Dynkin games via Dirichlet forms and singular control of one-dimensional diffusions*

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Abstract

We consider a zero-sum game of optimal stopping in which each of the opponents has the right to stop a one dimensional diffusion process. There are two types of costs. The first is accumulated continuously at the rate $H(X_t)$ where X_t is the current position of the process. In addition there is a cost associated with the stopping of the process. It is given by the function $f_1(x)$ for the first player and the function $f_2(x)$ for the second player, where x is the position of the process when the stopping option is exercised.

We study the solution of the free boundary problem associated with this game via Dirichlet forms on the appropriate functional space. Integrating the value function of the game we get a solution to another free boundary problem which yields the optimal return function for a singular stochastic control problem.

1 Introduction

The reflecting diffusion processes are interesting objects to be studied from a variety of different points of view . In particular, the reflecting Brownian motion on a one dimensional interval was characterized as a solution of a singular control problem ([HT 83, T 85]). More specifically, let (w_t, P) be a one dimensional standard Brownian motion starting at the origin and let

$$X_t = x + \sigma w_t + \mu t + A_t^{(1)} - A_t^{(2)} \qquad x \in \mathbb{R}, \tag{1.1}$$

where $\sigma \neq 0,\ \mu$ are constants and $\mathbb{S} = (A_t^{(1)}, A_t^{(2)})$ is a pair of non-anticipating increasing processes. $\mathbb{S}$ represents a strategy under which the cost function

$$k_x(\mathbb{S}) = E\left(\int_0^\infty e^{-\alpha t} h(X_t)dt + \int_0^\infty e^{-\alpha t}(r dA_t^{(1)} + \ell dA_t^{(2)})\right) \tag{1.2}$$

is to be minimized. Here, α, r, ℓ are preassigned positive constants and $h(x)$ is a given convex function taking its minimum at the origin. It was then shown by Taksar [T 85]

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that there exists an optimal strategy $\tilde{\mathbb{S}}$ such that

$$W(x) = \min_{\mathbb{S}} k_x(\mathbb{S}) = k_x(\tilde{\mathbb{S}}),$$

and that actually $\tilde{\mathbb{S}}$ is equal to (ℓ_t^a, ℓ_t^b) where ℓ^a, ℓ^b are local times at points a, b for uniquely determined $a, b,\ a < 0 < b$. Thus the corresponding optimal process (1.1) is the reflecting diffusion on the closed interval $[a, b]$. The proof in [T 85] was carried out by solving a related free boundary problem by making use of a solution of an optimal stopping game problem, which had been formulated by Gusein-Zade [G 69].

The purpose of the present paper is to extend those results in [T 85] by replacing the process $x + \sigma w_t + \mu t$ appearing in (1.1) on the one hand and constant costs r, ℓ appearing in (1.2) on the other, with a more general diffusion process governed by variable C^1-coefficients $\sigma(x), \mu(x)$ and with variable costs $f_1(x), f_2(x)$, respectively. To this end, we shall employ the Dynkin optimal stopping game and its Dirichlet form characterization due to Zabczyk [Z 84]. As will be explained in §2, the value function of the Dynkin game for a general symmetric Hunt process was identified in [Z 84] with the solution of a certain variational inequality in a regular Dirichlet space setting. Such an identification had been established by Nagai [N 78] for a one-sided optimal stopping problem. This sort of an analytic characterization of the stopping game was missing in [G 69], making the usage of [G 69] less simple.

We can then proceed along almost the same line as in [T 85] in getting the solution of our singular control problem. However, it is more useful to rewrite the infinitesimal generator $\frac{1}{2}\sigma(x)^2 \frac{d^2}{dx^2} + \mu(x)\frac{d}{dx}$ of the controlled diffusion in the Feller canonical form $\frac{d}{dm}\frac{d}{ds}$. The conditions on the data h, f_1, f_2 will be stated in terms of the intrinsic quantities s and m.

In §3, we shall apply the Dynkin game description of the solution V of a variational inequality presented in [Z 84] to a one dimensional diffusion with generator $\frac{d}{ds}\frac{d}{dm}$ in showing that an integral function W of V with respect to ds is a solution of a certain free boundary problem involving the operator $\frac{d}{dm}\frac{d}{ds}$, which will then be identified in §4 with the optimal return function of our singular control of the (σ, μ)-diffusion. The admissible processes X_t to be optimized will be formulated in §4 by SDE variants of the identity (1.1) and the optimal process will be shown to be the reflecting (σ, μ)-diffusion on the interval specified in the free boundary problem.

We emphasize that our Dirichlet form approach automatically guarantees the quasi-continuity (actually the absolute continuity in the present one-dimensional application) of the value function V, which, combined with the saddle point characterization of V, readily implies that its integral function W is the classical solution of the free boundary problem. As a result we get a classical solution to the one dimensional singular stochastic control problem as opposed to the viscosity solution guaranteed by a general theory (see [FS 93]).

A slight extension of [T 85] has been considered by Kawabata [K 98], where the costs r, ℓ were still kept constant however and the method of [Z 84] was not utilized.

In a recent paper [KW 01], Karatzas and Wang obtain the same relation as in our case between the value functions of a Dynkin game and a control problem of general bounded variation processes. The method in [KW 01] is more direct and pathwise, but the admissible process to be optimized is purely of bounded variation and the leading martingale part like in our case is absent.

In what follows, $C^k(I)$ (resp. $C_0^k(I)$) will denote the space of k-times continuously differentiable functions (resp. with compact support) on an interval $I \subset \mathbb{R}$, $k = 1, 2$.

2 Dynkin games via Dirichlet forms

Let X be a locally compact separable metric space and m be a positive Radon measure on X with full support. $L^2(X;m)$ denotes the real L^2-space with inner product $(\cdot,\cdot)$. We consider a Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(X;m)$. By definition, $\mathcal{E}$ is a closed symmetric form with domain $\mathcal{F}$ dense in $L^2(X;m)$ such that the unit contraction operates on it:

$$u \in \mathcal{F} \Longrightarrow v = 0 \vee u \wedge 1 \in \mathcal{F},\ \mathcal{E}(v,v) \leq \mathcal{E}(u,u).$$

Recall that a closed symmetric form is a Dirichlet form if and only if the associated L^2-semigroup $\{T_t, t > 0\}$ is Markovian in the sense that

$$0 \leq f \leq 1 \quad f \in L^2 \Longrightarrow 0 \leq T_t f \leq 1.$$

We let $\mathcal{E}_\alpha(u,v) = \mathcal{E}(u,v) + \alpha(u,v)$ for $\alpha > 0$. We assume that the Dirichlet form $(\mathcal{E},\mathcal{F})$ is regular in the sense that $\mathcal{F} \cap C_0(X)$ is $\mathcal{E}_1$-dense in $\mathcal{F}$ and uniformly dense in $C_0(X)$, where $C_0(X)$ denotes the space of continuous functions on X with compact support. There exists then a Hunt process (a right continuous, quasi-left continuous strong Markov process) $\mathbf{M} = (X_t, P_x)$ on X such that

$$p_t f(x) = E_x(f(X_t)), \qquad x \in X,$$

is a version of $T_t f$ for all $f \in C_0(X)$ [FOT 94].

In what follows, basic notions and relations concerning the regular Dirichlet form $(\mathcal{E},\mathcal{F})$ and the associated Hunt process $\mathbf{M}$ shall be taken from [FOT 94]. In particular, the L^2-resolvent $\{G_\alpha,\ \alpha > 0\}$ associated with the Dirichlet form $(\mathcal{E},\mathcal{F})$ satisfies

$$G_\alpha f \in \mathcal{F}, \quad \mathcal{E}_\alpha(G_\alpha f, v) = (f,v) \quad \forall f \in L^2(X;m),\ \forall v \in \mathcal{F},$$

and further the resolvent $\{R_\alpha\ \alpha > 0\}$ of the Hunt process $\mathbf{M}$ defined by

$$R_\alpha f(x) = E_x\left(\int_0^\infty e^{-\alpha t} f(X_t)dt\right) \qquad x \in X,$$

is a quasicontinuous modification of $G_\alpha f$ for any Borel function $f \in L^2(X;m)$. For $v \in \mathcal{F}$, $\tilde{v}$ will denote a quasicontinuous modification of v.

Given $\alpha > 0$, $H \in L^2(X;m)$ and $f_1, f_2 \in \mathcal{F}$ with $-f_1 \leq f_2$, we let

$$K = \{u \in \mathcal{F} : -f_1 \leq u \leq f_2 \quad m\text{–a.e.}\}. \tag{2.1}$$

One looks for a solution $V \in K$ of the inequality

$$\mathcal{E}_\alpha(V, u - V) \geq (H, u - V) \quad \forall u \in K. \tag{2.2}$$

Such a variational inequality arises in various contexts and it goes back to Stampacchia[S 64].

Proposition 2.1 *There exists a unique function $V \in K$ satisfying* (2.2)*.*

Proof. This is a well known fact but we reproduce a proof given by Nagai [N 78] in a way convenient for later use. First consider the special case that $H = 0$. We can then see the equivalence of the next inequalities holding for $V \in K$:

$$\mathcal{E}_\alpha(V, u - V) \geq 0 \qquad \forall u \in K, \tag{2.3}$$

$$\mathcal{E}_\alpha(V, V) \leq \mathcal{E}_\alpha(u, u) \qquad \forall u \in K. \tag{2.4}$$

In fact, (2.3) readily implies (2.4) by the Schwarz inequality. Conversely suppose (2.4). Take any $u \in K$ and put $w = u - V$. Since K is convex,

$$V + \epsilon w = (1 - \epsilon)V + \epsilon u \in K \quad \forall \epsilon \in (0, 1).$$

(2.4) then leads us to

$$\mathcal{E}_\alpha(V, V) \leq \mathcal{E}_\alpha(V + \epsilon w, V + \epsilon w)$$

and $2\mathcal{E}_\alpha(V, w) + \epsilon\mathcal{E}_\alpha(w, w) \geq 0$. We get (2.3) by letting $\epsilon \downarrow 0$.

Now (2.4) (and equivalently (2.3)) has a unique solution $V \in K$ by virtue of the closedness of the convex set K and the parallelogram law (see for instance the proof of [FOT 94, Lemma 2.1.2]).

Next consider a general $H \in L^2(X; m)$. By making use of the L^2-resolvent G_α, we can rewrite the inequality (2.2) as

$$\mathcal{E}_\alpha(V - G_\alpha H, (u - G_\alpha H) - (V - G_\alpha H)) \geq 0.$$

in concluding that the solution V of (2.1) and (2.2) is related to the solution V^0 of

$$K^0 = \{u \in \mathcal{F} : -h_1 \leq u \leq h_2 \quad m\text{–a.e.}\}, \quad h_1 = f_1 + G_\alpha H, \quad h_2 = f_2 - G_\alpha H, \tag{2.5}$$

$$V^0 \in K^0, \quad \mathcal{E}_\alpha(V^0, u - V^0) \geq 0, \quad \forall u \in K^0, \tag{2.6}$$

by the relation

$$V = V^0 + G_\alpha H. \tag{2.7}$$

□

J.Zabczyk has related the solution of the variational inequality (2.2) to the value function of an optimal stopping game (called a *Dynkin game* after [D 67]) for the associated Hunt process $\mathbf{M} = (X_t, P_x)$ in the following manner ([Z 84, Theorem 1]).

Theorem 2.1 (Zabczyk) *For any Borel function $H \in L^2(X;m)$ and for any $f_1, f_2 \in \mathcal{F}$ with $-f_1 \le f_2$, we put*

$$\begin{aligned} J_x(\tau,\sigma) &= E_x\left(\int_0^{\tau\wedge\sigma} e^{-\alpha t} H(X_t)dt\right) \\ &+ E_x\left(e^{-\alpha(\tau\wedge\sigma)}(-I_{\sigma\le\tau}\tilde{f}_1(X_\sigma) + I_{\tau<\sigma}\tilde{f}_2(X_\tau))\right) \end{aligned} \tag{2.8}$$

for $x \in X$ and for finite stopping times τ, σ. Then the solution of (2.1) *and* (2.2) *admits as its quasicontinuous version the value function of the game*

$$V(x) = \inf_\tau \sup_\sigma J_x(\tau,\sigma) = \sup_\sigma \inf_\tau J_x(\tau,\sigma), \qquad x \in X \setminus N, \tag{2.9}$$

where N is some properly exceptional set with respect to **M**.

Furthermore if we let

$$E_1 = \{x \in X - N : V(x) = -\tilde{f}_1(x)\}, \quad E_2 = \{x \in X - N : V(x) = \tilde{f}_2(x)\},$$

then the hitting times $\hat{\tau} = \sigma_{E_2}$, $\hat{\sigma} = \sigma_{E_1}$ are the saddle point of the game:

$$J_x(\hat{\tau},\sigma) \le J_x(\hat{\tau},\hat{\sigma}) \le J_x(\tau,\hat{\sigma}) \tag{2.10}$$

for any $x \in X - N$ and for any stopping times τ, σ. In particular

$$V(x) = J_x(\hat{\tau},\hat{\sigma}) \quad \forall x \in X \setminus N. \tag{2.11}$$

.

Actually this theorem was shown in [Z 84] only when $H = 0$. However, on account of the proof of Proposition 2.1, the statements of Theorem 2.1 for a general Borel function $H \in L^2(X;m)$ can be reduced to this special case. In fact, by what was proved in [Z 84], the solution of (2.5) and (2.6) admits a quasicontinuous version given by

$$V^0(x) = \inf_\tau \sup_\sigma J_x^0(\tau,\sigma) = \sup_\sigma \inf_\tau J_x^0(\tau,\sigma), \qquad x \in X \setminus N$$

where N is some properly exceptional set and

$$J_x^0(\tau,\sigma) = E_x\left(e^{-\alpha(\tau\wedge\sigma)}(-I_{\sigma\le\tau}\tilde{h}_1(X_\sigma) + I_{\tau<\sigma}\tilde{h}_2(X_\tau))\right), \ \ \tilde{h}_1 = \tilde{f}_1 + R_\alpha H, \ \tilde{h}_2 = \tilde{f}_2 - R_\alpha H.$$

In view of (2.7), the solution of (2.1) and (2.2) then admits a quasicontinuous version

$$V(x) = V^0(x) + R_\alpha H(x) \qquad x \in X \setminus N,$$

which in turn can be seen to satisfy the identity (2.9), because the Dynkin formula

$$R_\alpha H(x) - E_x\left(e^{-\alpha(\tau\wedge\sigma)} R_\alpha H(X_{\tau\wedge\sigma})\right) = E_x\left(\int_0^{\tau\wedge\sigma} e^{-\alpha t} H(X_t)dt\right),$$

leads us to

$$J_x^0(\tau,\sigma) + R_\alpha H(x) = J_x(\tau,\sigma).$$

The second statement of Theorem 2.1 is also an immediate consequence of that for V^0 and J^0.

We refer to [Z 84] for related literatures prior to [Z 84].

3 One dimensional Dynkin game and free boundary problems

When the underlying space X is one-dimensional, the solution V of the variational inequality (2.1),(2.2) can be described as a solution of a certain free boundary problem. The proof can be carried out using primarily its Dynkin game description (2.9) and (2.10).

More specifically, let $\dot{s}(x)$ and $\dot{m}(x)$ be strictly positive C^1-functions on $\mathbb{R}$. Denote the one-dimensional Lebesgue measure by dx and the measures $\dot{s}(x)dx$, $\dot{m}(x)dx$ by ds, dm respectively. We assume that both $-\infty$ and ∞ are natural (neither exit nor entrance) boundaries of $\mathbb{R}$ with respect to s, m in Feller's sense ([IM 74]):

$$\int_{-\infty<y<x<-1} ds(x)dm(y) = \infty \quad , \quad \int_{-\infty<y<x<-1} dm(x)ds(y) = \infty,$$
$$\int_{1<x<y<\infty} ds(x)dm(y) = \infty \quad , \quad \int_{1<x<y<\infty} dm(x)ds(y) = \infty. \tag{3.1}$$

For $A > 0$, we let

$$\begin{aligned} \mathcal{F} &= H^1((-A,A);dx) \\ &= \{u \in L^2((-A,A);dx) : u \text{ is absolutely continuous, } u' \in L^2((-A,A);dx)\} \end{aligned} \tag{3.2}$$

$$\mathcal{E}(u,v) = \int_{-A}^{A} u'(x)v'(x)\,\frac{1}{\dot{m}(x)}\,dx \qquad u,v \in \mathcal{F}. \tag{3.3}$$

We can and we shall regard $(\mathcal{E},\mathcal{F})$ as a regular local Dirichlet form on $L^2([-A,A];ds)$. The associated Hunt process $\mathbf{M} = (X_t, P_x)$ on the closed interval $[-A,A]$ is a conservative diffusion process, namely, a strong Markov process with continuous sample paths and infinite life time, and actually it is a reflecting barrier diffusion on $[-A,A]$ with infinitesimal generator $\frac{d}{ds}\frac{d}{dm}$. Since $\mathcal{F}$ is the ordinary Sobolev space $H^1(-A,A)$ on the one dimensional interval $(-A,A)$ and the metric $\mathcal{E}_1$ on it is equivalent to the square root of the Dirichlet integral plus L^2-norm, we see that each one point set $\{x\} \subset [-A,A]$ has a positive capacity, the quasicontinuity reduces to the ordinary continuity and $\mathbf{M}$ admits no non-empty exceptional set ([FOT 94, Example 2.1.2]).

We now let $V(x), x \in [-A,A]$, be the solution of the variational inequality (2.1),(2.2) for the present Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2([-A,A];ds)$ under the following assumptions on the data H, f_1, f_2:

Assumption 3.1 *$H(x)$ is a continuous function on $\mathbb{R}$ such that*

$$H(0) = 0,\ H(x) \text{ is strictly increasing,} \quad H(x) \to \pm\infty \text{ as } x \to \pm\infty.$$

$f_1,\ f_2$ are C^2-functions with $0 < f_1, f_2 \le M$ for some $M > 0$ and

$$f_1'(x) \ge 0,\ x \in \mathbb{R}, \quad \frac{d}{ds}\frac{d}{dm}f_1 - H \text{ is strictly decreasing on } (-\infty,0),$$

$$f_2'(x) \le 0,\ x \in \mathbb{R}, \quad \frac{d}{ds}\frac{d}{dm}f_2 + H \text{ is strictly increasing on } (0,\infty).$$

Remark 3.1 (i) The assumptions for f_1, f_2 are trivially satisfied by $f_1 = r, f_2 = \ell$ positive constant functions.
(ii) In the next section, we shall be concerned with controls of a diffusion with generator $\frac{d}{dm}\frac{d}{ds}$, a diffusion with scale ds and speed measure dm in the sense of W. Feller([IM 74]). For that purpose, we need to consider in the first part of this section a diffusion with the roles of ds and dm being interchanged.

Lemma 3.1 *There exists $A > 0$ such that the diffusion $\mathbf{M} = (X_t, P_x)$ on $[-A, A]$ associated with the Dirichlet form* (3.2),(3.3) *satisfies*

$$E_{\xi_1}\left(\int_0^{\sigma_0\wedge\sigma_{-A}} e^{-\alpha t}H(X_t)dt\right) < -2M, \; E_{\xi_2}\left(\int_0^{\sigma_0\wedge\sigma_{A}} e^{-\alpha t}H(X_t)dt\right) > 2M, \tag{3.4}$$

for some $\xi_1 \in (-A, 0)$ and $\xi_2 \in (0, A)$. Here σ_x denotes the hitting time of the one point set $\{x\}$.

Proof. For an open interval $I \subset \mathbb{R}$, we denote by $\mathbb{D}^I$ the absorbing diffusion on I with infinitesimal generator $\frac{d}{ds}\cdot\frac{d}{dm}$ and by R_α^I its resolvent operator. By virtue of the condition (3.1), $\mathbb{D}^{\mathbb{R}}$ is conservative and its α-order hitting probability $E_x\left(e^{-\alpha\sigma_c}\right)$ for any fixed point c tends to zero as $x \to \pm\infty$ ([IM 74]). Hence, by Dynkin's formula,

$$\lim_{x\to-\infty} R_\alpha^{(-\infty,c)}1(x) = \frac{1}{\alpha}, \quad \lim_{x\to\infty} R_\alpha^{(d,\infty)}1(x) = \frac{1}{\alpha}, \tag{3.5}$$

for any c and d. By assumption 3.1, we can take $\xi < 0$ such that

$$H(x) < -4\alpha M \qquad \forall x \le \xi.$$

By (3.5), $R_\alpha^{(-\infty,\xi)}1(\xi_1) > 1/(2\alpha)$ for some $\xi_1 < \xi$. Since $R_\alpha^{(-A,\xi)}1(\xi_1)$ increases to $R_\alpha^{(-\infty,\xi)}1(\xi_1)$ as $A \to \infty$, we have $R_\alpha^{(-A,\xi)}1(\xi_1) > 1/(2\alpha)$ for a sufficiently large A with $-A < \xi_1$.

For such A, let $\mathbf{M}$ be the diffusion on $[-A, A]$ governed by the Dirichlet form (3.2),(3.3). Since the process obtained from $\mathbf{M}$ by killing at time $\sigma_0 \wedge \sigma_{-A}$ coincides with $\mathbb{D}^{(-A,0)}$, the first expectation appearing in (3.4) equals $R_\alpha^{(-A,0)}H(\xi_1)$, which in turn is not greater than

$$R_\alpha^{(-A,\xi)}H(\xi_1) \le -4\alpha M \cdot R_\alpha^{(-A,\xi)}1(\xi_1) < -2M$$

proving the first inequality in (3.4). The second one can be shown in the same way. □

In what follows, we shall work with $A > 0$ for which (3.4) is satisfied.

Theorem 3.1 *There exist unique a, b such that $-A < a < 0 < b < A$ and*

$$-f_1(x) < V(x) < f_2(x), \qquad x \in (a, b), \tag{3.6}$$

$$V(x) = -f_1(x), \quad x \in [-A, a]; \qquad V(x) = f_2(x), \quad x \in [b, A], \tag{3.7}$$

$$V'(a) = -f_1'(a), \qquad V'(b) = f_2'(b). \tag{3.8}$$

Furthermore V is C^1 on $(-A, A)$, C^2 on (a, b) and

$$\begin{aligned} \alpha V(x) - \frac{d}{ds}\frac{d}{dm}V(x) &= H(x) \quad \forall x \in [a,b] \\ &> H(x) \quad \forall x \in (-A, a) \\ &< H(x) \quad \forall x \in (b, A). \end{aligned} \tag{3.9}$$

The theorem is divided into three propositions.

Proposition 3.1 (i) $-f_1(0) < V(0) < f_2(0)$.
(ii) *$V(x) > -f_1(x)$ for $x > 0$ and $V(x) < f_2(x)$ for $x < 0$.*
(iii) *Let*

$$E_1 = \{x \in [-A, A] : V(x) = -f_1(x)\}, \quad E_2 = \{x \in [-A, A] : V(x) = f_2(x)\} \tag{3.10}$$

and $a = \sup E_1, \quad b = \inf E_2$. Then

$$-A < a < 0 < b < A.$$

(iv) *If*

$$-f_1(x) < V(x) < f_2(x), \qquad \beta < x < \gamma,$$

for some interval $(\beta, \gamma) \subset (-A, A)$, then V is C^2 on (β, γ) and

$$\left(\alpha - \frac{d}{ds}\frac{d}{dm}\right) V(x) = H(x) \tag{3.11}$$

for $x \in (\beta, \gamma)$. In particular, this equation holds for $x \in (a, b)$.
(v) *If, for some $\beta \in (-A, 0)$,*

$$-f_1(x) < V(x), \qquad -A \le x < \beta,$$

then V is C^2 on $(-A, \beta)$, V satisfies the equation (3.11) on $(-A, \beta)$ and $V'(-A) = 0$.
(vi) *If, for some $\gamma \in (0, A)$,*

$$V(x) < f_2(x), \qquad \gamma < x \le A,$$

then V is C^2 on (γ, A), V satisfies the equation (3.11) on (γ, A) and $V'(A) = 0$.

Proof. Denote by σ_E the hitting time of the diffusion $\mathbf{M}$ for a set E. The hitting time for the one point set $\{x\}$ is simply denoted by σ_x. We let $\hat{\sigma} = \sigma_{E_1}, \quad \hat{\tau} = \sigma_{E_2}$ the hitting times for the sets E_1, E_2 defined by (3.10).
(i) We give the proof of the first inequality. The second one can be proved similarly. We have from (2.10) and (2.11) (the exceptional set N is now empty, as was explained in the

paragraph below (3.3)) that, for any positive $\epsilon < A$,

$$\begin{aligned} V(0) &\geq J_0(\hat\tau, \sigma_{-\epsilon}) = E_0\left(\int_0^{\hat\tau\wedge\sigma_{-\epsilon}} e^{-\alpha t}H(X_t)dt\right) \\ &- f_1(-\epsilon)E_0\left(e^{-\alpha\sigma_{-\epsilon}}; \sigma_{-\epsilon} < \hat\tau\right) + E_0\left(e^{-\alpha\hat\tau}f_2(X_{\hat\tau}); \sigma_{-\epsilon} \geq \hat\tau\right) \\ &\geq H(-\epsilon)E_0\left(\int_0^{\sigma_{-\epsilon}} e^{-\alpha t}dt\right) - f_1(0)E_0\left(e^{-\alpha\sigma_{-\epsilon}}\right) \\ &= -f_1(0) + (f_1(0) + \frac{H(-\epsilon)}{\alpha})\left(1 - E_0(e^{-\alpha\sigma_{-\epsilon}})\right), \end{aligned}$$

which is greater than $-f_1(0)$ for sufficiently small $\epsilon > 0$.
(ii) For $x > 0$,

$$\begin{aligned} V(x) &\geq J_x(\hat\tau, \sigma_0) = E_x\left(\int_0^{\hat\tau\wedge\sigma_0} e^{-\alpha t}H(X_t)dt\right) \\ &- f_1(0)E_x\left(e^{-\alpha\sigma_0}; \sigma_0 < \hat\tau\right) + E_x\left(e^{-\alpha\hat\tau}f_2(X_{\hat\tau}); \sigma_0 \geq \hat\tau\right) \\ &\geq -f_1(0)E_x\left(e^{-\alpha\sigma_0}\right) > -f_1(0) \geq -f_1(x). \end{aligned}$$

The second inequality can be proved similarly.
(iii) Suppose $V(x) > -f_1(x)$ for any $x \in (-A, 0)$. Then, by (i) and (ii), $P_x(\hat\sigma \geq \sigma_{-A}) = 1\ \forall x$. Further $P_x(\hat\tau > \sigma_0) = 1$ for any $x < 0$. Hence

$$P_x(\hat\sigma \wedge \hat\tau \geq \sigma_{-A} \wedge \sigma_0) = 1 \quad \forall x < 0,$$

which implies that the function $V(x) = J_x(\hat\tau, \hat\sigma)$ is H-α-harmonic on $(-A, 0)$ in the sense that, for $x \in (-A, 0)$,

$$V(x) = E_x\left(\int_0^{\sigma_0\wedge\sigma_{-A}} e^{-\alpha t}H(X_t)dt\right) + E_x\left(e^{-\alpha(\sigma_0\wedge\sigma_{-A})}V(X_{\sigma_0\wedge\sigma_{-A}})\right). \tag{3.12}$$

Since $V(x) \leq M$ for any $x \in [-A, A]$, we get from the above and (3.4)

$$V(\xi_1) < -2M + M = -M,$$

a contradiction. Hence $-A < a < 0$. The second inequality can be proved similarly.
(iv) As in the proof of (iii), V is then H-α-harmonic on the interval (β, γ) in the sense the identity (3.12) with $\sigma_0 \wedge \sigma_{-A}$ being replaced by $\sigma_\beta \wedge \sigma_\gamma$ holds for $x \in (\beta, \gamma)$, which is equivalent to the validity of the following equation ([FOT 94, §4.3,§4.4]):

$$\mathcal{E}_\alpha(V, v) = (H, v) \qquad \forall v \in C_0^1((\beta, \gamma)). \tag{3.13}$$

Since H is continuous, this equation in turn implies that V is C^2 on (β, γ) and an integration by parts yields the equation (3.11) on the same interval.
(v) In this case, the identity (3.12) with $\sigma_0 \wedge \sigma_{-A}$ being replaced by $\sigma_\beta \wedge \sigma_{-A}$ holds for $x \in [-A, \beta)$, which is equivalent to the validity of the equation (3.13) for any $v \in C_0^1([-A, \beta))$.

Again, an integration by parts gives the validity of (3.11) on $(-A, \beta)$ together with the stated boundary condition.
(vi) analogous to (v).

□

Before proceeding further, we prepare some notations. For $\xi \in (-A, A)$ and $\epsilon > 0$, we denote by $\tau_{\xi,\epsilon}$ the first exit time from the interval $I_{\xi,\epsilon} = (\xi - \epsilon, \xi + \epsilon)$, namely, $\tau_{\xi,\epsilon} = \sigma_{[-A,A]\setminus I_{\xi,\epsilon}}$. We then set

$$\begin{aligned} h_\alpha^-(\xi, \epsilon) &= E_\xi\left(e^{-\alpha\tau_{\xi,\epsilon}}; \sigma_{\xi-\epsilon} < \sigma_{\xi+\epsilon}\right) \\ h_\alpha^+(\xi, \epsilon) &= E_\xi\left(e^{-\alpha\tau_{\xi,\epsilon}}; \sigma_{\xi-\epsilon} \geq \sigma_{\xi+\epsilon}\right) \\ g_\alpha(\xi, \epsilon) &= 1 - E_\xi\left(e^{-\alpha\tau_{\xi,\epsilon}}\right) \end{aligned}$$

Lemma 3.2

$$\lim_{\epsilon\downarrow 0} h_\alpha^\pm(\xi, \epsilon) = \frac{1}{2}.$$

$$g_\alpha(\xi, \epsilon) = o(\epsilon) \quad \text{as} \quad \epsilon \downarrow 0.$$

Proof. The first identity for $\alpha = 0$ is evident because

$$h_0^-(\xi, \epsilon) = \frac{\int_\xi^{\xi+\epsilon} \dot{m}(x)dx}{\int_{\xi-\epsilon}^{\xi+\epsilon} \dot{m}(x)dx}. \tag{3.14}$$

Let u be a C^2-function vanishing at $-A$ and A such that

$$\frac{d}{ds}\frac{d}{dm}u(x) = -1 \quad x \in (\xi - \epsilon, \xi + \epsilon).$$

By Dynkin's formula applied to the 0-order resolvent of the process obtained from $\mathbf{M}$ by killing at time $\sigma_{-A} \wedge \sigma_A$,

$$E_\xi(\tau_{\xi,\epsilon}) = u(\xi) - h_0^-(\xi, \epsilon)u(\xi - \epsilon) - h_0^+(\xi, \epsilon)u(\xi + \epsilon),$$

which combined with (3.14) leads us to $E_\xi(\tau_{\xi,\epsilon}) = o(\epsilon)$.

The rest of the proof is obvious since

$$g_\alpha(\xi, \epsilon) = \alpha E_\xi\left(\int_0^{\tau_{\xi,\epsilon}} e^{-\alpha t}dt\right) \leq \alpha E_\xi(\tau_{\xi,\epsilon}).$$

□

Proposition 3.2 (i) *$V'(a) = -f_1'(a)$ and $V'(x)$ is right continuous at a.*
(ii) $V'(b) = f_2'(b)$ and $V'(x)$ is left continuous at b.

Proof. We only give a proof (i). The proof of (ii) is analogous. Take any $\epsilon > 0$ with $(a-\epsilon, a+\epsilon) \subset (-A, 0)$. Let θ_t be the shift operator on the probability space Ω for $\mathbf{M}$, that is $X_s(\theta_t \omega) = X_{s+t}(\omega), \forall \omega \in \Omega$ (cf [FOT 94]). If we let $\sigma = \tau_{a,\epsilon} + \hat{\sigma} \circ \theta_{\tau_{a,\epsilon}}$, then

$$\hat{\tau} \wedge \sigma = \tau_{a,\epsilon} + (\hat{\tau} \wedge \hat{\sigma}) \circ \theta_{\tau_{a,\epsilon}},$$

because $\hat{\tau} = \tau_{a,\epsilon} + \hat{\tau} \circ \theta_{\tau_{a,\epsilon}}$. Hence we have

$$V(a) \geq J_a(\hat{\tau}, \sigma) = E_a \left(\int_0^{\tau_{a,\epsilon}} e^{-\alpha t} H(X_t) dt \right) + h_\alpha^-(a,\epsilon) V(a-\epsilon) + h_\alpha^+(a,\epsilon) V(a+\epsilon),$$

and

$$\begin{aligned} & h_\alpha^-(a,\epsilon)V(a-\epsilon) + h_\alpha^+(a,\epsilon)V(a+\epsilon) - E_a\left(e^{-\alpha \tau_{a,\epsilon}}\right) V(a) \\ & \leq g_\alpha(a,\epsilon)V(a) - E_a \left(\int_0^{\tau_{a,\epsilon}} e^{-\alpha t} H(X_t) dt \right) \leq (V(a) - \frac{H(a-\epsilon)}{\alpha}) g_\alpha(a,\epsilon). \end{aligned}$$

Therefore

$$\begin{aligned} & h_\alpha^+(a,\epsilon)(-f_1(a+\epsilon) + f_1(a)) \leq h_\alpha^+(a,\epsilon)(V(a+\epsilon) - V(a)) \\ \leq \;& h_\alpha^-(a,\epsilon)(V(a) - V(a-\epsilon)) + (V(a) - \frac{H(a-\epsilon)}{\alpha}) g_\alpha(a,\epsilon) \\ \leq \;& h_\alpha^-(a,\epsilon)(-f_1(a) + f_1(a-\epsilon)) + (V(a) - \frac{H(a-\epsilon)}{\alpha}) g_\alpha(a,\epsilon). \end{aligned}$$

By dividing each side of the above inequality by ϵ and letting $\epsilon \to 0$, we get from the previous lemma the desired inequality

$$-D_+ f_1(a) \leq D_+ V(a) \leq D_- V(a) \leq -D_- f_1(a),$$

yielding the first half of (i). Since $V'(x)$ is easily seen to have the right limit at $x = a$ by virtue of Proposition 3.1 (iv), it is right continuous at a as well.

□

Proposition 3.3 *Let E_1, E_2 be the sets defined by* (3.10).
(i) $E_1 = [-A, a]$ *and*

$$\left(\alpha - \frac{d}{ds}\frac{d}{dm} \right) f_1(x) > H(x), \qquad \forall x \in [-A, a).$$

(ii) $E_2 = [b, A]$ *and*

$$\left(\alpha - \frac{d}{ds}\frac{d}{dm} \right) f_2(x) < H(x), \qquad \forall x \in (b, A].$$

Proof. We only give the proof of (i). (ii) can be proved similarly. Putting $x = a + \epsilon$ in (3.11) and letting $\epsilon \downarrow 0$, we get

$$\alpha V(a) - \frac{d+}{ds}\frac{dV}{dm}(a) = H(a), \tag{3.15}$$

where $\frac{d+}{ds}$ denotes the right derivative. On the other hand,

$$\frac{d+}{ds}\frac{dV}{dm}(a) \geq -\frac{d}{ds}\frac{d\,f_1}{dm}(a). \tag{3.16}$$

In fact, the function $F(x) = V(x) + f_1(x)$ satisfies $F(x) \geq 0, F(a) = 0$ and further $F'(a) = 0$, $F'(x)$ is right continuous at a by the preceding proposition. Taylor's theorem applies and

$$0 \leq \frac{F(a+\epsilon)}{\epsilon^2} = F''(a+\theta\epsilon) \to \frac{d+}{dx}F'(a) \quad \text{as} \quad \epsilon \downarrow 0.$$

Hence

$$\frac{d+}{ds}\frac{dF}{dm}(a) = \frac{1}{\dot{s}(a)\dot{m}(a)}\frac{d+}{dx}F'(a) - \frac{\dot{m}'(a)}{\dot{s}(a)\dot{m}(a)^2}F'(a) = \frac{1}{\dot{s}(a)\dot{m}(a)}\frac{d+}{dx}F'(a) \geq 0.$$

Now (3.15) and (3.16) and Assumption 3.1 lead us to the inequality

$$-\left(\alpha - \frac{d}{ds}\frac{d}{dm}\right) f_1(x) > H(x) \qquad \forall x \in [-A, a). \tag{3.17}$$

Turning to the proof of $E_1 = [-A, a]$ by reduction to a contradiction, we assume that there exists $x_0 \in [-A, a)$ such that $V(x_0) > -f_1(x_0)$. Then we have two possibilities:
(I) There exists $\beta, \gamma \in E_1$ such that $-A \leq \beta < x_0 < \gamma \leq a$ and $V(x) > -f_1(x) \quad \forall x \in (\beta, \gamma)$.
(II) There exists $\beta \in E_1$ such that $-A < x_0 < \beta$ and $V(x) > -f_1(x) \quad \forall x \in [-A, \beta)$.

Suppose case (I) occurs. By combining Proposition 3.1 (iv) with (3.17), we then see that the function $F = -f_1 - V$ satisfies

$$\left(\alpha - \frac{d}{ds}\frac{d}{dm}\right) F(x) > 0 \tag{3.18}$$

for any $x \in (\beta, \gamma)$. Since $F(\beta) = F(\gamma) = 0$, an integration by parts yields

$$\mathcal{E}_\alpha(F, v) \geq 0 \tag{3.19}$$

for any $v \in C_0^1((\beta, \gamma))$ such that $v \geq 0$. This means that (the restriction to (β, γ) of) F is α-excessive with respect to the part of the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on the interval (β, γ) ([FOT 94, Lem. 2.2.1, Th. 4.4.3]). In particular, $F(x_0) \geq 0$ a contradiction.

Suppose case (II) occurs. On account of Proposition 3.1 (v), (3.17) and Assumption 3.1, we see then that the function F satisfies inequality (3.18) holding for any $x \in (-A, \beta)$ as well as the inequality $F'(-A) \leq 0$. Therefore, an integration by parts leads us to the inequality (3.19) holding for any $v \in C_0^1([-A, \beta))$ such that $v \geq 0$. F is then α-excessive with respect to the part of $(\mathcal{E}, \mathcal{F})$ on the interval $[-A, \beta)$, arriving at a contradiction $F(x_0) \geq 0$ again.

□

By the preceding three propositions, the proof of Theorem 3.1 is complete.

The function V of Theorem 3.1 (the solution of (2.1), (2.2) for the Dirichlet form (3.3) on $L^2([-A,A],ds)$ under the assumption 3.1 for the data (H,f_1,f_2) gives rise to a solution of another type of free boundary problem stated below. Let us first extend the function V to whole $\mathbb{R}$ by setting

$$V(x) = -f_1(x) \quad x < -A, \qquad V(x) = f_2(x) \quad x > A. \tag{3.20}$$

In view of Assumption 3.1, we see that the extended function V still satisfies the first inequality of (3.9) on $(-\infty,a)$ and the second inequality on (b,∞).

We then let, for $x \in \mathbb{R}$,

$$h(x) = \int_0^x H(y)\dot{s}(y)dy + C. \tag{3.21}$$

where C is an arbitrarily taken fixed constant. We further let

$$W(x) = \int_a^x V(y)\dot{s}(y)dy + \frac{1}{\alpha}\left(-\frac{f_1'(a)}{\dot{m}(a)} + h(a)\right). \tag{3.22}$$

Theorem 3.2 *$W \in C^2(\mathbb{R})$ and there exist a,b with $a < 0 < b$ such that*

$$\begin{aligned} \alpha W(x) - \frac{d}{dm}\frac{d}{ds}W(x) &= h(x) \quad a < x < b \\ &< h(x) \quad x < a \ \text{ or } \ x > b, \end{aligned} \tag{3.23}$$

$$-f_1 < \frac{d}{ds}W < f_2 \qquad \text{on} \quad (a,b), \tag{3.24}$$

$$\frac{d}{ds}W = -f_1 \text{ on } (-\infty,a], \qquad \frac{d}{ds}W = f_2 \text{ on } [b,\infty) \tag{3.25}$$

$$\frac{d}{dx}\frac{d}{ds}W(a) = -f_1'(a), \quad \frac{d}{dx}\frac{d}{ds}W(b) = f_2'(b). \tag{3.26}$$

Proof. For the function

$$U(x) = \alpha W(x) - \frac{d}{dm}\frac{d}{ds}W(x) - h(x),$$

we have

$$\frac{1}{\dot{s}(x)}U'(x) = \alpha V(x) - \frac{d}{ds}\frac{d}{dm}V(x) - H(x).$$

Consider a,b of Theorem 3.1. Then, by Theorem 3.1 and the remark made before the statement of Theorem 3.2,

$$U(a) = 0; \quad U'(x) > 0,\ x < a; \quad U'(x) = 0,\ x \in (a,b); \quad U'(x) < 0,\ x > b,$$

which implies (3.23). The rest of the proof is obvious. □

4 A singular control of the (σ,μ)-diffusion

Let $\sigma(x)$ and $\mu(x)$ be C^1-functions on $\mathbb{R}$ with $\sigma(x) \neq 0,\ \forall x \in \mathbb{R}$. We are concerned with a diffusion on $\mathbb{R}$ with infinitesimal generator

$$Lu(x) = \frac{1}{2}\sigma(x)^2 \frac{d^2u}{dx^2}(x) + \mu(x)\frac{du}{dx}(x), \tag{4.1}$$

which can be converted into the Feller canonical form $\dfrac{d}{dm}\dfrac{du}{ds}(x)$ by setting

$$\dot{s}(x) = \exp\left(-\int_0^x \frac{2\mu(y)}{\sigma(y)^2}dy\right) \quad \dot{m}(x) = \frac{2}{\sigma(x)^2}\exp\left(\int_0^x \frac{2\mu(y)}{\sigma(y)^2}dy\right), \tag{4.2}$$

and $ds(x) = \dot{s}(x)dx,\ dm(x) = \dot{m}(x)dx$. We assume that $-\infty$ and ∞ are natural boundaries with respect to the operator (4.1) in the sense that condition (3.1) is satisfied by $\dot{s},\ \dot{m}$ of (4.2). Since $\dot{s},\ \dot{m}$ of (4.2) are strictly positive C^1-functions, all results of §3 apply.

Throughout this section, we fix $\sigma(x), \mu(x)$ as above and $\dot{s}(x), \dot{m}(x)$ are understood to be defined by (4.2). We call a triplet (S, X, A) *admissible policy* or just *admissible* if the following conditions are satisfied:

($\mathbb{A}$.1) S is a compact interval of $\mathbb{R}$.
($\mathbb{A}$.2) There is a filtered measurable space $(\Omega, \{\mathcal{F}_t\}_{t\geq 0})$ subject to usual conditions and probability measures $\{P_x\}_{x\in S}$ on it such that
$X = \{X_t\}_{t\geq 0}$ is an $\{\mathcal{F}_t\}$-adapted right continuous process and
$A = \{A_t\}_{t\geq 0}$ is an $\{\mathcal{F}_t\}$-adapted right continuous process of bounded variation satisfying

$$E_x\left(\int_{0-}^{\infty} e^{-\alpha t}dA_t^{(1)}\right) < \infty, \quad E_x\left(\int_{0-}^{\infty} e^{-\alpha t}dA_t^{(2)}\right) < \infty, \quad \forall x \in S, \tag{4.3}$$

where $A^{(1)}$ and $A^{(2)}$ are two $\{\mathcal{F}_t\}$-adapted right continuous increasing processes for which $A_t = A_t^{(1)} - A_t^{(2)}$ is the *minimal* decomposition of the bounded variation process A into a difference of two increasing processes.
($\mathbb{A}$.3) There is an $\{\mathcal{F}_t\}$-adapted standard Brownian motion $\{w_t\}_{t\geq 0}$ starting at the origin under P_x for any $x \in S$ such that the stochastic differential equation

$$X_t = x + \int_0^t \sigma(X_s)dw_s + \int_0^t \mu(X_s)ds + A_t^{(1)} - A_t^{(2)} \quad t \geq 0 \tag{4.4}$$

holds P_x-a.s. for each $x \in S$ and further

$$P_x(X_t \in S,\ \forall t \geq 0) = 1 \quad \forall x \in S. \tag{4.5}$$

We denote by $\mathbb{A}$ the totality of admissible triplets (S, X, A). In the sequel we will always represent A in terms of $A^{(1)}$ and $A^{(2)}$ and thus we will write (S, X, A) and $(S, X, A^{(1)}, A^{(2)})$ interchangeably.

Remark 4.1 (i) The probability space Ω with the filtration $\{\mathcal{F}_t\}$ in ($\mathbb{A}$.2) is not fixed a priori. It is a part of an admissible policy. The filtration $\{\mathcal{F}_t\}$ is assumed to be right continuous and $\mathcal{F}_0$ is assumed to contain every Ω-set which is P_x-negligible for any $x \in S$.
(ii) We shall use the notations

$$\Delta A_t^{(i)} = A_t^{(i)} - A_{t-}^{(i)}, \quad t \geq 0 \quad i = 1, 2,$$

$$\Delta X_t = X_t - X_{t-}, \quad \Delta u(X)_t = u(X_t) - u(X_{t-}), \quad t \geq 0.$$

Note that, due to the fact that $A^{(1)}$ and $A^{(2)}$ represent the *minimal* decomposition of A into two increasing processes, $\Delta A_t^{(1)} \Delta A_t^{(2)} = 0$ for each $t \geq 0$. By convention, we let

$$w_t = 0, \quad A_t^{(i)} = 0 \quad \forall t < 0, \quad i = 1, 2,$$

so that

$$\Delta A_0^{(i)} = A_0^{(i)}, \ i = 1, 2, \quad X_{0-} = x \quad P_x\text{-a.s.} \ \forall x \in S.$$

Further we define the continuous part of $A^{(i)}$ by

$$A_t^{(i),c} = A_t^{(i)} - \sum_{0 \leq s \leq t} \Delta A_s^{(i)}, \quad t \geq 0, \quad i = 1, 2.$$

(iii) The integrals in t in (4.3) involve the possible jump at 0 so that they are the sum of the integrals over $(0, \infty)$ and $A_0^{(i)}, \ i = 1, 2$.

Proposition 4.1 *Let $(S, X, A^{(1)}, A^{(2)}) \in \mathbb{A}$. Then, for any $u \in C^2(\mathbb{R})$, the following identity holds:*

$$\begin{aligned} u(x) &= E_x \left[\int_0^\infty e^{-\alpha t} \left(\alpha - \frac{d}{dm} \frac{d}{ds} \right) u(X_t) dt \right] \\ &+ E_x \left[\int_0^\infty e^{-\alpha t} \left(-\frac{du}{ds}(X_t) \dot{s}(X_t) dA_t^{(1),c} + \frac{du}{ds}(X_t) \dot{s}(X_t) dA_t^{(2),c} \right) \right] \\ &- E_x \left[\sum_{0 \leq t < \infty} e^{-\alpha t} \Delta u(X)_t \right]. \qquad x \in S. \end{aligned} \tag{4.6}$$

All expectations in the right side of (4.6) *exist and are finite.*

Proof. By a generalized Ito formula ([M 76, p 278], see also [HT 83, §4]) applied to the semimartingale (4.4), we have

$$\begin{aligned} e^{-\alpha t} u(X_t) &= u(X_0) - \alpha \int_0^t e^{-\alpha s} u(X_s) ds + \int_0^t e^{-\alpha s} u'(X_s) \sigma(X_s) dw_s \\ &+ \int_0^t e^{-\alpha s} u'(X_s) \mu(X_s) ds + \int_0^t e^{-\alpha s} u'(X_s) (dA_s^{(1),c} - dA_s^{(2),c}) \\ &+ \frac{1}{2} \int_0^t e^{-\alpha s} u''(X_s) \sigma(X_s)^2 ds + \sum_{0 < s \leq t} e^{-\alpha s} \Delta u(X)_s. \end{aligned} \tag{4.7}$$

Rewrite the sum of two terms in the right side of (4.7) as

$$u(X_0) + \sum_{0<s\leq t} e^{-\alpha s}\Delta u(X)_s = u(X_{0-}) + \sum_{0\leq s\leq t} e^{-\alpha s}\Delta u(X)_s,$$

then take the expectation of the both hand sides of (4.7) with respect to P_x and let $t \to \infty$ to get the identity (4.6). □

Lemma 4.1 *If* $(S, X, A^{(1)}, A^{(2)}) \in \mathbb{A}$*, then both* $A^{(1)}$ *and* $A^{(2)}$ *are non-trivial in the sense that, for each* $T > 0$,

$$P_x(A_t^{(i)} = A_0^{(i)},\ \forall t \in [0, T)) = 0, \quad \forall x \in S, \quad i = 1, 2. \tag{4.8}$$

Proof. (i) Since S is compact, the integrand of the first integral of the right hand side of (4.4) is bounded and is bounded away from zero, while the integrand of the second is bounded. If both $A^{(1)},\ A^{(2)}$ were trivial,the process X_t satisfying (4.4) hits therefore any point of $\mathbb{R}$ almost surely as the Brownian motion does ([IW 89, pp.85, pp.437]), a contradiction. If either $A^{(1)}$ or $A^{(2)}$ is trivial, the path of X_t can not be concentrated on a compact set, again a contradiction. □

Proposition 4.2 *For any finite* $\beta_1 < \beta_2$*, there exists* $([\beta_1, \beta_2], X, A^{(1)}, A^{(2)}) \in \mathbb{A}$ *such that*

$$A_t^{(i)} = \int_0^t I_{\{\beta_i\}}(X_s)dA_s^{(i)}, \quad \forall t \geq 0, \quad P_x - a.s.\ \forall x \in [\beta_1, \beta_2]. \quad i = 1, 2. \tag{4.9}$$

Such X_t *and* $A_t^{(i)},\ i = 1, 2$ *are necessarily continuous in* $t \geq 0$, P_x-a.s. *for any* $x \in [\beta_1, \beta_2]$. *Furthermore, the* P_x*-law of such* $(X, A^{(1)}, A^{(2)})$ *is unique for any* $x \in [\beta_1, \beta_2]$.

Proof. The equation (4.4) subjected to the conditions (4.5) and (4.9) is called the *Skorohod equation* for $[\beta_1, \beta_2]$.

Since $\sigma,\ \mu$ are C^1-functions, the existence and uniqueness of $(X, A^{(1)}, A^{(2)})$ satisfying (4.9) and all admissibility conditions except for the integrability (4.3) follow from Tanaka [Tana 79, Th. 4.1], where the unique existence of the strong solution of the Skorohod equation with Lipschitz continuous coefficients for a multidimensional convex domain was proved. It was also shown in [Tana 79] that the solution is necessarily continuous. The integrability (4.3) is then an automatic consequence of the equation (4.7) applied to C^2-function u such that $u'(\beta_1) = 1,\ u'(\beta_2) = 0$ (resp. $u'(\beta_1) = 0,\ u'(\beta_2) = 1$). □

The triple $(X, A^{(1)}, A^{(2)})$ of Proportion 4.2 is called a *reflecting* (σ, μ)*-diffusion* on the interval $[\beta_1, \beta_2]$.

We are now in the position to formulate our main theorem about a singular control problem for the admissible family $\mathbb{A}$.

Let h, f_1, f_2 be functions on $\mathbb{R}$ satisfying the following conditions:

Assumption 4.1 *$h(x)$ is a C^1 function on $\mathbb{R}$ such that*

$$h'(0)=0,\ \frac{dh}{ds}(x) \text{ is strictly increasing}, \quad \frac{dh}{ds}(x)\to\pm\infty \text{ as } x\to\pm\infty.$$

$f_1,\ f_2$ are C^2-functions with $0<f_1,f_2\le M$ for some $M>0$ and

$$f_1'(x)\ge 0,\ x\in\mathbb{R}, \quad \frac{d}{ds}\frac{d}{dm}f_1-\frac{dh}{ds} \text{ is strictly decreasing on } (-\infty,0),$$

$$f_2'(x)\le 0,\ x\in\mathbb{R}, \quad \frac{d}{ds}\frac{d}{dm}f_2+\frac{dh}{ds} \text{ is strictly increasing on } (0,\infty).$$

We note that then h can be expressed as (3.21) via a function H satisfying condition of Assumption 3.1 and further f_1, f_2 satisfy the condition of Assumption 3.1 for this function H. Therefore Theorem 3.2 applies to the present functions h, f_1, f_2.

For each $(S,X,A^{(1)},A^{(2)})\in\mathbb{A}$, the cost function k_x is defined, for $x\in S$, by

$$\begin{aligned} &k_x(S,X,A^{(1)},A^{(2)}) = E_x\left(\int_0^\infty e^{-\alpha t}h(X_t)dt\right)\\ &+E_x\left[\int_0^\infty e^{-\alpha t}\left(f_1(X_t)\dot{s}(X_t)dA_t^{(1),c}+f_2(X_t)\dot{s}(X_t)dA_t^{(2),c}\right)\right]\\ &+E_x\left[\sum_{0\le t<\infty} e^{-\alpha t}\left(\int_{X_{t-}}^{X_{t-}+\Delta A_t^{(1)}} f_1(y)ds(y)+\int_{X_{t-}-\Delta A_t^{(2)}}^{X_{t-}} f_2(y)ds(y)\right)\right]. \quad (4.10)\end{aligned}$$

Some remarks about the cost structure are due at this point. The first integral $\int_0^\infty e^{-\alpha t}h(X_t)dt$ in (4.10) represents the so-called *holding cost* associated with the position of the controlled process X_t. Other integrals represent the *control cost*, which is associated with the "efforts" to change the position of the controlled process. The cost associated with each of the control functionals $A^{(i)}, i=1,2$ is proportional to the displacement caused by each of these functionals, however the coefficient of the proportionality is a function of the position of the control process and is equal to $f_1(x)\dot{s}(x)$ if the controlled process is at the point x. Thus if $A^{(i)}$ is a continuous functional, we can write an approximation to the control cost as $\sum_j e^{-\alpha t_j} f_i(X_{t_j})\dot{s}(X_{t_j})\delta A_j^{(i)}$, where $\delta A_j^{(i)}$ is an increment of $A^{(i)}$ on the interval $[t_j,t_{j+1}]$. In a limit one gets $\int_0^\infty e^{-\alpha t}f_i(X_t)\dot{s}(X_t)dA_t^{(i)}$. When the control functional has a discontinuity at the point t, which results in a jump of the control process, then we represent this jump as if the real clock is stopped while a new clock is turned on and the controlled process is moving uniformly in the new time clock up or down from X_{t-} to $X_t = X_{t-}+\Delta A_t^{(i)}$. In such a representation the control cost of this displacement is equal to $\int_{X_{t-}}^{X_{t-}+\Delta A_t^{(i)}} f_i(y)\dot{s}(y)dy$, which corresponds to the last two terms in the right hand side of (4.10). The same expression in the right hand side of (4.10) would have been obtained as a limit if we had started with continuous functionals $A^{(i)}$ and then had approximated by them (via a monotone pointwise convergence) discontinuous control functionals.

Of course, when $f_i(x)\dot{s}(x)$ is equal to a constant r_i, the control cost associated with the functional $A^{(i)}$ can be written as $\int_0^\infty e^{-\alpha t} r_i dA_t^{(i)}$, without a need to have a special expression associated with the discontinuities of $A^{(i)}$. This was the case treated in [T 85]. We extend k_x outside the closed interval S denoted by $[\ell_1, \ell_2]$ as

$$\begin{aligned} k_x(S,X,A^{(1)},A^{(2)}) &= k_{\ell_1}(S,X,A^{(1)},A^{(2)}) + \int_x^{\ell_1} f_1(y)ds(y), \quad x < \ell_1, \\ &= k_{\ell_2}(S,X,A^{(1)},A^{(2)}) + \int_{\ell_2}^{x} f_2(y)ds(y), \quad x > \ell_2. \end{aligned} \tag{4.11}$$

Our problem is to find the function

$$W^*(x) = \inf_{(S,X,A^{(1)},A^{(2)})\in\mathbb{A}} k_x(S,X,A^{(1)},A^{(2)}), \quad x \in \mathbb{R} \tag{4.12}$$

called the *optimal return function* and find an optimal admissible quadruple $(S,X,A^{(1)},A^{(2)}) \in \mathbb{A}$ such that

$$W^*(x) = k_x(S,X,A^{(1)},A^{(2)}) \qquad \forall x \in \mathbb{R}.$$

The solution will be provided by the function W, the values a, b appearing in Theorem 3.2 and the reflecting (σ,μ)-diffusion on $[a,b]$ appearing in Proposition 4.2.

Here we introduce a subfamily $\mathbb{A}_0$ of $\mathbb{A}$ by

$$\mathbb{A}_0 = \{(S,X,A^{(1)},A^{(2)}) \in \mathbb{A} : A_0^{(i)} = 0 \ P_x\text{−a.s. } \forall x \in S, \ i = 1,2\}.$$

The reflecting (σ,μ)-diffusion on a compact interval appearing in Proposition 4.2 is a member of $\mathbb{A}_0$.

Theorem 4.1 *Under* Assumption 4.1 *for functions h, f_1, f_2, let W, a, b be the function and values in* Theorem 3.2. *Then,*

(i) $W(x) \le k_x(S,X,A^{(1)},A^{(2)})$, $\quad \forall x \in \mathbb{R}$, *for any* $(S,X,A^{(1)},A^{(2)}) \in \mathbb{A}$.

(ii) $W(x) = k_x(S,X,A^{(1)},A^{(2)})$, $\quad \forall x \in \mathbb{R}$, $\quad$ *for* $(S,X,A^{(1)},A^{(2)}) \in \mathbb{A}_0$ *if and only if*

$$S = [a,b], \quad (X,A^{(1)},A^{(2)}) \text{ is the reflecting } (\sigma,\mu)\text{-diffusion on the interval } [a,b]. \tag{4.13}$$

Proof. (i) Take any $(S,X,A^{(1)},A^{(2)}) \in \mathbb{A}$. Subtracting from (4.10) the identity (4.6) for $u = W$, we have

$$k_x(S,X,A^{(1)},A^{(2)}) - W(x) = E_x(I_1 + I_2 + I_3 + I_4) \qquad x \in S. \tag{4.14}$$

where

$$I_1 = \int_0^\infty e^{-\alpha t}\left\{\left(\frac{d}{dm}\frac{d}{ds} - \alpha\right)W(X_t) + h(X_t)\right\}dt,$$

$$I_2 = \int_0^\infty e^{-\alpha t}\left\{\frac{dW}{ds}(X_t) + f_1(X_t)\right\}\dot{s}(X_t)dA_t^{(1),c},$$

$$I_3 = \int_0^\infty e^{-\alpha t} \left\{ -\frac{dW}{ds}(X_t) + f_2(X_t) \right\} \dot{s}(X_t) dA_t^{(2),c},$$

$$I_4 = \sum_{0 \le t < \infty} e^{-\alpha t} \left(\Delta W(X)_t + \int_{X_{t-}}^{X_{t-}+\Delta A_t^{(i)}} f_1(y) ds(y) + \int_{X_{t-}-\Delta A_t^{(2)}}^{X_{t-}} f_2(y) ds(y) \right).$$

The integrands I_1, I_2, I_3 are non-negative by virtue of Theorem 3.2. To see that I_4 is non-negative, let

$$\Gamma_+ = \{t \ge 0 : \Delta A_t^{(1)} > 0\}, \quad \Gamma_- = \{t \ge 0 : \Delta A_t^{(2)} > 0\}.$$

Since $\Gamma_+ \cap \Gamma_- = \phi$ by Remark 4.1, we have for $t \in \Gamma_+$,

$$\Delta X_t = \Delta A_t^{(1)}, \quad \Delta W(X)_t = W(X_{t-} + \Delta A_t^{(1)}) - W(X_{t-}),$$

and consequently the sum in I_4, taken over all $t \in \Gamma_+$ equals

$$\int_{X_{t-}}^{X_{t-}+\Delta A_t^{(1)}} (\frac{dW}{ds}(y) + f_1(y)) ds(y).$$

We have a similar expression for $t \in \Gamma_-$ and we get eventually

$$I_4 = \sum_{t \in \Gamma_+} e^{-\alpha t} \int_{X_{t-}}^{X_t} (\frac{dW}{ds}(y) + f_1(y)) ds(y) + \sum_{t \in \Gamma_-} e^{-\alpha t} \int_{X_t}^{X_{t-}} (-\frac{dW}{ds}(y) + f_2(y)) ds(y). \quad (4.15)$$

which is non-negative by Theorem 3.2.

We have seen that $k_x \ge W(x),\ x \in S$. This inequality extends to $\mathbb{R}$ by the definition (4.11) and Theorem 3.2.

(ii) Suppose $k_x(S, X, A^{(1)}, A^{(2)}) = W(x),\ \forall x \in S$ for some $(S, X, A^{(1)}, A^{(2)}) \in \mathbb{A}_0$. Then all P_x-expectations of I_1, I_2, I_3, I_4 must vanish for any $x \in S$. Notice further that $X_0 = x\ P_x$−a.s.$\forall x \in S$, because $A_0^{(i)} = 0,\ P_x$−a.s. $\forall x \in S,\ i = 1, 2.$ We let $S = [\beta, \gamma]$.

Suppose that $\beta < a$ (resp.$b < \gamma$). Then $E_x(I_1) > 0$ for $x \in (\beta, a)$(resp. (b, γ)) by (3.23) and the right continuity of X_t. Therefore we have that $[\beta, \gamma] \subset [a, b]$.

In view of Lemma 4.1, both $A^{(1)},\ A^{(2)}$ are non-trivial. If $a < \beta$(resp. $\gamma < b$), then $\frac{dW}{ds} + f_1 \left(\text{resp. } -\frac{dW}{ds} + f_2 \right)$ is strictly positive on S by (3.24),(3.25) and hence either I_2 or the first sum of (4.15) (resp. either I_3 or the second sum of (4.15)) has a positive P_x-expectation for any $x \in S$. We have proven that $S = [a, b]$.

Then, by virtue of (3.24), we see that X_t or equivalently $A_t^{(i)}$ $i = 1, 2$, must be continuous in $t \ge 0$ P_x-a.s. for any $x \in S$ in order to make the expectation of I_4 expressed as (4.15) to be zero. Finally, using (3.24) and (3.25), we see that $A^{(i)} = A^{(i),c},\ i = 1, 2$, must satisfy the relations (4.9) for $\beta_1 = a,\ \beta_2 = b$ in order to make both expectations of I_2, I_3 to be zero. This means that $(X, A^{(1)}, A^{(2)})$ must be the reflecting (σ, μ)-diffusion on the interval $[a, b]$.

Conversely the cost function k_x of the reflecting (σ, μ)-diffusion on the interval $[a, b]$ is obviously identical with $W(x)$ on $\mathbb{R}$ in view of (4.14). □

Corollary 4.1 *Under* Assumption 4.1 *for functions h, f_1, f_2, the solution $W \in C^2(\mathbb{R})$ and values a, b $(a < b)$, of the free boundary problem* (3.23), (3.24) *and* (3.25) *are unique. The solution $W(x)$, $x \in \mathbb{R}$, coincides with the optimal return function $W^*(x)$ given by* (4.12).

Proof. In the proof of Theorem 4.1, we have seen that any function W satisfying (3.23),(3.24) and (3.25) for some a, b $(a < b)$, coincides with the function defined by (4.12). Further this function determines a, b uniquely according to (3.23). □

References

[BL 82] A. Bensuoussan and J.L. Lions, *Applications of variational inequalities in stochastic control*, North-Holland, Amsterdam and New York, 1982

[D 67] E.B. Dynkin, Game variant of a problem on optimal stopping, Soviet mathematics Doklady 10(1967), 270-274

[FS 93] W. H. Fleming and H. M. Soner ,*Controlled Markov Processes and Viscosity Solutions*, Springer - Verlag, 1993

[F 99] M. Fukushima, On semi-martingale characterizations of functionals of symmetric Markov processes, Electronic J. of Probability, 4(1999), Paper no. 18, pages 1-32

[FOT 94] M. Fukushima,Y.Oshima and M.Takeda, *Dirichlet forms and symmetric Markov processes*, Walter de Gruyter, Berlin New York, 1994

[G 69] S.M. Gusein-Zade, On a game connected with the Wiener process, Theory of Prbability and Its Applications, XIV4(1969), 701-704

[HT 83] M. Harrison and M. Taksar, Instantaneous control of Brownian motion, Math. Operations Research 8(1983), 439-453

[IW 89] N. Ikeda and S. Watanabe, *Stochastic differential equations and diffusion processes*, Second Edition, North-Holland/Kodansha, 1989

[IM 74] K. Ito and H.P. McKean, *Diffusion processes and their sample paths*, Springer, Berlin, Heidelberg, New York, 1974

[K 98] T. Kawabata, On a singular control problem for a time changed distorted Brownian motion, Doctoral Thesis, Gradute School of Engineering Sci. Osaka Univ., 1998

[KW 01] I. Karatzas and H. Wang, Connection between bounded-variation control and Dynkin games, *Optimal Control and Partial Differential Equations*, J.L. Menaldi et al. (Eds), IOS Press, 2001, 363-373.

[KI 00] Y. Kifer, Game options, *Finance and Stochastics* 4(2000), 443-463.

[M 76] P.A. Meyer, Un cours sur les intégrales stochastiques, In *Séminaire de Probabilités X*, Lecture Notes in Mathematics 511, Springer, Berlin, Heidelberg, New york, 1976

[N 78] H. Nagai, On an optimal stopping problem and a variational inequality, J. Math. Soc. Japan 30(1978), 303-312

[S 64] G. Stampacchia, Forms bilinéaires coercitives sur les ensembles convexes, C.R.Acad.Sc. Paris, t.258(1964), 4413-4416

[T 85] M. Taksar, Average optimal singular control and a related stopping problem, Math. Opertions Research 10(1985), 63-81

[Tana 79] H. Tanaka, Stochastic differential equations with reflecting boundary condition in convex regions, Hiroshima Math. J. 9(1979), 163-177

[Z 84] J. Zabczyk, Stopping games for symmetric Markov processes, Prob. Math. Statistics 4(1984), 185-196