and a positive constant $\kappa$ as a random driving function, and called the family of random growing hulls driven by $B(\kappa t)$

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*stochastic Loewner evolution* (often called *Schramm-Loewner evolution*), which is denoted by $\mathrm{SLE}_\kappa$.

For several values of $\kappa$, $\mathrm{SLE}_\kappa$s have been identified with the scaling limits of various critical lattice models in statistical physics, and their significant features have been thereby revealed. W. Werner and S. Smirnov were awarded Fields Prizes for their related works ([32, 33, 39, 43]). It was proved by S. Rohde and O. Schramm [37] that, with probability 1, $\mathrm{SLE}_\kappa$ is generated by a continuous curve $\gamma$ in $\overline{\mathbb{H}}$, which is simple when $0 < \kappa \le 4$, self-intersecting but not filling $\overline{\mathbb{H}}$ when $4 < \kappa < 8$, and a Peano-curve filling $\overline{\mathbb{H}}$ when $\kappa \ge 8$.

One of the main concerns of the researchers in the SLE theory was its extensions to multiply connected domains, and several papers on it have appeared ([3, 4, 31, 44]). About extensions of the Loewner equation itself, there were two pioneering works by Yusaku Komatu: an extension to annulus [26] and an extension to circularly slit annulus [29]. The equation in [26] was later extended by Goluzin [20] to the annulus with the interior boundary being the unit circle, which continues to be studied under a more general framework ([12]).

The equation in [29] seems to have been left unnoticed for almost half a century until R.O. Bauer-R.M. Friedrich [3, 4] rewrote it in the three cases of a circularly slit disk, a circlularly slit annulus, and a standard slit domain and called them *Komatu-Loewner equations*. In particular, for a standard slit domain, namely, for a domain obtained from the upper half-plane $\mathbb{H}$ by removing several mutually disjoint line segments parallel to the $x$-axis called *slits*, the kernel appearing on the right-hand side of the equation in [4] bears a significant potential theoretic meaning. In an electronic mail sent from Roland Friedrich to the present author at the end of 2013, he recalled that when he found and read the article [29] in the archive of the library at the Institute of Princeton in 2004, he felt like having found a treasure.

However most of the differential equations mentioned above were incomplete in that they were determined only in the left derivative sense, and the continuity of some relevant important quantities with respect to the parameter were left unproven. The first aim of this exposition is to explain in some detail from section 3 to section 7 how to establish these Komatu-Loewner equations (K-L equations in brief) as genuine ordinary differential equations and solve the relevant continuity problems by making use of a method in stochastic analysis, a method of interior variations in PDE, and several methods in complex analysis along the lines of two recent joint papers [9] with Zhen-Qing Chen and Steffen Rohde and [16] with Hiroshi Kaneko.

On the other hand, in the case of the standard slit domain, a Jordan arc $\gamma$ with parameter $t$ induces via the K-L equation not only a motion $\xi(t)$ on $\partial\mathbb{H}$ but also a motion $\mathbf{s}(t)$ of slits. Further, by taking the trace of the K-L equation on the slits, one can derive a differential equation for $\mathbf{s}(t)$ containing $\xi(t)$ in it, and the vector field of its right-hand side is locally Lipschitz continuous due to the result stated in section 5. Consequently, given conversely an arbitrary continuous function $\xi(t)$ with values on $\partial\mathbb{H}$, one can define a slit motion $\mathbf{s}(t)$ as a unique solution of this slit equation and then generate via the K-L equation a family $\{g_t\}$ of conformal maps and a family $\{F_t\}$ of growing hulls driven by the pair $(\xi(t), \mathbf{s}(t))$. $\{F_t\}$ is called a *Komatu-Loewner evolution.* The second aim of this exposition is to give a brief account of these developments in sections 8 and 9 along the lines of a recent joint work [8] with Zhen-Qing Chen.

As will be explained at the end of section 9, [8] actually shows that, as a possible random driving process $(\xi(t), \mathbf{s}(t))$, it is natural to adopt the path of the strong solution of a system of Markov type stochastic differential equations with coefficients $\alpha$, $b$ of $\xi(t)$ being homogeneous functions of specific degrees. The family $\{F_t\}$ of random growing hulls it generates is called in [8] a *stochastic Komatu-Loewner evolution* and is designated by $\mathrm{SKLE}_{\alpha,b}$. Many interesting properties of $\mathrm{SKLE}_{\alpha,b}$ extending those for $\mathrm{SLE}_\kappa$ are being exploited ([8], [10]), although we will not touch upon them in this exposition.

We end the introduction by noting the following remarkable potential theoretic feature of the Loewner equation (1.1): if we put $\Psi(z,\zeta) = -1/\pi(z-\zeta)$, then the right-hand side of (1.1) is expressed as $-2\pi\Psi(g_t(z), \xi(t))$, and the imaginary part $y/\pi[(x-\zeta)^2 + y^2]$ of $\Psi(z,\zeta)$ is nothing but the Poisson kernel for the integral representation of harmonic functions on the upper half-plane $\mathbb{H}$. In this sense, $\Psi(z,\zeta)$ is the *complex Poisson kernel for the absorbing Brownian motion on* $\mathbb{H}$.

As will be explained in the next section, for a multiply connected domain $D$, a major role is played by a stochastic process called a *Brownian motion with darning* (BMD in brief) living on the quotient topological space obtained by regarding each bounded connected component of $\mathbb{C} \setminus D$ as a single point. The kernel in the right-hand side of the K-L equation for the standard slit domain obtained by [4] turns out to be the complex Poisson kernel for the BMD on the image domain, while the kernels in the right-hand sides of the K-L equations for the annulus and circularly slit annulus obtained by [16] are the *BMD Schwarz kernels* in the sense that their real parts are the Poisson kernels for the BMDs on the respective image domains. However, except for those parts directly linked to BMDs, our considerations will be deterministic and non-probabilistic.

## 2. Multiply connected domains and BMD

A connected closed subset of the complex plane $\mathbb{C}$ containing at least two points is called a *continuum*. For $N \geq 0$, a domain $D \subset \mathbb{C}$ is said to be of $N+1$-*connectivity* if $\mathbb{C} \setminus D = \bigcup_{k=1}^{N+1} A_k$ for mutually disjoint continuum $A_k$ and $A_1, \dots, A_N$ are compact, while $A_{N+1}$ is unbounded.

As was introduced in [28] (see also [1, §5.3]), the following potential theoretic construction of a conformal map from a multiply connected domain $D$ to a parallel slit domain goes back to D. Hilbert [23]. For $1 \leq k \leq N$, a continuous function $\varphi^{(k)}(z)$ on $\mathbb{C}$ that is harmonic in $z \in D$, equal to 1 for $z \in A_k$, and 0 for $z \in \bigcup_{j \neq k} A_j$ has been called a *harmonic measure* since the period when the Lebesgue measure theory was not yet popularized. For a fixed $z_0 \in D$, let $G(z, z_0)$ be the Green function on $D$ and define $v^*(z) = -\partial G(z, z_0)/\partial y_0 + \sum_{k=1}^{N} \lambda_k \varphi^{(k)}(z)$, $y_0 = \Im z_0$. One can determine real constants $\lambda_k$ in such a way that the period of $v^*$ around each $A_k$ (see below) equals zero so that there exists an analytic function $f$ on $D$ satisfying $\Im f = v^*$ uniquely up to an additional real constant. This function $f$ gives the conformal map from $D \setminus \{z_0\}$ onto a parallel slit domain with the property $f(z_0) = \infty$. Notice that every real function appearing here takes a constant value on each $A_k$, suggesting strongly that it is natural to develop the analysis for the multiply connected domain $D$ by regarding each hole $A_k$, $1 \leq k \leq N$, as a single point.

Accordingly we put $E = \mathbb{C} \setminus A_{N+1}$ and we consider the identification space (the quotient topological space)

$$D^* = D \cup K^*, \qquad K^* = \{c_1^*, \dots, c_N^*\},$$

obtained from $E$ by regarding each $A_k$, $1 \le k \le N$, as a single point $c_k^*$. Denote by $m$ the Lebesgue measure on $D$ and extend it to $D^*$ by setting $m(K^*) = 0$. Let $Z^0 = (Z_t^0, \mathbb{P}_z^0)$ be the absorbing Brownian motion on $D$. A diffusion process $Z^*$ on $D^*$ is called a *symmetric diffusion extension of* $Z^0$ if the transition function of $Z^*$ is symmetric with respect to $m$ and the subprocess of $Z^*$ obtained by killing upon the hitting time for $K^*$ is identical in law with $Z^0$. According to a general theory [6, §7.8], there exists a unique symmetric diffusion extension $Z^* = (Z_t^*, \mathbb{P}_z^*)$ of $Z^0$ admitting no killing on $K^*$. $Z^*$ is said to be the *Brownian motion with darning* (BMD) on $D^*$ or for the multiply connected domain $D$.

The first construction of BMD for $N = 1$ was carried out in the joint paper [18] with Hiroshi Tanaka by using Itô's method in [24]. It was constructed by piecing together the $Z^0$-excursion paths around $A_1$ (namely, the Brownian paths starting at $\partial A_1$ and returning to $\partial A_1$ passing only through $D$) by means of an excursion-valued Poisson point process. Itô dealt with the case where $A_1$ consists of a single point. But his method was robust enough to be applicable to the case where $A_1$ is a set of points. Patrick J. Fitzsimmons suggested to the author the use of the term *darning*, which sounds appropriate in that the above-mentioned procedure of the construction resembles repairing a hole by thread.

Actually a version of BMD had been constructed in an older paper of the author [15] by means of a regularization of a Dirichlet form. Indeed BMD can be directly constructed by a regular Dirichlet form as follows: For $E = \mathbb{C} \setminus A_{N+1}$, we let $H_0^1(E)$ be the closure of $C_c^\infty(E)$ in the Sobolev space $H^1(E) = \{u \in L^2(E) : |\nabla u| \in L^2(E)\}$ of order 1 and put

$$\begin{cases} \mathcal{F}^* = \{u \in H_0^1(E); u \text{ is constant q.e. on each } A_k\}, \\ \mathcal{E}^*(u, v) = \frac{1}{2} \int_D \nabla u(x) \cdot \nabla v(x) dx, \quad u, v \in \mathcal{F}^*. \end{cases}$$

Then $(\mathcal{E}^*, \mathcal{F}^*)$ is a strongly local regular Dirichlet form on $L^2(D^*; m)$, and the associated diffusion process on $D^*$ is just the BMD for $D$ ([9, Thm. 2.3]). Here "q.e." means "quasi everywhere (except for a set of zero capacity)" and every function in $H_0^1(E) (\subset L^2(E))$ is assumed to be chosen to be its quasi-continuous modification.

The most useful property of BMD, $Z^* = (Z_t^*, \mathbb{P}_z^*)$, is the zero-period property of any BMD-harmonic function. Let $G$ be a connected open subset of $D^*$. A real continuous function $u$ defined on $G$ is called *BMD-harmonic* if, for any relatively compact open set $G_1$ with $\overline{G}_1 \subset G$, the probabilistic averaging property

$$\mathbb{E}_z^* \left[ u(Z_{\tau_{G_1}}^*) \right] = u(z), \qquad z \in G_1,$$

holds, where $\tau_{G_1}$ is the exit time from $G_1$. In this case, the restriction of $u$ to $G \cap D$ is clearly harmonic in the ordinary sense.

For a harmonic function $u$ on $D$, the value of the line integral

$$\int_\gamma \frac{\partial u(\zeta)}{\partial \mathbf{n}_\zeta} ds(\zeta)$$

along a smooth Jordan curve $\gamma$ surrounding $A_k$ ($\mathbf{n}$ is the inward unit normal vector) does not depend on the choice of $\gamma$ and is called the *period* of $u$ around $A_k$.

**Theorem 2.1** ([9, Thm. 3.4]). *If $u$ is BMD-harmonic on a connected open set $G \subset D^*$, then, for any $k$ with $c_k^* \in G$, the period of $u$ around $A_k$ equals* 0.

This theorem can be proved by rewriting a general theorem established in [6, §7.8] that the generator of the strongly continuous semigroup on $L^2$ determined by BMD is characterized by the zero flux condition at every $c_k^*$. Consequently, if $u$ is BMD-harmonic on $D^*$, then there exists an analytic function on $D$ whose real part (resp., imaginary part) equals $u\big|_D$ uniquely up to an additional imaginary (resp., real) constant ([11, Thm. 15.1.2]).

From the next section to section 7, we survey the joint works [9] with Z.-Q. Chen and S. Rhode and [16] with H. Kaneko on Komatu-Loewner equations being done by invoking BMDs. In fact, they were strongly motivated by an earlier work of G. Lawler [31]. In [31], a stochastic process quite similar to BMD called *excursion reflected Brownian motion* (ERBM) had been introduced independently, and thereby the Komatu-Loewner equation for the standard slit domain analytically derived by Bauer-Friedrich [4] had been rewritten, and the investigations in this direction were took over by Lawler's student S. Drenning [13] (which is unpublished, unfortunately).

While [31] only presents several properties of an ERBM in a descriptive manner, [13] gave its rigorous characterization when $N = 1$, which has enabled us to identify the ERBM with the BMD in a recent joint paper [7]. When $N > 1$, however, no characterization of the ERBM is given, and so we can only say at present that those properties of ERBM described in [13, 31] remain valid also for the BMD. Nevertheless, a related concept of a *boundary Poisson kernel* being used in [13, 31] can be identified with the *Feller kernel* formulated in [6], and we need to notice the roles it plays.

## 3. BMD Complex Poisson kernel and K-L left-differential equation on standard slit domain

In the following three sections, we consider a standard slit domain $D = \mathbb{H} \setminus K$, $K = \bigcup_{j=1}^N C_j$. $C_j$, $1 \le j \le N$, are mutually disjoint line segments contained in $\mathbb{H}$ paralell to the $x$-axis and called *slits*.

Let $D^* = D \cup K^*$, $K^* = \{c_1^*, \dots, c_N^*\}$ be the quotient topological space obtained from $\mathbb{H}$ by regarding each set $C_j$ as a single point $c_j^*$ and let $Z^* = (Z_t^*, \zeta^*, \mathbb{P}_z^*)$ be the BMD on it. We consider the absorbing Brownian motion $Z^0 = (Z_t^0, \zeta^0, \mathbb{P}_z^0)$ on $D$, denote its 0-order resolvent density function (the classical Green function multiplied by $1/\pi$) by $G(z,\zeta)$, and put

$$\varphi^{(j)}(z) = \mathbb{P}_z^0(Z^0_{\zeta^0-} \in \partial C_j), \quad z \in D, \quad \varphi^{(j)}(c_i^*) = \delta_{ij}, \quad 1 \le i, j \le N,$$

which is a harmonic function in $z \in D$ and a continuous function on $D^*$ ([36]). $Z^*$ is transient and its 0-order resolvent density function $G^*(z,\zeta)$, $z \in D^*$, $\zeta \in D$, can be shown to admit the following expression:

$$G^*(z,\zeta) = G(z,\zeta) + 2\Phi(z) \cdot \mathcal{A}^{-1} \cdot {}^t\Phi(\zeta), \quad z \in D^*, \ \zeta \in D. \tag{3.1}$$

Here $\Phi(z)$ is the $N$-vector with component $\varphi^{(j)}(z)$, and $\mathcal{A}$ is an $N \times N$-matrix with $(i,j)$ component being the period of $\varphi^{(i)}$ around $C_j$. Since $G^*(z,\zeta)$ is BMD-harmonic in $z \in D^* \setminus \{\zeta\}$, (3.1) follows immediately from Theorem 2.1.

The *BMD Poisson kernel* is defined as $K^*(z,\zeta) = -2^{-1}\partial G^*(z,\zeta)/\partial \mathbf{n}_\zeta$ by the outward unit normal vector $\mathbf{n}_\zeta$ at $\zeta \in \partial\mathbb{H}$, and we get the next expression of it from (3.1):

$$K^*(z,\zeta) = -\frac{1}{2}\frac{\partial}{\partial \mathbf{n}_\zeta} G(z,\zeta) - \Phi(z)\cdot \mathcal{A}^{-1}\cdot \frac{\partial}{\partial \mathbf{n}_\zeta}{}^t\Phi(\zeta), \quad z\in D^*, \quad \zeta\in\partial\mathbb{H}. \tag{3.2}$$

The following integral representation of the expectation by BMD ([9, Lem. 5.2]) legitimates the term "BMD Poisson kernel" for $K^*$: for any bounded continuous function on $\partial\mathbb{H}$,

$$\mathbb{E}_z^*[g(Z^*_{\zeta^*-})] = \int_{\partial\mathbb{H}} K^*(z,\zeta)g(\zeta)ds(\zeta), \quad z\in D^*. \tag{3.3}$$

As $K^*(z,\zeta)$, $\zeta\in\partial\mathbb{H}$, is BMD-harmonic in $z\in D^*$, Theorem 2.1 implies again that there exists an analytic function $\Psi(z,\zeta)$ on $D$ satisfying $\Im\Psi(z,\zeta)=K^*(z,\zeta)$, $z\in D$, uniquely under the normalization condition

$$\lim_{z\to\infty}\Psi(z,\zeta)=0. \tag{3.4}$$

$\Psi(z,\zeta)$ is called the *BMD complex Poisson kernel* on the standard slit domain $D$.

We now consider a Jordan arc:

$$\gamma:[0,t_\gamma] \mapsto \overline{D}, \qquad \gamma(0)\in\partial\mathbb{H}, \qquad \gamma(0,t_\gamma]\subset D. \tag{3.5}$$

For each $t\in[0,t_\gamma]$, there exists a unique conformal map $g_t$ sending $D\setminus\gamma[0,t]$ onto some standard slit domain $D_t=\mathbb{H}\setminus\bigcup_{j=1}^N C_j(t)$ satisfying the hydrodynamic normalization condition

$$g_t(z) = z+\frac{a_t}{z}+o(|z|^{-1}), \qquad z\to\infty. \tag{3.6}$$

The tip $\gamma(t)$ of $\gamma[0,t]$ is sent by $g_t$ to a point $\xi(t)\in\partial\mathbb{H}$:

$$\xi(t) = g_t(\gamma(t))(=\lim_{z\to\gamma(t),\ z\in D\setminus\gamma[0,t]} g_t(z))\in\partial\mathbb{H}. \tag{3.7}$$

**Theorem 3.1** ([9, Cor. 6.3, Thm. 6.4])**.**

(i) *$a_t$ is a strictly increasing function of $t$ and $a_0=0$.*

(ii) *For $t\in(0,t_\gamma]$, $z\in D$, $g_t(z)$ is left differentiable with respect to $a_t$ and satisfies*

$$\frac{\partial^- g_t(z)}{\partial a_t} = -\pi\Psi_t(g_t(z),\xi(t)), \qquad g_0(z)=z. \tag{3.8}$$

*Here, the left-hand side denotes the left derivative $\lim_{s\uparrow t}\frac{g_t(z)-g_s(z)}{a_t-a_s}$, and $\Psi_t$ is the complex Poisson kernel for $D_t$.*

This theorem can be proved essentially in the same way as the proof by Komatu ([26, 29]) for the annulus and circularly slit annulus. For $0<s<t<t_\gamma$, we consider the conformal map

$$g_{t,s}=g_s\circ g_t^{-1} \ :\ D_t \ \mapsto\ D_s\setminus g_s(\gamma[s,t])$$

and use the homeomorphic extension of $g_{t,s}$ to the set of prime ends in determining two unique points $\beta_0(t,s),\beta_1(t,s)\in\partial\mathbb{H}$ satisfying the following properties:

$$\beta_0(t,s)<\xi(t)<\beta_1(t,s), \qquad g_{t,s}(\beta_0(t,s))=g_{t,s}(\beta_1(t,s))=\xi(s),$$
$$\beta_0(t,s)\uparrow\xi(t), \qquad \beta_1(t,s)\downarrow\xi(t), \qquad s\uparrow t.$$

We can then derive the next two equations:

$$a_t-a_s = \frac{1}{\pi}\int_{\beta_0(t,s)}^{\beta_1(t,s)} \Im g_{t,s}(x+i0+)dx, \tag{3.9}$$

$$g_s(z) - g_t(z) = \int_{\beta_0(t,s)}^{\beta_1(t,s)} \Psi_t(g_t(z), x) \Im g_{t,s}(x + i0+) dx. \tag{3.10}$$

$a_t$ appearing in (3.6) is called the *half-plane capacity* and is calculated by $a_t = \lim_{z\to\infty} z(g_t(z) - z)$. Extending $g_{t,s}(z)$ to the lower half-plane by the Schwarz reflection and putting $f(w) = [g_{t,s}(1/w) - (1/w)]/w$, we see that $\mathbf{0}$ is a removable singularity of $f$ and $f(\mathbf{0}) = a_t - a_s$. We then express $f(\mathbf{0})$ as a sum of line integrals on several closed curves by Cauchy's integral formula and arrive at (3.9) after some manipulation. Equation (3.10) can be readily obtained by noting that $\Im(g_{t,s}(z)-z)$ is constant on each slit of $D_t$ with zero period around it and by using the explicit expression (3.2) of $\Im\Psi_t$ together with the normalization condition (3.4).

Dividing both sides of (3.10) by those of (3.9), noting the continuity of $\Psi_t(g_t(z), x)$ in $x$ (see Lemma 3.2 below), using the mean value theorem, and letting $s \uparrow t$, we are led to Theorem 3.1. If we let $t \downarrow s$ by replacing $g_{t,s}$ with its inverse map, we would attain no immediate result. Note that even in the derivation of the classical Loewner equation for a disk, the right derivative requires more care than the left one ([2, §6-2]).

If $a_t$ were continuous in $t$, we could make $a_t$ into $2t$ by changing the parameter $t$ of the Jordan arc $\gamma(t)$ into $(a^{-1})_{2t}$. This procedure is called the *half-plane capacity reparametrization.* By this change, the equation (3.8) is converted into

$$\frac{\partial^- g_t(z)}{\partial t} = -2\pi\Psi_t(g_t(z), \xi(t)), \qquad g_0(z) = z. \tag{3.11}$$

If further the right-hand side of (3.11) were continuous in $t$, then $g_t(z)$ would become differentiable in $t$, and so (3.11) would become a genuine ordinary differential equation ([30, Lem. 4.3]).

In the following two sections, we shall show that the two "if" assumptions stated above actually hold true. To this end, let us consider the totality $\mathcal{D}$ of labelled standard slit domains. For instance, $\mathbb{H} \setminus \{C_1, C_2, C_3, \ldots, C_N\}$ and $\mathbb{H} \setminus \{C_2, C_1, C_3, \ldots, C_N\}$ are regarded as different elements of $\mathcal{D}$ although they correspond to the same subset $\mathbb{H} \setminus \bigcup_{i=1}^N C_i$ of $\mathbb{H}$. For $D$, $\widetilde{D} \in \mathcal{D}$, their distance $d(D, \widetilde{D})$ is defined by

$$d(D, \widetilde{D}) = \max_{1\le i\le N}(|z_i - \widetilde{z}_i| + |z_i' - \widetilde{z}_i'|). \tag{3.12}$$

Here, for $D = \mathbb{H} \setminus \bigcup_{i=1}^N C_i$, the left and right endpoints of the line segment $C_i$ are denoted by $z_i$, $z_i'$, respectively. $\widetilde{z}_i$, $\widetilde{z}_i'$ are defined analogously for $\widetilde{D}$. $\{D_t\}$ is a subfamily of $\mathcal{D}$ parametrized by $t$. The BMD-complex Poisson kernel $\Psi$ is a *domain function* with the defining set $\mathcal{D}$ in the sense that it is uniquely determined by $D \in \mathcal{D}$.

Under the above setting, our problems will be threefold:

**(I)** Local uniform continuity of $g_t(z)$ in $t$.

**(II)** Continuity of $a_t$, $D_t \in \mathcal{D}$, $\xi(t)$ in $t$.

**(III)** Lipschitz continuity of the correspondence $D \in \mathcal{D} \mapsto \Psi$.

**(I)** will be shown in section 4 by means of a probabilistic repesentation of $\Im g_t(z)$ in terms of BMD. As a consequence **(II)** will be derived by complex analytic considerations. **(III)** will be shown in section 5 by using a method of interior variations in partial differential equations. By combining this with **(I), (II)**, the right-hand side of (3.11) becomes continuous in $t$, establishing (3.11) as a genuine ordinary differential equation.

Before moving ahead, we give a remark on an important property of the BMD complex Poisson kernel $\Psi(z,\zeta)$.

**Lemma 3.2** ([9, Lem. 6.1]). *For $1 \le k \le N$, we denote by $C_k^+$, $C_k^-$ the upper part and the lower part of the slit $C_k$, respectively, and by $\partial_p C_k$ the set $C_k^+ \cup C_k^-$ with the path-distance topology on $\mathbb{H}\setminus C_k$. We put $\partial_p K = \bigcup_{k=1}^N \partial_p C_k$. For any bounded closed interval $J$ in $\partial\mathbb{H}$, $\Psi(z,\zeta)$ can then be extended continuously to $D \cup \partial_p K \cup (\partial\mathbb{H}\setminus J) \times J$ as a function of two variables $z, \zeta$.*

## 4. Representation of $\Im g_t(z)$ by BMD and uniform continuity of $g_t$

Let $D = \mathbb{H}\setminus K$, let $K = \bigcup_{j=1}^N C_j$ be a standard slit domain, and let $\gamma$ be a Jordan arc satisfying (3.5) as in the preceding section. Fix $t \in [0, t_\gamma]$ and put $F_t = \gamma[0,t]$. Let $Z^* = (Z_t^*, \mathbb{P}_z^*)$ be the BMD on $D^* = D \cup K^*$, $K^* = \{c_1^*, \dots, c_N^*\}$. For $r > 0$, consider the line $\Gamma_r = \{z = x + iy : y = r\}$ parallel to the $x$-axis and let

$$v_t^*(z) = \lim_{r\to\infty} r \cdot \mathbb{P}_z^*(\sigma_{\Gamma_r} < \sigma_{F_t}), \quad z \in D^* \setminus F_t, \tag{4.1}$$

where $\sigma_A$ denotes the hitting time of the set $A$. $Z^{\mathbb{H}} = (Z_t^{\mathbb{H}}, \mathbb{P}_z^{\mathbb{H}})$ will denote the absorbing Brownian motion on $\mathbb{H}$.

**Theorem 4.1** ([9, Thms. 7.1, 7.2]).

(i) *The function $v_t^*$ defined by (4.1) is a BMD-harmonic function on $D^* \setminus F_t$ and admits the following expression:*

$$v_t^*(z) = v_t(z) + \sum_{j=1}^N \mathbb{P}_z^{\mathbb{H}}\left(\sigma_K < \sigma_{F_t},\ Z_{\sigma_K}^{\mathbb{H}} \in C_j\right) v_t^*(c_j^*), \ z \in D \setminus F_t, \tag{4.2}$$

*where*

$$v_t(z) = \Im z - \mathbb{E}_z^{\mathbb{H}}[\Im Z_{\sigma_{F_t}}^{\mathbb{H}};\ \sigma_{F_t} < \sigma_K]\ (\ge 0), \tag{4.3}$$

$$v_t^*(c_i^*) = \sum_{j=1}^N \frac{M_{ij}(t)}{1 - R_j^*(t)} \int_{\eta_j} v_t(z)\nu_j(dz), \quad 1 \le i \le N. \tag{4.4}$$

*Here $\eta_1, \dots, \eta_N$ are smooth Jordan curves surrounding $C_1, \dots, C_N$, respectively, and*

$$\nu_i(dz) = \mathbb{P}_{c_i^*}^*\left(Z_{\sigma_{\eta_i}}^* \in dz\right), \quad 1 \le i \le N, \tag{4.5}$$

$$R_i^*(t) = \int_{\eta_i} \mathbb{P}_z^{\mathbb{H}}\left(\sigma_K < \sigma_{F_t},\ Z_{\sigma_K}^{\mathbb{H}} \in C_i\right) \nu_i(dz), \quad 1 \le i \le N. \tag{4.6}$$

*$M_{ij}(t)$ denotes the $(i,j)$-component of the matrix $M(t) = \sum_{n=0}^\infty (Q^*(t))^n$ where $Q^*(t)$ is a matrix possessing the following components:*

$$q_{ij}^*(t) = \begin{cases} \mathbb{P}_{c_i^*}^*\left(\sigma_{K^*} < \sigma_{F_t},\ Z_{\sigma_{K^*}}^* = c_j^*\right)/(1 - R_i^*(t)) & \text{if } i \ne j, \\ 0 & \text{if } i = j, \end{cases} \quad 1 \le i,j \le N. \tag{4.7}$$

(ii) *$v_t^*\big|_{D\setminus F_t}$ is the imaginary part of $g_t$:*

$$v_t^*(z) \ = \ \Im g_t(z), \qquad z \in D \setminus F_t. \tag{4.8}$$

As $v_t^*$ is BMD-harmonic, it is the imaginary part of an analytic function $f$ sending $D \setminus F_t$ into a parallel slit domain by Theorem 2.1, and $f$ can be verified to satisfy the hydrodynamic normalization condition by using the above expression of $v_t^*$. By invoking a degree theorem for a proper analytic map ([9, Lem. 11.1]), it can be further shown that $f$ is a conformal map onto a standard slit domain, yielding $f = g_t$. The probabilistic expression of $\Im g_t(z)$ like (4.1) was first asserted by G. Lawler [31] in terms of ERBM instead of BMD.

The expression of $v_t^*(z)$ in the above theorem looks rather involved. But it contains the hitting time $\sigma_{F_t}$ of the set $F_t$ so explicitly that it enables us to derive from the stochastic continuity

$$\mathbb{P}_z^*(\lim_{s\to t} \sigma_{F_s} = \sigma_{F_t}) = 1, \ \mathbb{P}_z^{\mathbb{H}}(\lim_{s\to t} \sigma_{F_s} = \sigma_{F_t}) = 1, \ \mathbb{P}_z^{\mathbb{H}}(\lim_{s\to t} \sigma_{F_s\cup K} = \sigma_{F_t\cup K}) = 1$$

the desired local uniform continuity of $v_t^*(z)$ and $g_t(z)$ directly. In this way we obtain the following theorem from Theorem 4.1.

**Theorem 4.2** ([9, Thms. 8.1, 8.2, 8.3])**.**
(i) *For a fixed* $t \in (0, t_\gamma]$, *it holds that* $\lim_{s\uparrow t} g_{s,t}(z) = z$ *uniformly in* $z$ *in each compact subset of* $D_t \cup \partial_p K_t \cup (\partial\mathbb{H} \setminus \{\xi(t)\})$.
(ii) *For a fixed* $s \in (0, t_\gamma]$, *it holds that* $\lim_{t\downarrow s} g_{s,t}^{-1}(z) = z$ *uniformly in* $z$ *in each compact subset of* $D_s \cup \partial_p K_s \cup (\partial\mathbb{H} \setminus \{\xi(s)\})$.
(iii) *For a fixed* $s \in (0, t_\gamma]$, $g_t(z)$ *is continuous in* $(t, z) \in [0, s] \times ((D \times \partial_p K \cup \partial\mathbb{H}) \setminus \gamma[0, s])$.

The next theorem follows from this.

**Theorem 4.3** ([9, Thms. 8.4, 8.5, 8.6])**.**
(i) $a_t$ *is a continuous function of* $t \in [0, t_\gamma]$.
(ii) $D_t \in \mathcal{D}$ *is continuous in* $t \in [0, t_\gamma]$.
(iii) $\xi(t) \in \partial\mathbb{H}$ *is continuous in* $t \in [0, t_\gamma]$.

(ii) follows immediately from the definition of the distance (3.12) for $\mathcal{D}$ and Theorem 4.2. (iii) can be shown in the same way as in P.L. Duren [14, p. 85] by using Theorem 4.2. As for (i), the left continuity of $a_t$ follows from Theorem 4.2(i) combined with Theorem 3.1. On the other hand, as in the previous derivation of (3.9), we express $a_t - a_s$ as a sum of line integrals of $[g_{t,s}^{-1}(1/w) - (1/w)]/w$ on several closed curves and we let $t \downarrow s$. By taking Theorem 4.2(ii) into account, we can get the right continuity of $a_t$.

## 5. Lipschitz continuity of BMD complex Poisson kernel and differentiability of $g_t(z)$

Each $D \in \mathcal{D}$ determines uniquely the BMD complex Poisson kernel $\Psi(z, \zeta)$, $z \in D$, $\zeta \in \partial\mathbb{H}$, which can be extended continuously to $((D \cup \partial_p K \cup (\partial\mathbb{H} \setminus J)) \times J$ as a function of two variables $(z, \zeta)$, where $J$ is an arbitrary bounded closed interval in $\partial\mathbb{H}$. Recall that the distance $d$ of $\mathcal{D}$ is being defined by (3.12).

**Theorem 5.1** ([9, Thm. 9.1])**.** *The correspondence* $D \in \mathcal{D} \mapsto \Psi(z, \zeta)$ *is Lipschitz continuous in the following sense.*

*Let* $U_j$, $V_j$, $1 \le j \le N$, *be a family of relatively compact open subsets of* $\mathbb{H}$ *satisfying*

$$\overline{U}_j \subset V_j \subset \overline{V}_j \subset \mathbb{H}, \quad 1 \le j \le N, \quad \overline{V}_j \cap \overline{V}_k = \emptyset, \quad j \ne k. \tag{5.1}$$

*We fix any* $a > 0,\ b > 0$ *such that the subfamily* $\mathcal{D}_0$ *of* $\mathcal{D}$ *defined by*
(5.2)

$$\mathcal{D}_0 = \{\mathbb{H} \setminus \bigcup_{j=1}^{N} C_j \in \mathcal{D} : C_j \subset U_j,\ |z_j - z'_j| > a,\ \mathrm{dist}(C_j, \partial U_j) > b,\ 1 \le j \le N\}$$

*is non-empty. There exists then* $\epsilon_0 > 0$ *admitting the following:*

*For any* $\varepsilon \in (0, \varepsilon_0)$, *any* $D \in \mathcal{D}_0$, *and any* $\widetilde{D} \in \mathcal{D}$ *satisfying* $d(D, \widetilde{D}) < \varepsilon$, *there exists a diffeomorphism* $\widetilde{f}_\varepsilon$ *from* $\mathbb{H}$ *onto* $\mathbb{H}$ *satisfying the following* (i), (ii), (iii).

(i) $\widetilde{f}_\varepsilon$ *sends* $D$ *onto* $\widetilde{D}$, *linear on* $\bigcup_{j=1}^N U_j$, *and the identity map on* $\mathbb{H} \setminus \bigcup_{j=1}^N \overline{V}_j$.

(ii) *For some positive constant* $L_1$ *independent of* $\varepsilon \in (0, \varepsilon_0)$ *and* $D \in \mathcal{D}_0$,

$$|z - \widetilde{f}_\varepsilon(z)| < L_1 \cdot \varepsilon, \quad z \in \mathbb{H}. \tag{5.3}$$

(iii) *For any compact subset* $Q$ *of* $\overline{\mathbb{H}}$ *containing* $\bigcup_{j=1}^N U_j$ *and for any compact subset* $J$ *of* $\partial\mathbb{H}$,

$$|\Psi(z, \zeta) - \widetilde{\Psi}(\widetilde{f}_\varepsilon(z), \zeta)| < L_{Q,J} \cdot \varepsilon, \quad z \in (Q \setminus K) \cup \partial_p K,\ \zeta \in J, \tag{5.4}$$

*where* $\widetilde{\Psi}$ *denotes the BMD complex Poisson kernel for* $\widetilde{D}$ *and* $L_{Q,J}$ *is a positive constant independent of* $\varepsilon \in (0, \varepsilon_0)$, $D \in \mathcal{D}_0$, *and* $\widetilde{D} \in \mathcal{D}$.

To prove Theorem 5.1, we consider the stated sets $U_j,\ V_j$. For any $\varepsilon > 0,\ D \in \mathcal{D}_0$, we take $\widetilde{D} \in \mathcal{D}$ satisfying $d(D, \widetilde{D}) < \varepsilon$. The quantities for $\widetilde{D}$ will be designated with a $\widetilde{\ }$. For $1 \le j \le N$, let $\delta_j \in \mathbb{R},\ b_j \in \mathbb{C}$ be the constants determined uniquely by

$$\begin{cases} \widetilde{z}_j - z_j = \delta_j z_j + b_j, \\ \widetilde{z}'_j - z'_j = \delta_j z'_j + b_j. \end{cases}$$

Here $z_j = x_{j1} + i x_{j2}$ $(z'_j = x'_{j1} + i x'_{j2})$ is the left (right) endpoint of the slit $C_j$. We define a linear map by

$$F_{j,\varepsilon}(z) = \frac{1}{\varepsilon}(\delta_j z + b_j), \quad 1 \le j \le N. \tag{5.5}$$

Then these coefficients are uniformly bounded with respect to $\varepsilon > 0,\ D \in \mathcal{D}_0,\ \widetilde{D} \in \mathcal{D}$. Choose a smooth function $q(x_1, x_2),\ z = x_1 + i x_2 \in \mathbb{H}$, taking the value of $[0, 1]$ such that

$$q(x_1, x_2) = \begin{cases} 1 & \text{if } x_1 + i x_2 \in U_j, \quad 1 \le j \le N, \\ 0 & \text{if } x_1 + i x_2 \in \mathbb{H} \setminus \bigcup_{j=1}^N \overline{V}_j, \end{cases}$$

and define a map $\widetilde{f}_\varepsilon$ by

$$\begin{cases} \widetilde{f}_\varepsilon(z) = z + \varepsilon F_\varepsilon(x_1, x_2), \\ F_\varepsilon(x_1, x_2) = q(x_1, x_2) \sum_{j=1}^N \mathbf{1}_{V_j}(z) F_{j,\varepsilon}(z), \quad z = x_1 + i x_2. \end{cases} \tag{5.6}$$

**Lemma 5.2** ([9, Lem. 9.2]). *There exists* $\varepsilon_0 > 0$ *such that, for any* $\varepsilon \in (0, \varepsilon_0)$ *and for any* $D \in \mathcal{D}_0$ *and for any* $\widetilde{D} \in \mathcal{D}$ *with* $d(D, \widetilde{D}) < \varepsilon$, *the map* $\widetilde{f}_\varepsilon$ *defined by* (5.6) *satisfies* (i), (ii) *of* Theorem 5.1.

We can prove that $\widetilde{f}_\varepsilon$ also satisfies (iii) of Theorem 5.1 by making use of the perturbation formulae of the Green function $G(z, w)$ of the domain $D \in \mathcal{D}$ that will be stated below. Let $\widetilde{G}(\widetilde{z}, \widetilde{w})$ be the Green function of $\widetilde{D}$ and put

$$g(z, w, \varepsilon) = \widetilde{G}(\widetilde{f}_\varepsilon(z), \widetilde{f}_\varepsilon(w)), \quad z, w \in D. \tag{5.7}$$

Define a self-adjoint second order elliptic partial differential operator $A_\varepsilon$ by

$$
\begin{cases} (A_\varepsilon u)(x_1,x_2) = \displaystyle\sum_{k,\ell=1}^{2} \frac{\partial}{\partial x_k}\left(A_{k\ell}^{(\varepsilon)} \frac{\partial u}{\partial x_\ell}\right), \\ A_{k\ell}^{(\varepsilon)} = \displaystyle\frac{1}{2}\frac{\partial(\widetilde{x}_1,\widetilde{x}_2)}{\partial(x_1,x_2)} \sum_{j=1}^{2} \frac{\partial x_k}{\partial \widetilde{x}_j}\frac{\partial x_\ell}{\partial \widetilde{x}_j}, \quad 1 \le k,\ell \le 2. \end{cases} \tag{5.8}
$$

**Proposition 5.3** ([9, Appen. 3]). (i) *$g(z,w,\varepsilon)$ is the fundamental solution of $A_\varepsilon$ in the following sense: for any $f \in C_c(D)$,*

$$A_\varepsilon(g_\varepsilon f)(z) = -f(z), \quad z \in D, \tag{5.9}$$

*where $(g_\varepsilon f)(z) = \int_D g(z,w,\varepsilon) f(w) dw_1 dw_2$.*

(ii) *$A_\varepsilon = (1/2)\Delta + \varepsilon B^{(\varepsilon)}$, where*

$$B^{(\varepsilon)} = \sum_{k,\ell=1}^{2} b_{k\ell}^{(\varepsilon)} \frac{\partial^2}{\partial x_k \partial x_\ell} + \sum_{k,\ell=1}^{2} \frac{\partial b_{k\ell}^{(\varepsilon)}}{\partial x_k}\frac{\partial}{\partial x_\ell}. \tag{5.10}$$

*Here $b_{k\ell}^{(\varepsilon)}$, $1 \le k,\ell \le 2$, are smooth functions on $\mathbb{H}$ with $b_{k\ell}^{(\varepsilon)} = b_{\ell k}^{(\varepsilon)}$, vanishing on $(\mathbb{H} \setminus \bigcup_{i=1}^{2} \overline{V}_i) \cup (\bigcup_{i=1}^{N} U_i)$, which together with their derivatives are bounded on $\mathbb{H}$ uniformly in $\varepsilon \in (0,\varepsilon_0)$, $D \in \mathcal{D}_0$, and $\widetilde{D} \in \mathcal{D}$.*

(iii) *Put $F = \bigcup_{i=1}^{N} (\overline{V}_i \setminus U_i)$. Then, for any $\zeta \in \overline{\mathbb{H}} \setminus F$ and $w \in \overline{\mathbb{H}}$,*

$$g(\zeta,w,\varepsilon) - G(\zeta,w) = \varepsilon \int_F B_z^{(\varepsilon)} G(z,\zeta) g(z,w,\varepsilon) dx_1 dx_2, \quad z = x_1 + ix_2,\ \varepsilon \in (0,\varepsilon_0). \tag{5.11}$$

(iv) *There exists $\widetilde{\varepsilon}_0 \in (0,\varepsilon_0]$ independent of $D \in \mathcal{D}$ such that for any $\zeta \in \overline{\mathbb{H}} \setminus F$ and $w \in \overline{\mathbb{H}}$,*

$$g(\zeta,w,\varepsilon) - G(\zeta,w) = \varepsilon \int_F B_z^{(\varepsilon)} G(z,\zeta)(G(z,w) + \varepsilon \eta^{(\varepsilon)}(z,w)) dx_1 dx_2, \quad \varepsilon \in (0,\widetilde{\varepsilon}_0), \tag{5.12}$$

*where $\eta^{(\varepsilon)}$ is a continuous function on $\overline{\mathbb{H}} \times \overline{\mathbb{H}}$ that is uniformly bounded in $\varepsilon \in (0,\widetilde{\varepsilon}_0)$, $D \in \mathcal{D}_0$, and $\widetilde{D} \in \mathcal{D}$.*

For the proof of (iii) and (iv), we first construct an appropriate parametrix for the second order elliptic differential operator $A_\varepsilon$ by following the interior variation method in P.R. Garabedian [19, §15.1] and then solve the corresponding Fredholm type integral equation to obtain the perturbation formulae (5.11) and (5.12).

In view of the concrete expression (3.2) of the BMD complex Poisson kernel $\Psi(z,\zeta)$ for the standard slit domain $D$, we see that $\Psi(x,\zeta)$ can be obtained by repeating the following three operations on the Green function $G(z,w)$ of $D$:

(a) taking the normal derivatives at $\partial\mathbb{H}$,
(b) taking the periods around slits,
(c) taking the line integrals of the normal derivatives along smooth curves.

By applying those operations to the perturbation formulae (5.11) and (5.12) of the Green function, we can finally get the desired Lipschitz type estimate (5.4).

Theorems 4.3 and 5.1 lead us to the following final conclusion:

**Theorem 5.4** ([9, Thm. 9.9]). *The Jordan curve $\gamma$ admits the half-plane capacity reparametrization $a_t = 2t,\ t \in [0, t_\gamma]$. Under this parametrization, $g_t(z)$ is differentiable in $t \in [0, t_\gamma]$ and satisfies, for each $z \in (D \cup \partial_p K) \setminus \gamma[0, t_\gamma]$, the Komatu-Loewner differential equation*

$$\frac{\partial g_t(z)}{\partial t} = -2\pi \Psi_t(g_t(z), \xi(t)), \quad g_0(z) = z, \quad 0 < t \le t_\gamma. \tag{5.13}$$

## 6. BMD Schwarz kernel and K-L left-differential equation for circularly slit annulus

In the last three sections, we have dealt with the Komatu-Loewner equation for a standard slit domain. Now we shall consider it for a circularly slit annulus. We let

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\},\ \mathbb{D}_q = \{z \in \mathbb{C} : |z| < q\},\ \mathbb{A}_q = \{z \in \mathbb{C} : q < |z| < 1\},\ q \in (0, 1).$$

$\mathbb{A}_q$ is an annulus. For $N \ge 1$, a domain $D$ expressed as $D = \mathbb{A}_q \setminus \bigcup_{j=1}^{N-1} C_j$ is called a *circularly slit annulus*, where $C_j,\ 1 \le j \le N-1$, are mutually disjoint concentric circular slits contained in $\mathbb{A}_q$. Denote by $\mathcal{D}$ the collection of all circularly slit annuli.

For $D = \mathbb{A}_q \setminus \bigcup_{j=1}^{N-1} C_j \in \mathcal{D}$, let $D^* = D \cup K^*,\ K^* = \{c_0^*, c_1^*, \ldots, c_{N-1}^*\}$ be the quotient topological space obtained from $\mathbb{D}$ by regarding each of the sets $\overline{\mathbb{D}}_q, C_1, \ldots, C_{N-1}$ as a single point and let $Z^*$ be the BMD on $D^*$. $Z^*$ is uniquely determined as a symmetric diffusion extension of the absorbing Brownian motion $Z^0$ on $D$ admitting no killing on $K^*$. The Poisson kernel for $Z^*$, namely, the BMD Poisson kernel $K^*(z, \zeta),\ z \in D^*,\ \zeta \in \partial\mathbb{D}$, for the circularly slit annulus $D$ admits the expression

$$K^*(z, \zeta) = -\frac{1}{2} \frac{\partial}{\partial \mathbf{n}_\zeta} G(z, \zeta) - \Phi(z) \cdot \mathcal{A}^{-1} \cdot \frac{\partial}{\partial \mathbf{n}_\zeta} {}^t\Phi(\zeta), \quad z \in D^*, \quad \zeta \in \partial\mathbb{D}, \tag{6.1}$$

exactly in the same way as (3.2). Here, $\mathbf{n}_\zeta$ is the outward unit normal vector at $\zeta \in \partial\mathbb{D}$, and $G,\ \Phi,\ \mathcal{A}$ are determined from the absorbing Brownian motion $Z^0$ on $D$ precisely in the same way as (3.1).

Since the BMD Poisson kernel $K^*(z, \zeta)$ is BMD harmonic, there exists by Theorem 2.1 an analytic function $S_D(z, \zeta)$ on $D$ satisfying $\Re S_D(z, \zeta) = K^*(z, \zeta)$ uniquely for each $\zeta \in \partial\mathbb{D}$ up to an additional imaginary constant. We call it the *BMD Schwarz kernel* for the circularly slit annulus. The reason for this name is that it is analogous to the classical Schwarz kernel $1/(2\pi) \cdot (\zeta + z)/(\zeta - z)$ for the unit disk $\mathbb{D}$. The BMD Schwarz kernel $\widehat{S}_D$ satisfying

$$\Im \widehat{S}_D(q, \zeta) = 0 \qquad \forall \zeta \in \partial\mathbb{D} \tag{6.2}$$

is called the *normalized Schwarz kernel* for $D$. For any BMD Schwarz kernel $S_D(z, \zeta)$,

$$\widehat{S}_D(z, \zeta) = S_D(z, \zeta) - i \Im S_D(q, \zeta), \quad z \in D, \zeta \in \partial\mathbb{D}, \tag{6.3}$$

determines the normalized BMD Schwarz kernel $\widehat{S}_D(z, \zeta)$.

For a fixed $D = \mathbb{A}_Q \setminus \bigcup_{j=1}^{N-1} C_j \in \mathcal{D}$, let us consider a Jordan arc $\gamma : [0, t_\gamma] \mapsto \overline{D}$ satisfying $\gamma(0) \in \partial\mathbb{D},\ \gamma(0, t_\gamma] \subset D$. We can then find an increasing function $\alpha : [0, t_\gamma] \mapsto [Q, Q_\gamma],\ (\alpha(t_\gamma) = Q_\gamma < 1)$ such that, for $q = \alpha(t)$, there exists a unique conformal map

$$g_q: \; D \setminus \gamma[0,t] \;\mapsto\; D_q = \mathbb{A}_q \setminus \bigcup_{j=1}^{N-1} C_j(q) \in \mathcal{D}, \quad \text{with} \quad g_q(Q) = q,$$

that associates the outer component of $\partial(D \setminus \gamma[0,t])$ (resp., its inner component $\partial\mathbb{D}_Q$) with $\partial\mathbb{D}$ (resp., $\partial\mathbb{D}_q$) (cf. [11, Thm. 15.5.1]).

We have the following theorem analogous to Theorem 3.1.

**Theorem 6.1** ([16, Thm. 6.1]). *$q = \alpha(t)$ is strictly increasing and left continuous in $t \in (0, t_\gamma]$. $g_q(z)$ is left-differentiable in $q$ and satisfies the following for $z \in D\setminus\gamma[0,t]$:*

$$\frac{\partial^- \log g_q(z)}{\partial \log q} = 2\pi \widehat{S}_q(g_q(z), \lambda(q)), \quad q \in \alpha(0, t_\gamma] \subset (Q, Q_\gamma], \; g_Q(z) = z, \tag{6.4}$$

*where the left-hand side denotes the left derivative. $\widehat{S}_q(z,\zeta)$ is the normalized BMD Schwarz kernel for $D_q$, and $\lambda(q) = g_q(\gamma(t)) \in \partial\mathbb{D}$.*

To prove this theorem, we take $0 \le t^* < t \le t_\gamma$ and put $q = \alpha(t)$, $q^* = \alpha(t^*)$. $g_{q^*q} = g_{q^*} \circ g_q^{-1}$ is a conformal map from $D_q$ onto $D_{q^*} \setminus S_{q^*q}$ (where $S_{q^*q} = g_{q^*}\gamma[t^*, t]$) satisfying

$$g_{q^*q}(q) = q^*. \tag{6.5}$$

$g_{q^*q}^{-1}(S_{q^*q})$ equals a subarc $\{e^{i\theta} : \; \beta_1(t^*, t) < \theta < \beta_2(t^*, t)\}$ of the outer circle of $D_q$ and it contains the point $\lambda(q) = g_q(\gamma(t))$. We then have

$$\log \frac{q^*}{q} = \frac{1}{2\pi} \int_{\beta_0(t^*,t)}^{\beta_1(t^*,t)} \log |g_{q^*q}(e^{i\varphi})| d\varphi, \tag{6.6}$$

$$\log \frac{g_{q^*q}(z)}{z} = \int_{\beta_0(t^*,t)}^{\beta_1(t^*,t)} \log |g_{q^*q}(e^{i\varphi})| \widehat{S}_q(z, e^{i\varphi}) d\varphi. \tag{6.7}$$

Observe that the branch $f(z)$ of $\log[g_{q^*q}(z)/z]$ with $f(q) = \log \frac{q^*}{q}$ is a single-valued analytic function on $D_q$. The Cauchy integral theorem applied to the analytic function $\frac{1}{z} f(z)$ on $D_q$ then yields (6.6) just as in [29, p. 30]. Equation (6.7) can be derived, exactly analogously to the derivation of (3.10) in [9, §6.3], by noting that the real part $\log |g_{q^*q}(z)/z|$ of the analytic function $f(z)$ is constant on each slit and on the inner circle of $D_q$ with zero period around each of them and by using the concrete expression (6.1) of $\Re S_q$ together with the normalization condition (6.5). Substituting $z = g_q(w)$ into (6.7), dividing both sides of the resulting identity by those of (6.6), and letting $t^* \uparrow t$, we arrive at (6.4). Since the integrand of the right-hand side of (6.6) is uniformly bounded in $t^*$, $\alpha$ is left continuous.

The first extension of the Loewner equation to a circularly slit annulus goes back to Y. Komatu [29], and the resulting Komatu-Loewner equation for $g_q$ was rewritten in [4] and then in [16] as Theorem 6.1 above. But the problem of the continuity of $\alpha(t)$ and the differentiability of $g_q(z)$ in Theorem 6.1 remain open for $N > 1$, although Komatu [29] tried to solve the problem by an induction in $N \ge 1$ not quite successfully.

Recently C. Boehm-W. Lauf [5] derived a genuine Komatu-Loewner differential equation for a circularly slit disk by making use of a generalized Carathéodory convergence theorem. An analogous method might work for a circularly slit annulus.

Othewise, we have to repeat arguments similar to those in the preceding two sections of the present exposition to solve the problem.

When $N = 1$, namely, in the case of annulus $\mathbb{A}_q$, the above problem can be solved by means of a method suggested already by Y. Komatu [26, 28], as will be explained in the next section.

## 7. Villat's kernel and K-L differential equation for annulus

First of all, let us observe that the BMD Poisson kernel $K^*(z,\zeta)$, $z \in \mathbb{A}_q$, $\zeta \in \partial\mathbb{D}$, for the annulus $\mathbb{A}_q$ admits a simple expression,

$$K^*(z, e^{i\theta}) = -\frac{1}{2}\frac{d}{dr}G(z, re^{i\theta})\big|_{r=1} - \varphi(z)p^{-1}\frac{d}{dr}\varphi(re^{i\theta})\big|_{r=1}, \tag{7.1}$$

by putting $N = 1$ in (6.1). Here $G$ is the 0-order resolvent density function of the absorbing Brownian motion on $\mathbb{A}_q$, $\varphi$ is the hitting probability of the absorbing Brownian motion on the unit disk $\mathbb{D}$ to the inner disk $\mathbb{D}_q$, and $p$ is the period of $\varphi$ around $\mathbb{D}_q$. Due to the rotation invariance, the second term on the right-hand side does not depend on $\theta$, so that, for each $\theta \in [0, 2\pi)$, $K^*(z,\zeta)$ is harmonic on $\mathbb{A}_q$ and takes a constant value $1/(2\pi)$ on $\partial\mathbb{D}_q$ as a function of $z$. In particular, if we take a function $\phi \in C(\partial\mathbb{D}, \mathbb{R})$ with

$$\int_0^{2\pi} \phi(e^{i\theta})d\theta = 1 \tag{7.2}$$

and put $(K^*\phi)(z) = \int_0^{2\pi} K^*(z, e^{i\theta})\phi(e^{i\theta})d\theta$, $z \in \mathbb{A}_q$, then $K^*\phi$ is a harmonic function on $\mathbb{A}_q$ taking a constant $1/(2\pi)$ on $\partial\mathbb{D}_q$, and its value at $e^{i\theta} \in \partial\mathbb{D}$ equals $\phi(\theta)$.

We now consider *Villat's kernel* on the annulus $\mathbb{A}_q$ defined by

(7.3)

$$\mathcal{K}_q(z,\zeta) = \mathcal{K}_q(z/\zeta),\ z \in \mathbb{A}_q,\ \zeta \in \partial\mathbb{A}_q, \quad \text{where} \quad \mathcal{K}_q(z) = \lim_{N\to\infty}\sum_{n=-N}^{N}\frac{1+q^{2n}z}{1-q^{2n}z}.$$

This kernel with slightly different forms was introduced by H. Villat [41, 42] to give an integral representation of an analytic function $f$ on the annulus by its boundary values, which led through further manipulation to the celebrated *Villat's integral formula* for $f$ in terms of Weierstrass elliptic functions. The simple expression of the kernel $\mathcal{K}_q$ using the principal value as (7.3) is taken from G.M. Goluzin [20].

**Lemma 7.1** ([16, Prop. 2.2(ii)])**.** *For any function $\phi \in C(\partial\mathbb{D}, \mathbb{R})$ satisfying* (7.2)*, let*

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} \mathcal{K}_q(z, e^{i\theta})\phi(e^{i\theta})d\theta.$$

*Then $f$ is analytic on $\mathbb{A}_q$, and $\lim_{r\downarrow q}\Re f(re^{i\eta}) = 1/(2\pi)$, $\lim_{r\uparrow 1}\Re f(re^{i\theta}) = \phi(e^{i\theta})$, $\forall\eta,\ \theta \in [0, 2\pi)$.*

For $f$ of Lemma 7.1, $\Re f$ coincides with the above-mentioned $K^*\phi$ and consequently,

**Corollary 7.2.** $(1/(2\pi))\mathcal{K}_q(z,\zeta)$, $z \in \mathbb{A}_q, \zeta \in \partial\mathbb{D}$, *is a BMD Schwarz kernel for the annulus $\mathbb{A}_q$.*

We define the *normalized Villat's kernel* $\widehat{\mathcal{K}}_q(z,\zeta)$ by

$$\widehat{\mathcal{K}}_q(z,\zeta) = \mathcal{K}_q(z,\zeta) - i\Im\mathcal{K}_q(q,\zeta), \quad z \in \mathbb{A}_q, \quad \zeta \in \partial\mathbb{D}. \tag{7.4}$$

Let us fix an annulus $\mathbb{A}_Q$, $0 < Q < 1$, and a Jordan arc $\gamma = \{\gamma(t) : 0 \le t \le t_\gamma\}$ satisfying $\gamma(0) \in \partial\mathbb{D}$, $\gamma(0, t_\gamma] \subset \mathbb{A}_Q$. There exists then an increasing function $\alpha : [0, t_\gamma] \mapsto [Q, Q_\gamma]$ $(\alpha(t_\gamma) = Q_\gamma < 1)$ with the following property: if $\alpha(t) = q$, then there is a unique conformal map

$$g_q : \mathbb{A}_Q \setminus \gamma[0,t] \ \mapsto \ \mathbb{A}_q, \quad g_q(Q) = q.$$

**Theorem 7.3** ([16, Thm. 3.1]). *$q = \alpha(t)$ is a strictly increasing continuous function from $[0, t_\gamma]$ onto $[Q, Q_\gamma]$.*

*$g_q(z)$ is continuously differentiable in $q$ and satisfies the following equation for each $z \in \mathbb{A}_Q \setminus \gamma[0,t]$ :*

$$\frac{\partial \log g_q(z)}{\partial \log q} = \widehat{\mathcal{K}}_q(g_q(z), \lambda(q)), \quad Q \le q \le Q_\gamma, \quad g_Q(z) = z. \tag{7.5}$$

*Moreover $\lambda(q) = g_q(\gamma(t)) \in \partial\mathbb{D}$ is continuous in $q$.*

Among the assertions in this theorem, the left continuity of $\alpha$, the left differentiability of $g_q(z)$, and the validity of the equation (7.5) as the left differential equation follow from Theorem 6.1 and Corollary 7.2. The rest of the assertions can be proved mostly by using the following proposition.

**Proposition 7.4** ([16, Cor. 7.2]). *For $0 < q^* < 1$, let $\{q_n\}$ be a sequence of real numbers with $q^* < q_n < 1$, $n \ge 1$, and let $\{h_n\}$ be a sequence of univalent functions satisfying the following conditions:*

(i) *$h_n$ is a map from a subdomain $E_n$ of $\mathbb{A}_{q^*}$ onto $\mathbb{A}_{q_n}$.*

(ii) *$E_n \subset E_{n+1}$, $n \ge 1$, $\bigcup_{n=1}^{\infty} E_n = \mathbb{A}_{q^*}$.*

(iii) *Each $E_n$ has $\partial\mathbb{D}_{q^*}$ as one of its boundary components.*

(iv) *For each $n$, $h_n(q^*) = q_n$.*

*Then it holds that $\lim_{n\to\infty} q_n = q^*$ and $\{h_n\}$ converges as $n \to \infty$ to the identity map locally uniformly on $\mathbb{A}_{q^*}$.*

[26, p. 6] and [28] stated without proof an annulus variant of the Carathéodory kernel convergence theorem for a disk (cf. [21, p. 55]). [16, §7] gave a proof of a version of this annulus variant, and the above proposition was obtained as its corollary.

Keeping the notation used in the explanation of Theorem 6.1, we put $h_{q^*q} = g_{q^*q}^{-1}$. For $t_n$ with $t_n \downarrow t^*$, we can apply Proposition 7.4 to $q_n = \alpha(t_n)$, $h_n = h_{q^*q_n}$, $E_n = \mathbb{A}_{q^*} \setminus S_{q^*q_n}$ in getting the right continuity of $\alpha$, $\lim_{t\downarrow t^*} q = q^*$, and local uniform right convergence $\lim_{t\downarrow t^*} h_{q^*q}(z) = z$, $z \in \mathbb{A}_{q^*}$. By combining the last property with the continuity of Villat's kernel $\mathcal{K}_q$, we can obtain the desired right differentiability of $g_q(z)$.

Komatu [26,28] derived the equation (7.5) for $g_q \circ g_Q^{-1}$ in place of $g_q$ by using the Weierstrass elliptic function and Jacobi elliptic function in place of Villat's kernel $\mathcal{K}_q$. But no rigorous proof of the right continuity of $\alpha$ and the right differentiability of $g_q$ were presented. As extensions of [26], we would like to mention the works by Goluzin [20] and N.A. Lebedev [34]. Recently the Loewner equation for annulus is being investigated under a much more general setting ([12]), which is so general that the equation seems to be derived only for those parameters outside a set of Lebesgue measure zero.

[16, Thm. 4.1] further considers, in place of the family $\{\gamma(0,t]\}$ of growing Jordan arcs a more general right continuous family$\{F_t\}$ of growing hulls, as appears in Theorem 9.2(iv) below, and derives the right continuity of $q = \alpha(t)$ and the equation (7.5) in the right derivative sense by making use of Proposition 7.4.

Since $\alpha$ in Theorem 7.3 is continuous, we can reparametrize the Jordan arc $\gamma$ by using $q$ as $\{\gamma(q) : Q \le q \le Q_\gamma\}$, with $g_q(z)$ being a conformal map from $\mathbb{A}_Q \setminus \gamma[0,q]$ onto $\mathbb{A}_q$ satisfying the above-mentioned normalization condition.

We can further put $P = -\log Q$, $P_\gamma = -\log Q_\gamma$, $\mathbf{S}_p(z,\zeta) = \mathcal{K}_{e^{-p}}(z,\zeta)$ and change the parameter from $q$ to $s$ by $q = e^{s-P}$, $0 \le s \le s_\gamma = P - P_\gamma$. Then the equation (7.5) is converted into

$$\frac{\partial \log g_s(z)}{\partial s} = \mathbf{S}_{P-s}(g_s(z), \lambda(s)) - i\Im \mathbf{S}_{P-s}(e^{s-P}, \lambda(s)), \quad g_0(z) = z, \tag{7.6}$$

for $z \in \mathbb{A}_Q \setminus \gamma[0,s]$ and $s \in [0, s_\gamma]$. Here $g_s$ is a conformal map from $\mathbb{A}_Q \setminus \gamma[0,s]$ onto $\mathbb{A}_{Qe^s}$ satisfying $g_s(Q) = Qe^s$ and $\lambda(s) = g_s(\gamma(s))$.

D. Zhan [44, Prop.2.1] stated that the equation (7.6) holds for a family $\{F_s\}$ of growing hulls on $\mathbb{A}_Q$ satisfying a Pommerenke type condition [32] and a normalization condition on a relevant capacity of the set $F_s$. Accordingly, an SLE$_\kappa$ for the annulus was formulated in [44] based on the equation (7.6) but with the second term of its righthand side being dropped off. In this connection, see section 6.1 of my joint paper [10] with Z.-Q.Chen and H.Suzuki.

## 8. Equation of slit motions and its unique solution for a given $\xi(t)$

In the rest of this exposition, we return to the setting adopted in sections 3, 4 and 5 and consider a standard slit domain $D = \mathbb{H} \setminus \bigcup_{j=1}^N C_j$. Let $\gamma$ be a Jordan arc satisfying (3.5), and for each $t \in [0, t_\gamma]$, let $g_t$ be a conformal map from $D \setminus \gamma[0,t]$ onto some standard slit domain $\mathbb{H} \setminus \bigcup_{j=1}^N C_j(t)$ satisfying the hydrodynamic normalization condition (3.6). By virtue of Theorem 5.4, $\gamma$ admits the half-plane capacity reparametrization making $a_t$ of (3.6) equal to $2t$. Under this parametrization, $g_t(z)$ is, for each $z \in (D \cup \partial_p K) \setminus \gamma[0, t_\gamma]$, differentiable in $t \in [0, t_\gamma]$ and satisfies the Komatu-Loewner differential equation

$$\frac{\partial g_t(z)}{\partial t} = -2\pi \Psi_t(g_t(z), \xi(t)), \quad g_0(z) = z, \quad 0 < t \le t_\gamma. \tag{8.1}$$

**Theorem 8.1** ([8, Thm. 2.3])**.** *For each $1 \le j \le N$, the two endpoints $z_j(t) = x_j(t) + iy_j(t)$, $z_j'(t) = x_j'(t) + iy_j(t)$ of the slit $C_j(t)$ satisfy the following equations:*

$$\begin{cases} dy_j(t)/dt = -2\pi \Im \Psi_t(z_j(t), \xi(t)), \\ dx_j(t)/dt = -2\pi \Re \Psi_t(z_j(t), \xi(t)), \\ dx_j'(t)/dt = -2\pi \Re \Psi_t(z_j'(t), \xi(t)). \end{cases} \tag{8.2}$$

Denote the endpoints of $C_j$ by $z_j$, $z_j'$. If it holds that

$$g_t(z_j) = z_j(t), \qquad g_t(z_j') = z_j'(t), \tag{8.3}$$

then (8.2) immediately follows from (8.1) by substituting $z = z_j$, $z = z_j'$. Equations (8.3) are not true in general however. Theorem 8.1 can be proved as a consequence of the Komatu-Loewner equation (8.1) after a more detailed consideration described below.

Since $g_t$ can be extended to a homeomorphic map between $\partial_p C_j$ and $\partial_p C_j(t)$, there exists, for each $t \in [0, t_\gamma]$, $1 \le j \le N$, unique points $\widetilde{z}_j(t)$, $\widetilde{z}'_j(t)$ satisfying

$$\begin{cases} \widetilde{z}_j(t) = \widetilde{x}_j(t) + iy_j \in \partial_p C_j, & g_t(\widetilde{z}_j(t)) = z_j(t), \\ \widetilde{z}'_j(t) = \widetilde{x}'_j(t) + iy_j \in \partial_p C_j, & g_t(\widetilde{z}'_j(t)) = z'_j(t). \end{cases}$$

The proof of the theorem is carried out in three cases about the location of $\widetilde{z}_j(t)$:

$$\textbf{(i)}\ \widetilde{z}_j(t) \in C_j^+ \setminus \{z_j, z'_j\},$$
$$\textbf{(ii)}\ \widetilde{z}_j(t) \in C_j^- \setminus \{z_j, z'_j\},$$
$$\textbf{(iii)}\ \widetilde{z}_j(t) \in \partial_p C_j \cap B(z_j, \varepsilon), z'_j \notin B(z_j, \varepsilon).$$

The proof for the case **(i)** is as follows.

For the left and right endpoints $z_j = a + ic$, $z'_j = b + ic$ of $C_j$, consider rectangles

$$R_+ = \{z : a < x < b,\ c < y < c + \delta\},\ R_- = \{z : a < x < b,\ c - \delta < y < c\},\ \delta > 0,$$

satisfying $R_+ \cup R_- \subset D \setminus \gamma[0, t_\gamma]$ and put $R = R_+ \cup C_i \cup R_-$. As $\Im g_t(z)$ takes a constant value on $C_j$, $g_t$ can be extended to an analytic function $g_t^+$ on $R$ from $R_+$ across $C_j$ by the reflection principle. By making use of Theorems 4.2(iii), 4.4, and the Cauchy integral formula, we can show that $\partial_t g_t^+(z)$, $(g_t^+)'(z)$, $(g_t^+)''(z)$ are continuous in $(t, z) \in [0, t_\gamma) \times R$ and further, by repeating analogous computations to section 3, that $(g_t^+)'(z)$ is differentiable in $t$ and $\partial_t (g_t^+)'(z)$ is continuous in $(t, z) \in [0, t_\gamma) \times R$. In particular, $h(t, z) = (g_t^+)'(z)$ becomes a $C^1$-class function of $(t, z) \in (0, t_\gamma) \times R$.

On the other hand, as $z_j(t)$ is the endpoint of the slit $C_j(t)$, it follows that $g_t^+(z) - z_j(t)$ has a zero of order 2 at $\widetilde{z}_j(t) \in C_j \setminus \{z_j, z'_j\}$:

$$(g_t^+)'(\widetilde{z}_j(t)) = 0, \qquad (g_t^+)''(\widetilde{z}_j(t)) \neq 0. \tag{8.4}$$

Accordingly we can apply the implicit function theorem to $h(t, z)$ in concluding that $\widetilde{z}(t)$ is of $C^1$-class in a neighborhood of $t$. By noting (8.1), (8.4), we thus have

$$\frac{d}{dt} z_j(t) = \frac{d}{dt}(g_t^+)(\widetilde{z}_j(t)) = \partial_t g_t^+(\widetilde{z}_j(t)) + (g_t^+)'(\widetilde{z}_j(t)) \frac{d}{dt}\widetilde{z}_j(t) = -2\pi \Psi_t(z_j(t), \xi(t)).$$

The same proof works for the case **(ii)**. In the case **(iii)**, we map this region by $\psi(z) = (z - z_j)^{1/2}$ onto $B(0, \sqrt{\varepsilon}) \cap \mathbb{H}$ and extend $f_t(z) = g_t \circ \psi^{-1}(z) = g_t(z^2 + z_j)$ to $B(0, \sqrt{\varepsilon}) \setminus \{0\}$ by the reflection principle. As $\Im f_t(z)$ is bounded in a neighborhood of the origin, $f_t$ becomes analytic on $B(0, \sqrt{\varepsilon}) \cap \mathbb{H}$. We can then make the same argument as above for $(f_t, B(0, \sqrt{\varepsilon}))$ in place of $(g_t^+, R)$.

The equation (8.2) of slit motions was first obtained by Bauer-Friedrich [3, 4] in the cases **(i)**, **(ii)** by taking for granted the $C^1$-class property of $h(t, z)$. As (8.2) is eventually a consequence of (8.1), we call (8.2) the *Komatu-Loewner equation for slits*.

Now let us recall the collection $\mathcal{D}$ of all labelled standard slit domains introduced in the second half of section 3. $\mathcal{D}$ is a metric space with distance (3.12). It is convenient to consider an open subset $\mathcal{S}$ of $\mathbb{R}^{3N}$ defined by

$$\begin{aligned} \mathcal{S} \;=\; \Big\{ &\mathbf{s} := (\mathbf{y}, \mathbf{x}, \mathbf{x}') \in \mathbb{R}^{3N} :\ \mathbf{y},\ \mathbf{x},\ \mathbf{x}' \in \mathbb{R}^N,\ \mathbf{y} > \mathbf{0},\ \mathbf{x} < \mathbf{x}', \\ &\text{either } x'_j < x_k \text{ or } x'_k < x_j \text{ whenever } y_j = y_k,\ j \neq k \Big\}. \end{aligned} \tag{8.5}$$

The space $\mathcal{D}$ can be identified with $\mathcal{S}$ as a topological space. The element of $\mathcal{S}$ corresponding to $D \in \mathcal{D}$ will be denoted by $\mathbf{s}(D)$, while the element of $\mathcal{D}$ corresponding to $\mathbf{s} \in \mathcal{S}$ will be denoted by $D(\mathbf{s})$. In particular, for $\mathbf{s} = (\mathbf{y}, \mathbf{x}, \mathbf{x}')$, $z_j = x_j + iy_j$, $z_j' = x_j' + iy_j$ are left and right endpoints of the $j$-th slit $C_j$ of $D(\mathbf{s}) \in \mathcal{D}$.

For $\mathbf{s} = (\mathbf{s}_1, \ldots, \mathbf{s}_{3N}) \in \mathcal{S}$, we denote by $\Psi_{\mathbf{s}}(z, \xi)$ the BMD complex Poisson kernel of $D(\mathbf{s})$. We can then rewrite the Komatu-Loewner equation (8.2) for slits as follows:

$$\mathbf{s}_j(t) - \mathbf{s}_j(0) = \int_0^t b_j(\xi(s), \mathbf{s}(s))ds, \quad t \geq 0, \quad 1 \leq j \leq 3N, \tag{8.6}$$

where, for $\xi \in \mathbb{R}$, $\mathbf{s} \in \mathcal{S}$,

$$b_j(\xi, \mathbf{s}) = \begin{cases} -2\pi\Im\Psi_{\mathbf{s}}(z_j, \xi), & 1 \leq j \leq N, \\ -2\pi\Re\Psi_{\mathbf{s}}(z_j, \xi), & N+1 \leq j \leq 2N, \\ -2\pi\Re\Psi_{\mathbf{s}}(z_j', \xi), & 2N+1 \leq j \leq 3N. \end{cases} \tag{8.7}$$

We let $\widetilde{b}_j(\mathbf{s}) = b_j(0, \mathbf{s})$, $1 \leq j \leq 3N$. Then the function $b_j(\xi, \mathbf{s})$ has the following homogeneity in the direction of the $x$-axis. For $\xi \in \mathbb{R}$, denote by $\widehat{\xi}$ the $3N$-vector with the first $N$ components equal to 0 and the next $2N$ components equal to $\xi$. Then we have

$$b_j(\xi, \mathbf{s}) = \widetilde{b}_j(\mathbf{s} - \widehat{\xi}), \quad \xi \in \mathbb{R}, \quad \mathbf{s} \in \mathcal{S}, \tag{8.8}$$

which follows from the shift invariance of BMD in the $x$-direction and the characterization (3.3) of the BMD Poisson kernel. Thus the equation (8.2) for slits can be further rewritten in terms of the function $\widetilde{b}_j(\mathbf{s})$ on $\mathcal{S}$ as

$$\mathbf{s}_j(t) - \mathbf{s}_j(0) = \int_0^t \widetilde{b}_j(\mathbf{s}(s) - \widehat{\xi}(s))ds, \quad t \geq 0, \quad 1 \leq j \leq 3N. \tag{8.9}$$

Let us consider the following local Lipschitz condition for a real function $f = f(\mathbf{s})$ on $\mathcal{S}$:

**(L)** For any $\mathbf{s}^{(0)} \in \mathcal{S}$ and for any finite open interval $J \subset \mathbb{R}$, there exist a neighborhood $U(\mathbf{s}^{(0)}) \subset \mathcal{S}$ of $\mathbf{s}^{(0)}$ and a constant $L > 0$ such that

$$|f(\mathbf{s}^{(1)} - \widehat{\xi}) - f(\mathbf{s}^{(2)} - \widehat{\xi})| \leq L\,|\mathbf{s}^{(1)} - \mathbf{s}^{(2)}| \quad \forall \mathbf{s}^{(1)}, \mathbf{s}^{(2)} \in U(\mathbf{s}^{(0)}) \quad \forall \xi \in J. \tag{8.10}$$

**Proposition 8.2** ([8, Lem. 4.1])**.**

(i) $\widetilde{b}_j(\mathbf{s})$, $1 \leq j \leq 3N$, *satisfy the condition* **(L)**.

(ii) *Given any real continuous function* $\xi(t)$ *on* $[0, \infty)$ *and any* $\mathbf{s}^{(0)} \in \mathcal{S}$, *there exists a unique solution* $\mathbf{s}(t)$ *of the Komatu-Loewner equation* (8.9) *for slits satisfying* $\mathbf{s}(0) = \mathbf{s}^{(0)}$.

(i) follows immediately from Theorem 5.1. (ii) is a consequence of (i) [22].

## 9. Komatu-Loewner evolution driven by a continuous function $\xi(t)$

Given an arbitrary real continuous function $\xi(t)$, $t \in [0, \infty)$, let $\mathbf{s}(t)$, $t \in [0, \zeta)$ be the unique solution of the Komatu-Loewner equation (8.9) for slits with the right maximal interval $[0, \zeta)$ of existence according to Proposition 8.2. We write $D_t = D(\mathbf{s}(t)) \in \mathcal{D}$, $t \in [0, \zeta)$, and put

$$G = \bigcup_{t \in [0,\zeta)} \{t\} \times D_t. \tag{9.1}$$

Since $t \mapsto D_t$ is continuous, $G$ is a domain of $[0, \zeta) \times \mathbb{H}$. For $\tau \in [0, \zeta)$, $z_0 \in D_\tau$, we consider the solution of the Cauchy problem

$$\frac{d}{dt} z(t) = -2\pi \Psi_{\mathbf{s}(t)}(z(t), \xi(t)), \tag{9.2}$$

$$z(\tau) = z_0 \in D_\tau. \tag{9.3}$$

**Lemma 9.1** ([8, Prop. 5.1]).

(i) $\Psi_{\mathbf{s}(t)}(z, \xi(t))$ *is continuous in* $(t, z) \in G$.

(ii) $\Psi_{\mathbf{s}(t)}(z, \xi(t))$ *is locally Lipschitz continuous in* $z$ *in the following sense: for any* $(\tau, z_0) \in G$, *there exist* $t_0 > 0$, $\rho > 0$, *and* $L > 0$ *such that*

$$V = [(\tau - t_0)^+, \tau + t_0] \times \{z : |z - z_0| \le \rho\} \subset G$$

$$|\Psi_{\mathbf{s}(t)}(z_1, \xi(t)) - \Psi_{\mathbf{s}(t)}(z_2, \xi(t))| \le L\, |z_1 - z_2| \quad \forall\, (t, z_1),\ (t, z_2) \in V. \tag{9.4}$$

(iii) *For any* $\tau \in [0, \zeta)$ *and any* $z_0 \in D_\tau$, *there exists a unique solution* $\{z(t); t \in (\tau - t_0, \tau + t_0) \cap [0, \zeta)\}$ *of* (9.2) *satisfying* (9.3).

(i) follows from the continuity of $t \mapsto D_t$ and Theorem 5.1 concerning a Lipschitz continuity of $\Psi$. (ii) follows from (i) together with the Cauchy integral formula. (iii) is a consequence of (ii).

This lemma assures the unique existence of the local solution of the equation (9.2) passing through the open region (9.1). In order to know the properties of the solution with the maximal interval of existence, we further make a detailed study of behaviors of the local solution of (9.2) passing through the domain $\widehat{G}$ broader than (9.1), where, for $D_t = \mathbb{H} \setminus K(t)$, $K(t) = \bigcup_{j=1}^N C_j(t)$,

$$\widehat{G} = \bigcup_{t \in [0, \zeta)} \{t\} \times (D_t \cup \partial_p K(t) \cup (\partial \mathbb{H} \setminus \{\xi(t)\})).$$

As a consequence, it turns out that the solution $z(t)$ passing through $G$ never approaches $\partial_p K(t)$. By noting that $\Im z(t)$ is a decreasing function of $t$ due to (9.2), this leads us to the first assertion of the following theorem. We denote $D_0 = D(\mathbf{s}(0)) \in \mathcal{D}$ by $D$.

**Theorem 9.2** ([8, Thms. 5.5, 5.8, 5.12]).

(1) *For each* $z \in D$, *the equation*

$$\partial_t g_t(z) = -2\pi \Psi_{\mathbf{s}(t)}(g_t(z), \xi(t)), \quad g_0(z) = z \in D \tag{9.5}$$

*admits a unique solution* $g_t(z)$, $t \in [0, t_z)$, *passing through* $G$. *Here* $[0, t_z)$, $t_z > 0$, *is its right maximal interval of existence. Furthermore*

$$\lim_{t \uparrow t_z} \Im g_t(z) = 0; \quad \textit{if } t_z < \zeta, \quad \text{then} \quad \lim_{t \uparrow t_z} |g_t(z) - \xi(t_z)| = 0. \tag{9.6}$$

(2) *Let* $F_t = \{z \in D : t_z \le t\}$, $t > 0$. $F_t$ *is a half-plane hull in the following sense:* $F_t$ *is a bounded relatively closed set of* $\mathbb{H}$, *and* $\mathbb{H} \setminus F_t$ *is simply connected.*

(3) $g_t$ *is a conformal map from* $D \setminus F_t$ *onto* $D_t$ *satisfying the hydrodynamic normalization condition* (3.6) *with the half-plane capacity* $a_t = 2t$.

(4) *The family $\{F_t\}$ of growing hulls enjoys the right continuity property with limit $\xi(t)$ in the following sense:*

$$\bigcap_{\delta>0} \overline{g_t(F_{t+\delta} \setminus F_t)} = \{\xi(t)\}, \quad t \in [0, \zeta). \tag{9.7}$$

The family $\{F_t\}$ of growing hulls in Theorem 9.2 is called the *Komatu-Loewner evolution driven by* the continuous function $\xi(t)$. We have produced a pair $(\xi(t), \mathbf{s}(t))$ first by giving $\xi(t)$ and then by solving the equation (8.9) of slits in $s(t)$.

Instead of taking this procedure, we can first give real functions $\alpha, b$ on $\mathcal{S}$ satisfying the Lipschitz condition **(L)** and find a strong solution $(\xi(t), \mathbf{s}(t))$ of the system of stochastic differential equations that is obtained by combining the equation

$$\xi(t) = \xi + \int_0^t \alpha(\mathbf{s}(s) - \widehat{\xi}(s)) dB_s + \int_0^t b(\mathbf{s}(s) - \widehat{\xi}(s)) ds, \qquad B_s \text{ is a Brownian motion,} \tag{9.8}$$

with the equation (8.9) for slits. We can then create the family $\{F_t\}$ of random growing hulls driven by the random process $(\xi(t), \mathbf{s}(t))$ via its substitution into (9.5).

This random family is denoted by $\mathrm{SKLE}_{\alpha,b}$ and is called a *stochastic Komatu-Loewner evolution* ([8, 10]). In relation to the requirements of the domain Markov property and a conformal invariance of the distribution for the random growing hulls, it is shown in [8, §3] that the diffusion coefficient $\alpha$ (resp., drift coefficient $b$) ought to be a homogeneous function of degree 0 (resp., $-1$).

## References


[1] L. V. Ahlfors, *Complex Analysis*, McGraw-Hill, 1979 MR510197

[2] L.V. Ahlfors, *Conformal Invariants*, AMS, 2010 MR2730573

[3] R. O. Bauer and R. M. Friedrich, On radial stochastic Loewner evolution in multiply connected domains, *J. Funct. Anal.* **237**(2006), 565-588 MR2230350

[4] R. O. Bauer and R. M. Friedrich, On chordal and bilateral SLE in multiply connected domains, *Math. Z.* **258**(2008), 241-265 MR2357634

[5] C. Boehm and W. Lauf, A Komatu-Loewner equation for multiple slits, *Computational Methods in Function Theory* **14**(2014), 639-660, Springer MR3274893

[6] Z.-Q. Chen and M. Fukushima, *Symmetric Markov Processes, Time Change, and Boundary Theory*, Princeton University Press, 2012 MR2849840

[7] Z.-Q. Chen and M. Fukushima, One-point reflection, *Stochastic Process Appl.* **125**(2015), 1368-1393 MR3310351

[8] Z.-Q. Chen and M. Fukushima, *Stochastic Komatu-Loewner evolutions and BMD domain constant*, Stochastic Process. Appl. **128** (2018), no. 2, 545–594, DOI 10.1016/j.spa.2017.05.007. MR3739508

[9] Z.-Q. Chen, M. Fukushima and S. Rhode, Chordal Komatu-Loewner equation and Brownian motion with darning in multiply connected domains, *Trans. Amer. Math. Soc.* **368**(2016), 4065-4114 MR3453365

[10] Z.-Q. Chen, M. Fukushima and H. Suzuku, Stochastic Komatu-Loewner evolutions and SLEs, *Stochastic Processes and their Applcations* **127**(2017), 2068-2087 MR3646440

[11] J. B. Conway, *Functions of One Complex Variable* II, Springer, 1995 MR1344449

[12] M. D. Contreras, S. Díaz-Madrigal, and P. Gumenyuk, *Loewner theory in annulus I: Evolution families and differential equations*, Trans. Amer. Math. Soc. **365** (2013), no. 5, 2505–2543, DOI 10.1090/S0002-9947-2012-05718-7. MR3020107

[13] S. Drenning, *Excursion reflected Brownian motion and loewner equations in multiply connected domains*, ProQuest LLC, Ann Arbor, MI, 2011. Thesis (Ph.D.)–The University of Chicago. MR2992562

[14] P. L. Duren, *Univalent Functions*, Springer, 1983 MR708494

[15] M. Fukushima, On boundary conditions for multi-dimensional Brownian motions with symmetric resolvent densities, *J. Math. Soc. Japan* **21**(1969), 58-93 MR236998

[16] M. Fukushima and H. Kaneko, On Villat's kernels and BMD Schwarz kernels in Komatu-Loewner equations, in: *Stochastic Analysis and Applications 2014*, Springer Proc. in Math. and Stat. Vol. 100 (Eds.) D. Crisan, B. Hambly, T. Zariphopoulous, 2014, pp. 327-348 MR3332718

[17] M. Fukushima, Y. Oshima, and M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*, De Gruyter, 1994, Second Extended Edition, 2011 MR2778606

[18] M. Fukushima and H. Tanaka, Poisson point processes attached to symmetric diffusions. *Ann. Inst. H. Poincare Probab. Statist.* **41** (2005),419-459 MR2139028

[19] P. R. Garabedian, *Partial Differential Equations*, AMS Chelsia, 2007, republication of 1964 edition MR1657375

[20] G.M. Goluzin, On the parametric representation of functions univalent in a ring (in Russian), *Math. Sbornik N.S.* **29(71)** (1951), 469-476 MR0047780

[21] G.M. Goluzin, *Geometric Theory of Functions of a Complex Variable*, American Mathematical Society Translations 26, Providence, 1969 MR0247039

[22] P. Hartman, *Ordinary Differentail Equations*, Birkhäuser, 1982 MR658490

[23] D. Hilbert, Zur Theorie der konformen abbildung, N̂achr. Ges. Wiss. Göttingen (1906), 314-323

[24] K. Itô, Poisson point processes attached to Markov processes, Proc. Sixth Berkeley Symp. Math. Stat. Probab. **vol 3**, 1970, pp. 225–239. MR0402949

[25] Y. Komatu, Über einen Satz von Herren Löwner, *Proc. Imp. Akad. Tokyo* **16**(1940), 512-514 MR3811

[26] Y. Komatu, Untersuchungen über konforme Abbildung von zweifach zusammenhängenden Gebieten, *Proc. Phys.-Math. Soc. Japan* **25**(1943), 1-42. MR16129

[27] Y. Komatu, *Theory of Conformal Mappings* I (in Japanese), Kyoritsu Pub. Co., 1944

[28] Y. Komatu, *Theory of Conformal Mappings* II (in Japanese), Kyoritsu Pub. Co., 1949

[29] Y. Komatu, On conformal slit mapping of multiply-connected domains, *Proc. Japan Acad.* **26**(1950), 26-31 MR46437

[30] G. F. Lawler, Conformally Invariant Processes in the Plane, Mathematical Surveys and Monographs, AMS, 2005 MR2129588

[31] G. F. Lawler, The Laplacian-$b$ random walk and the Schramm-Loewner evolution, *Illinois J. Math.* **50** (2006), 701-746 (Special volume in memory of Joseph Doob) MR2247843

[32] G. Lawler, O. Schramm, and W. Werner, Values of Brownian intersectiong exponents, *Acta Mathematica* **187**(2001), 237-273 MR1879850

[33] G. Lawler, O. Schramm, and W. Werner, Conformal restriction: the chordal case, *J. Amer. Math. Soc.* **16**(2003), 917-955 MR1992830

[34] N.A. Lebedev, On parametric representation of functions regular and univalent in a ring (in Russian), *Dokl. Akad. Nauk SSSR* **103**(1955), 767-768 MR0072950

[35] K. Löwner, Untersuchungen über schlichte konforme Abbildungen des Einheitskreises I, *Math. Ann.* **89**(1923), 103-121 MR1512136

[36] S. C. Port and C. J. Stone, *Brownian Motion and Classical Potential Theory*, Academic Press, 1978 MR0492329

[37] S. Rohde and O. Schramm, Basic properties of SLE, *Ann. Math.* **161**(2005), 879-920 MR2153402

[38] O. Schramm, Scaling limits of loop-erased random walks and uniform spanning trees, *Israel J. Math.* **118**(2000), 221-288 MR1776084

[39] S. Smirnov, Critical percolation in the plane: Conformal invariance, Cardy's formula, scaling limits, *C.R. Acad. Sci. Paris Sér. I Math.* **333**(2001), 239-244 MR1851632

[40] M. Tsuji, *Potential Theory in Modern Function Theory*, Marzen, Tokyo, 1959 MR0114894

[41] H. Villat, Le problème de Dirichlet dans une aire anulare, *Rend. de. Circ. mat. di Palermo* **33**(1912), 134-157

[42] H. Villat, *Lecons sur l'hydrodynamique*, Gautiers-Villas, 1929, Paris

[43] W. Werner, *Random Planar Curves and Schramm-Loewner Evolutions*, Lecture Notes in Math. **1840**, Springer, 2004 MR2079672

[44] D. Zhan, Stochastic Loewner evolution in doubly connected domains, *Probab. Theory Relat. Fields* **129**(2004), 340-380 MR2128237

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