On general boundary conditions for one-dimensional diffusions and symmetry

Masatoshi Fukushima, Osaka University
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Lecture 1: Minimal diffusion X^0 and symmetry (§1, §2, §3)
Lecture 2: From Dirichlet forms to C_b-generators (§4, §5)
Lecture 3: All possible diffusion extensions of X^0 (§6)

1 Minimal diffusion X^0

$I = (r_1, r_2) \subset \mathbb{R}$: a one-dimensional open interval.

A strictly increasing continuous function s on I is called a **canonical scale**.

A positive Radon measure m on I with full topological support is called a **canonical measure**.

1.1 General expression of C_b-generator of X^0

A Markov process $X^0 = (X_t^0, \zeta^0, \mathbf{P}_x^0)$ on I is called a **minimal diffusion** if

(d.1) X^0 is a Hunt process on I,

(d.2) X^0 is a diffusion process: X_t^0 is continuous in $t \in (0, \zeta^0)$ almost surely,

(d.3) X^0 is irreducible: $\mathbf{P}_x^0(\sigma_y < \infty) > 0$ for any $x, y \in I$.

Denote the one-point compactification of I by $I_\partial = I \cup \{\partial\}$.
X_t^0 takes value in I_∂. For $B \subset I_\partial$, we define

$$\sigma_B = \inf\{t > 0 : X_t^0 \in B\}, \quad \inf \emptyset = \infty, \quad \tau_B = \sigma_{I_\partial \setminus B}.$$

We write σ_B as σ_b when $B=\{b\}$ a one point set.
$\{\partial\}$ plays the role of cemetery for X^0: $\zeta^0=\sigma_\partial, \quad X^0_t=\partial$ for any $t\geq\zeta^0$.
Condition **(d.1)** means that X^0 is a strong Markov process whose sample path X^0_t is right continuous and has the left limit on $[0,\infty)$ and absorbed upon approaching $\{\partial\}$: $\lim_{n\to\infty}\tau_{J_n}=\zeta^0$ whenever $\{J_n\}$ are subintervals of I with $\overline{J}_n\subset I,\ J_n\uparrow I$. X^0 is **minimal** in this sense.
Under **(d.1)** and **(d.2)**, the condition **(d.3)** is equivalent to the requirement for each point $a\in I$ to be **regular** in the sense that, for $\alpha>0$,
$\mathbf{E}_a[e^{-\alpha\sigma_{a+}}]=\mathbf{E}_a[e^{-\alpha\sigma_{a-}}]=1 \quad$ where $E_a[e^{-\alpha\sigma_{a\pm}}]=\lim_{b\to\pm a}E_a[e^{-\alpha\sigma_b}]$.
$\{R^0_\alpha;\alpha>0\}$: the **resolvent** of a minimal diffusion X^0:
$R^0_\alpha f(x)=\mathbf{E}^0_x\left[\int_0^\infty e^{-\alpha t}f(X^0_t)dt\right]$.
Denote by $\mathcal{B}_b(I)$ (resp. $C_b(I)$) the space of all bounded Borel measurable (resp. continuous) function in I.
Then $R^0_\alpha(\mathcal{B}_b(I))\subset C_b(I)$ due to the above regularity of each point of I.
R^0_α is a one-to-one map from $C_b(I)$ into itself because of the resolvent equation
$R^0_\alpha-R^0_\beta+(\alpha-\beta)R^0_\alpha R^0_\beta=0$ and $\lim_{\alpha\to\infty}\alpha R^0_\alpha f(x)=f(x),\ x\in I, f\in C_b(I)$.
Thus the generator $\mathcal{G}^0$ of X^0 is well defined by

$$\begin{cases}\mathcal{D}(\mathcal{G}^0)=R^0_\alpha(C_b(I)),\\ (\mathcal{G}^0u)(x)=\alpha u(x)-f(x) \quad \text{for } u=R^0_\alpha f,\ f\in C_b(I),\ x\in I,\end{cases} \tag{1.1}$$

$\mathcal{G}^0$ so defined is independent of $\alpha>0$ by the resolvent equation.
Let us call $\mathcal{G}^0$ the $\mathbf{C_b}$**-generator** of X^0.
For X^0, the fine continuity is equivalent to the ordinary continuity so that $C_b(I)$ is the space of all bounded finely continuous functions on I.
With this interpretation, the above definition of the C_b-generator is well formulated for a general right process.
In Chapter 4 of Itô-McKean's book [IM2], it was proved that,
for a given minimal diffusion X^0, there exist
a canonical scale s, a canonical measure m and a positive Radon measure k called a **killing measure** on I such that

$$(\mathcal{G}^0u)(x)=\frac{dD_su-udk}{dm}(x) \qquad x\in I, \quad \text{for any } u\in\mathcal{D}(\mathcal{G}^0), \tag{1.2}$$

in the sense that the Radon Nikodym derivative appearing on the right hand side has a version belonging to $C_b(I)$ which coincides with the left hand side.
In particular, we have for $u=R^0_\alpha f,\ f\in C_b(I)$,

$$\alpha u(x)-\frac{dD_su-udk}{dm}(x)=f(x) \qquad x\in I. \tag{1.3}$$

The triplet (s, m, k) is unique up to a multiplicative constant
in the sense that, for another such triplet $(\widetilde{s}, \widetilde{m}, \widetilde{k})$, there exists a constant $c > 0$ such that $d\widetilde{s} = cds,\ d\widetilde{m} = c^{-1}dm$ and $d\widetilde{k} = c^{-1}dk$.

We call (s, m, k) satisfying (1.2) to be a **triplet attached to** the minimal diffusion X^0.
(1.2) can be called a **generalized second order elliptic differential operator** because any operator of the form

$$\mathcal{A}^0 u(x) = \frac{1}{2}a(x)u''(x) + b(x)u'(x) + c(x)u(x),\ x \in I,\ a,\ b,\ c \in C_b(I),\ a > 0,\ c \leq 0,$$

can be converted into (1.2) by

$$ds = \exp\left(-\int \frac{2b(\xi)}{a(\xi)}d\xi\right)dx,\ dm = \frac{2}{a(x)}\exp\left(-\int \frac{2b(\xi)}{a(\xi)}d\xi\right)dx,\ dk = -c(x)dx.$$

The triplet (s, m, k) attached to a minimal diffusion $X^0 = (X_t^0, \zeta^0, \mathbf{P}_x)$ was constructed in [IM2] as follows: for $r_1 < a < b < r_2$ and $J = (a, b)$,
consider the hitting probabilities and mean exit time

$$p_{ab}(\xi) = \mathbf{P}_\xi^0(\sigma_a < \sigma_b),\ p_{ba}(\xi) = \mathbf{P}_\xi^0(\sigma_b < \sigma_a),\ e_{ab}(\xi) = \mathbf{E}_\xi^0[\tau_J], \quad \xi \in J,$$

and define

$$\begin{cases} s(d\xi) = s_{ab}(d\xi) = p_{ab}(\xi)p_{ba}(d\xi) - p_{ba}(\xi)p_{ab}(d\xi) \\ k(d\xi) = k_{ab}(d\xi) = D_s p_{ab}(d\xi)/p_{ab}(\xi) \\ m(d\xi) = m_{ab}(d\xi) = -\left\{D_s e_{ab}(d\xi) - e_{ab}(\xi)k_{ab}(d\xi)\right\}, \quad a < \xi < b. \end{cases}$$

For another choice of $\widetilde{a}, \widetilde{b}$ with $r_1 < \widetilde{a} < a < b < \widetilde{b} < r_2$, we have

$$s_{\widetilde{a}\widetilde{b}}(d\xi) = cs_{ab}(d\xi),\ k_{\widetilde{a}\widetilde{b}}(d\xi) = c^{-1}k_{ab}(d\xi),\ m_{\widetilde{a}\widetilde{b}}(d\xi) = c^{-1}m_{ab}(d\xi),\ a < \xi < b,$$

for a constant $c > 0$ depending on $a, b, \widetilde{a}, \widetilde{b}$, so that a universal triplet (s, m, k) can be introduced on I.

Problem
Identification of the domain $\mathcal{D}(\mathcal{G}^0) \subset C_b(I)$ of C_b-generator of X^0.

To this end, we first prove the m-symmetry of X^0
and determine its Dirichlet form.

2 m-symmetry of X^0 and its Dirichlet form

2.1 a key lemma

$J = (j_1, j_2)$ with $r_1 < j_1 < j_2 < r_2$: a subinterval of I
$R_\alpha^J f(x) = \mathbf{E}_x^0 \left[\int_0^{\tau_J} e^{-\alpha t} f(X_t^0)dt\right]$: the resolvent of the part process of X^0 on J.

Lemma 2.1 *Let $u = R_\alpha^J f$ for $f \in C_b(I)$. Then*

$$u \in C_c(I), \quad \alpha u - \frac{dD_s u - udk}{dm} = f \quad on \quad J, \tag{2.1}$$

for a triplet (s, m, k) attached to X^0. Moreover

$$u(j_1+) = u(j_2-) = 0. \tag{2.2}$$

Indeed, u can be expressed as
$u = R_\alpha^0 f + c_1 R_\alpha^0 g_1 + c_2 R_\alpha^0 g_2$ for $g_1,\ g_2 \in C_b(I)$ vanishing on J and strictly positive on $(r_1, j_1),\ (j_2, r_2)$, respectively, and for some constant $c_1,\ c_2$.
(2.1) follows from this and (1.3).
(2.2) can be shown using the continuity of $R_\alpha^0 g$ for $g \in C_b(I)$.

2.2 m-symmetry of X^0

Let $u = R_\alpha^J f,\ v = R_\alpha^J g$ for $f, g \in C_c(I)$. We then get from (2.1)

$$-\int_J v dD_s u + \int_J uvdk + \alpha \int_J uvdm = \int_J vfdm.$$

By (2.2), $v(j_1+)D_s u(j_1+) - v(j_2-)D_s u(j_2-) = 0$ so that an integration by parts gives

$$\int_J (D_s u)(D_s v)ds + \int_J uvdk + \alpha \int_J uvdm = \int_J vfdm. \tag{2.3}$$

Thus

$$\int_J f\ R_\alpha^J g dm = \int_J R_\alpha^J f\ g dm,$$

which implies the **m-symmetry**

$$\int_I f\ R_\alpha^0 g dm = \int_I R_\alpha^0 f\ g dm$$

of the resolvent of X^0 by letting $J \uparrow I$ for $f \geq 0, g \geq 0$.

2.3 The Dirichlet form of X^0 on $L^2(I;m)$

Define the space $(\mathcal{F}^{(s)}, \mathcal{E}^{(s)})$ by

$$\mathcal{F}^{(s)} = \{u : u \text{ is absolutely continuous in } s \text{ and } \mathcal{E}^{(s)}(u,u) < \infty\}. \tag{2.4}$$

$$\mathcal{E}^{(s)}(u,v) = \int_I D_s u(x) D_s v(x) \; ds(x) \tag{2.5}$$

An elementary inequality holds:

$$(u(b) - u(a))^2 \leq |s(b) - s(a)| \mathcal{E}^{(s)}(u,u), \quad a, b \in I, \quad u \in \mathcal{F}^{(s)}. \tag{2.6}$$

We call the boundary r_i **approachable** if $|s(r_i)| < \infty, \; i = 1, 2$.
If r_i is approachable, then any $u \in \mathcal{F}^{(s)}$ admits a finite limit $u(r_i)$ in view of (2.6).
Let us introduce the space

$$\mathcal{F}_0^{(s)} = \{u \in \mathcal{F}^{(s)} : u(r_i) = 0 \text{ whenever } r_i \text{ is approachable}\}. \tag{2.7}$$

We further write $(u,v)_k = \int_I uvdk, \; (u,v) = \int_I uvdm$, and let

$$\begin{cases} \mathcal{F}^{(s),k} = \mathcal{F}^{(s)} \cap L^2(I;k), \quad \mathcal{F}_0^{(s),k} = \mathcal{F}_0^{(s)} \cap L^2(I;k), \\ \mathcal{E}^{(s),k}(u,v) = \mathcal{E}^{(s)}(u,v) + (u,v)_k, \quad u, v \in \mathcal{F}^{(s),k}, \\ \mathcal{E}_\alpha^{(s),k}(u,v) = \mathcal{E}^{(s),k}(u,v) + \alpha(u,v), \; \alpha > 0, \; u, v \in \mathcal{F}^{(s),k} \cap L^2(I;m). \end{cases} \tag{2.8}$$

We will be concerned with the form $(\mathcal{E}^0, \mathcal{F}^0)$ defined by

$$\mathcal{F}^0 = \mathcal{F}_0^{(s),k} \cap L^2(I;m), \quad \mathcal{E}^0(u,v) = \mathcal{E}^{(s),k}(u,v), \; u, v \in \mathcal{F}^0, \tag{2.9}$$

which can be readily shown to be a regular Dirichlet form on $L^2(I;m)$.
Further, each one point of I is of positive capacity with respect to the form (2.9) because (2.6) implies for any finite closed interval $K \subset I$

$$\sup_{x \in K} u(x)^2 \leq C_K \; \mathcal{E}_1^0(u,u), \quad u \in \mathcal{F}^0. \tag{2.10}$$

We say that the boundary r_i is **regular**
if r_i is approachable and $m + k$ is finite in a neighborhood of r_i.
If r_i is approachable but non-regular, then any function in $\mathcal{F}^{(s),k} \cap L^2(I;m)$ vanishes at r_i.
In particular, $\mathcal{F}^0$ can be rewritten as

$$\mathcal{F}^0 = \{u \in \mathcal{F}^{(s),k} \cap L^2(I;m) : u(r_i) = 0 \text{ whenever } r_i \text{ is regular}\}. \tag{2.11}$$

Theorem 2.2 (i) *X^0 is m-symmetric.*
(ii) *The Dirichlet form of X^0 on $L^2(I;m)$ coincides with $(\mathcal{E}^0, \mathcal{F}^0)$ defined by* (2.9) *in terms of the attached triplet (s, m, k).*
(iii) *Conversely, for an arbitrary triplet (s, m, k), define the regular Dirichlet form $(\mathcal{E}^0, \mathcal{F}^0)$ on $L^2(I;m)$ by* (2.9)*. Then the associated Hunt process on I is a minimal diffusion on I possessing (s, m, k) as its attached triplet.*

Proof. (i) was shown already.
To see (ii), let $(\mathcal{E}, \mathcal{F})$ be the Dirichlet form of X^0 on $L^2(I;m)$. By a general theory (cf. [FOT]), $\{R_\alpha^J f,\ f \in C_c(I),\ \overline{J} \subset I\}$ is dense in $\mathcal{F}$ and

$$\mathcal{E}(u,v) + \alpha \int_J uv dm = \int_J vf dm, \quad \text{for } u = R_\alpha^J f,\ v = R_\alpha^J g,\ f, g \in C_c(I). \tag{2.12}$$

By comparing this with (2.3), we have

$$\mathcal{F} \subset \mathcal{F}^0, \quad \mathcal{E} = \mathcal{E}^0 \quad \text{on} \quad \mathcal{F} \times \mathcal{F}.$$

We also have the identity (2.3) for $u = R_\alpha^J f,\ v \in \mathcal{F}^0 \cap C_c(I)$, which means that $\mathcal{F}^0 \cap C_c(I) \subset \mathcal{F}$. Since $(\mathcal{E}^0, \mathcal{F}^0)$ is regular, we get $\mathcal{F} = \mathcal{F}^0$.

(iii) Given a triplet (s, m, k), the Dirichlet form defined by (2.9) is not only regular but also local and irreducible. Since each one point set of I is of positive capacity, the associated Hunt process X^0 on I is a minimal diffusion I and m-symmetric.

Let $(\widetilde{s}, \widetilde{m}, \widetilde{k})$ be a triplet attached to X^0.
Then X^0 is $\widetilde{m}$-symmetric by (i) above, and consequently $\widetilde{m} = m$ up to a constant multiplication due to the uniqueness theorem of Ying and Zhao [YZ].

We may assume $\widetilde{m} = m$.

By (ii), the Dirichlet form $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ of X^0 on $L^2(I;m)$ is given by (2.9) with $(\widetilde{s}, \widetilde{k})$ in place of (s, k).

Since $\widetilde{\mathcal{E}} = \mathcal{E}$, we have $\widetilde{s} = s,\ \widetilde{k} = k$. □

2.4 L^2-generator of X^0

X^0: a minimal diffusion on I with the triplet (s, m, k) attached to it.
$\mathcal{A}^0$: the generator of the strongly continuous contraction semigroup of X^0 on $L^2(I; m)$:

$$u \in \mathcal{D}(\mathcal{A}^0) \quad \text{and} \quad \mathcal{A}^0 u = f \in L^2(I; m)$$

if and only if

$$u \in \mathcal{F}^0, \quad \mathcal{E}^0(u, v) = (f, v) \quad \text{for any} \quad v \in \mathcal{F}^0 \cap C_c(I), \tag{2.13}$$

on account of the regularity of $(\mathcal{E}^0, \mathcal{F}^0)$.
$\mathcal{A}^0$ is smply called the $\mathbf{L^2}$**-generator** of X^0. We write $u(r_i) = \lim_{x \to r_i,\, x \in I} u(x)$.
The following is immediate from Theorem 2.2 (ii) and (2.11):

Corollary 2.3 *$u \in \mathcal{D}(\mathcal{A}^0)$ if and only if*

$$\begin{cases} u \in \mathcal{F}^{(s),k} \cap L^2(I : m), \quad \dfrac{dD_s u - udk}{dm} \in L^2(I; m), \quad \text{and} \\ u(r_i) = 0 \quad \text{whenever} \quad r_i \quad \text{is regular.} \end{cases} \tag{2.14}$$

In this case,

$$\mathcal{A}^0 u = \frac{dD_s u - udk}{dm}, \quad u \in \mathcal{D}(\mathcal{A}^0). \tag{2.15}$$

3 Reflecting extension X^r of X^0

3.1 Reflected Dirichlet space $(\mathcal{F}^r, \mathcal{E}^r)$ of $(\mathcal{E}^0, \mathcal{F}^0)$

Given a triplet (s, m, k) on an interval $I = (r_1, r_2)$, recall the form $(\mathcal{E}^{(s)}, \mathcal{F}^{(s)})$ defined by (2.4), (2.5) and the form $(\mathcal{E}^{(s),k}, \mathcal{F}^{(s),k})$ defined by (2.8). We write

$$\mathcal{F}^r = \mathcal{F}^{(s),k} \cap L^2(I; m), \quad \mathcal{E}^r(u, v) = \mathcal{E}^{(s),k}(u, v), \quad u, v \in \mathcal{F}^r. \tag{3.1}$$

We denote by I^* the interval obtained from I by adding r_i if it is regular , $i = 1, 2$, (for the triplet (s, m, k)).
We know from (2.6) that any function in $\mathcal{F}^{(s)}$ can be continuously extended to I^*.
The canonical measure m is extended to I^* by setting $m(I^* \setminus I) = 0$.
$L^2(I^*; m)$ can be identified with $L^2(I; m)$.

Theorem 3.1 (i) $(\mathcal{E}^r, \mathcal{F}^r)$ *is a regular, local and irreducible Dirichlet form on* $L^2(I^*; m)$ *for which each one point of* I^* *has a positive capacity.*
(ii) *Define* $(\mathcal{E}^0, \mathcal{F}^0)$ *by* (2.9) *which is a regular Dirichlet form on* $L^2(I; m)$. $(\mathcal{F}^r, \mathcal{E}^r)$ *is then the active reflected Dirichlet space of* $(\mathcal{F}^0, \mathcal{E}^0)$.

The last statement of (i) follows from the inequality (2.10) holding for any compact subset of I^* and $\mathcal{E}^r$, $\mathcal{F}^r$, in place of I, $\mathcal{E}^0$, $\mathcal{F}^0$. (ii) is shown in Chapter 6 of [CF].
By (i), there exists uniquely an m-diffusion $X^r = (X_t^r, \mathbf{P}_x^r)$ on I^* whose Dirichlet form on $L^2(I^*; m)$ equals $(\mathcal{E}^r, \mathcal{F}^r)$. X^r is strongly irreducible in the sense that

$$\mathbf{P}_x^r(\sigma_y < \infty) > 0, \quad \text{for any} \quad x, y \in I^*. \tag{3.2}$$

In view of (2.9), X^r is an **m-symmetric diffusion extension** of X^0 in the sense that the part process of X^r on I, namely, the process obtained from it by killing upon hitting $I^* \setminus I$ coincides with X^0.
On account of (ii), we may call X^r the **reflecting extension** of X^0.

3.2 L^2-generator of X^r

$\mathcal{A}^r$: the generator of the strongly continuous contraction semigroup of X^r on $L^2(I; m)$.
$u \in \mathcal{D}(\mathcal{A}^r)$ and $\mathcal{A}^r u = f \in L^2(I; m)$ if and only if
(2.13) holds for $\mathcal{F}^r$, $\mathcal{E}^r$, $C_c(I^*)$ in place of $\mathcal{F}^0$, $\mathcal{E}^0$, $C_c(I)$.

Proposition 3.2 $u \in \mathcal{D}(\mathcal{A}^r)$ *if and only if*

$$\begin{cases} u \in \mathcal{F}^{(s),k} \cap L^2(I : m), \quad \dfrac{dD_s u - u dk}{dm} \in L^2(I; m), \quad \text{and} \\ D_s u(r_i) = 0 \quad \text{whenever} \quad r_i \quad \text{is regular.} \end{cases} \tag{3.3}$$

In this case,

$$\mathcal{A}^r u = \frac{dD_s u - u dk}{dm}, \quad u \in \mathcal{D}(\mathcal{A}^r). \tag{3.4}$$

The second condition in (3.3) can be deduced either by integration by parts or by a general condition that the flux of u at r_i equals zero formulated in Chapter 7 of [CF].

4 C_b-generators of X^0 and X^r

4.1 Boundary classification and behaviors of α-harmonic functions

For a given triplet (s, m, k), we adopt Feller's classification of the boundary: we write $j = m + k$ and we let for $r_1 < c < r_2$

$$\lambda_1 = \int_{r_1}^{c} s(dx) \int_{x}^{c} j(dy), \quad \mu_1 = \int_{r_1}^{c} j(dx) \int_{x}^{c} s(dy), \ r_1 < c < r_2.$$

The left boundary r_1 of I is called

regular	if	$\lambda_1 < \infty,$	$\mu_1 < \infty,$
exit	if	$\lambda_1 < \infty,$	$\mu_1 = \infty,$
entrance	if	$\lambda_1 = \infty,$	$\mu_1 < \infty,$
natural	if	$\lambda_1 = \infty,$	$\mu_1 = \infty.$

An analogous classification of r_2 is in force.
r_i is regular in Feller's sense if and only if it is regular in the previous sense, namely, it is approachable and j is finite in a neighborhood of r_i.
Moreover, if r_i is exit, then it is approachable but non-regular, and so

$$u(r_i) = 0 \quad \text{for any } u \in \mathcal{F}^r \quad \text{whenever } r_i \text{ is exit.} \tag{4.1}$$

For a given triplet (s, m, k) on I, consider a homogeneous equation

$$\alpha u(x) - \frac{dD_s u - u dk}{dm}(x) = 0, \quad x \in I, \quad \alpha > 0. \tag{4.2}$$

whose solution is called α-**harmonic**. It is known that there exist a positive strictly increasing (resp. decreasing) solution u_1 (resp. u_2) of (4.2).
When r_i is regular, there are many solutions u_i; among them are
the minimal one $\underline{u}_i$ with $\underline{u}_i(r_i) = 0,\ D_s \underline{u}_i(r_i) < 0$
and the maximal one $\overline{u}_i$ with $D_s \overline{u}_i(r_i) = 0,\ \overline{u}_i(r_i) > 0$.
Otherwise u_i is unique up to a multiplicative positive constant.

The following table on the behaviors of u_i for the right boundary r_2 is taken from Itô-McKean's book [IM]:

	regular	exit	entrance	natural
$u_1(r_2)$	$\in (0,\infty)$	$\in (0,\infty)$	$= \infty$	$= \infty$
$D_s u_1(r_2)$	$\in (0,\infty)$	$= \infty$	$\in (0,\infty)$	$= \infty$
$u_2(r_2)$	$< \infty$	$= 0$	$\in (0,\infty)$	$= 0$
$-D_s u_2(r_2)$	$< \infty$	$\in (0,\infty)$	$= 0$	$= 0$

(4.3)

4.2 The C_b-generator of X^0

Let X^0 be a minimal diffusion on I with an attached triplet (s, m, k).
By Theorem 1.2, X^0 is m-symmetric and its Dirichlet form on $L^2(I; m)$ is $(\mathcal{E}^0, \mathcal{F}^0)$ given by (2.9).
Due to (2.10), the Hilbert space $(\mathcal{F}^0, \mathcal{E}^0_\alpha)$ admits a **reproducing kernel** $g^0_\alpha(x, y),\ x, y \in I$: for each $y \in I$,

$$g^0_\alpha(\cdot, y) \in \mathcal{F}^0, \quad \mathcal{E}^0_\alpha(g^0_\alpha(\cdot, y), v) = v(y), \quad \text{for any } v \in \mathcal{F}^0. \tag{4.4}$$

It follows from the first property of (2.9) and (4.1) that

$$g^0_\alpha(r_i, y) = 0, \quad \text{whenever } r_i \text{ is either regular or exit.} \tag{4.5}$$

Lemma 4.1 (i) $g^0_\alpha(x, y)$ *admits an expression*

$$g^0_\alpha(x, y) = \begin{cases} W(u_1, u_2)^{-1}\, u_1(x)u_2(y) & \text{if } x \le y, \quad x, y \in I, \\ W(u_1, u_2)^{-1}\, u_2(x)u_1(y) & \text{if } x \ge y \quad x, y \in I, \end{cases} \tag{4.6}$$

where $W(u_1, u_2)(x) = D_s u_1(x)u_2(x) - D_s u_2(x)u_1(x)$ *is the* **Wronskian** *of* u_1, u_2 *which is positive and independent of* $x \in I$. *Here* u_i *should be chosen to be*

$$u_i = \underline{u}_i, \quad \text{whenever } r_i \text{ is regular,} \tag{4.7}$$

(ii) $g^0_\alpha(x, y)$ *is a density function of the resolvent kernel* R^0_α *of* X^0 *with respect to* m*:*

$$R^0_\alpha f(x) = \int_I g^0_\alpha(x, y) f(y) m(dy), \quad x \in I, \quad f \in C_b(I). \tag{4.8}$$

Notice that (4.5) and (4.6) imply that

$$u_i(r_i) = 0, \quad \text{whenever } r_i \text{ is exit,} \tag{4.9}$$

and we conclude from (4.7) and (4.9) that, for $f \in C_b(I)$,

$$R^0_\alpha f(r_i) = 0, \quad \text{if } r_i \text{ is either regular or exit.} \tag{4.10}$$

We now give a complete characterization of the C_b-generator $\mathcal{G}^0$ of the minimal diffusion X^0 on I.

Theorem 4.2 $u \in \mathcal{D}(\mathcal{G}^0)$ *if and only if*

$$\begin{cases} u \in C_b(I), \quad \dfrac{dD_s u - u dk}{dm} \in C_b(I), \quad \text{and} \\ u(r_i) = 0 \quad \text{if} \quad r_i \quad \text{is either regular or exit.} \end{cases} \tag{4.11}$$

In this case,

$$\mathcal{G}^0 u = \frac{dD_s u - udk}{dm}, \quad u \in \mathcal{D}(\mathcal{G}^0). \tag{4.12}$$

Proof. Take any $u \in \mathcal{D}(\mathcal{G}^0)$ so that $u = R^0_\alpha f$ for some $f \in C_b(I)$.
Then u satisfies the boundary condition in (4.11) by (4.10).
$\mathcal{G}^0 u = \alpha u - f$, while it follows from (4.6) that $\alpha u - f = \frac{dD_s u - udk}{dm}$.
Conversly, take any u satisfying condition (4.11) and let $f = \alpha u - \frac{dD_s u - udk}{dm}$, $v = R^0_\alpha f$ and $w = u - v$. Then w is a bounded α-harmonic function and vanishes whenever r_i is regular or exit.
Write $w = C_1 u_1 + C_2 u_2$. If both $r_1,\ r_2$ are either regular or exit, then $w(r_1) = w(r_2) = 0$ and we get $C_1 = C_2 = 0$ because $u_1(r_1)u_2(r_2) - u_1(r_2)u_2(r_1) < 0$. If r_1 is either regular or exit but r_2 is either entrance or natural, then $u_1(r_2) = \infty$ by the table (4.3) so that $C_1 = 0$ and $0 = w(r_1) = C_2 u_2(r_1)$, yielding $C_2 = 0$ because $u_2(r_1) > 0$. If both $r_1,\ r_2$ are either entrance or natural, we have $C_1 = C_2 = 0$. □

4.3 The C_b-generator of X^r

X^r: the reflecting extension of minimal diffusion X^0 with an attached triplet (s, m, k).
R^r_α: the resolvent of X^r;

$$R^r_\alpha f(x) = \mathbf{E}^r_x \left[\int_0^\infty e^{-\alpha t} f(X^r_t) dt \right], \quad R^r_\alpha f(x) = \int_{I^*} R^r_\alpha(x, dy) f(y),\ x \in I^*.$$

X^r has the strong irreducibility $\mathbf{P}^r_a(\sigma_b < \infty) > 0,\ \forall a, b \in I^*$ so that

$$\mathbf{E}^r_a \left[e^{-\alpha \sigma_{a\pm}} \right] = 1,\ \forall a \in I,\ \mathbf{E}^r_{r_1} \left[e^{-\alpha \sigma_{r_1 +}} \right] = 1, \text{ if } r_1 \in I^*,\ \mathbf{E}^r_{r_2} \left[e^{-\alpha \sigma_{r_2 -}} \right] = 1, \text{ if } r_2 \in I^*.$$

Therefore if we define

$$C_b(I^*) = \{u \in C_b(I) : u(r_i) = \lim_{x \to r_i,\ x \in I} u(x) \text{ whenever } r_i \in I^*\}, \tag{4.13}$$

then $R^r_\alpha(\mathcal{B}_b(I)) \subset C_b(I^*)$ and the C_b-*generator* $\mathcal{G}^r$ of X^r is well defined by

$$\begin{cases} \mathcal{D}(\mathcal{G}^r) = R^r_\alpha(C_b(I^*)), \\ (\mathcal{G}^r u)(x) = \alpha u(x) - f(x),\ \text{for } u = R^r_\alpha f,\ f \in C_b(I^*),\ x \in I^*. \end{cases} \tag{4.14}$$

The Dirichlet form $(\mathcal{E}^r, \mathcal{F}^r)$ of X^r on $L^2(I^*; m)$ admits a reproducing kernel $g^r_\alpha(x, y),\ x, y \in I^*$; for each $y \in I^*$,

$$g^r_\alpha(\cdot, y) \in \mathcal{F}^r, \quad \mathcal{E}^r_\alpha(g^r_\alpha(\cdot, y), v) = v(y), \quad \text{for any } v \in \mathcal{F}^r. \tag{4.15}$$

This implies that

$$\begin{cases} D_s g_\alpha^r(r_i, y) = 0, & \text{for each } y \in I, \quad \text{whenever } r_i \text{ is regular} \\ g_\alpha^r(r_i, y) = 0, & \text{for each } y \in I, \quad \text{whenever } r_i \text{ is exit} \end{cases} \tag{4.16}$$

Analogously to Lemma 4.1, g_α^r can be shown to be a density function of the resolvent kernel R_α^r with respect to m and admit a similar expression to (4.6)
but with $\overline{u}_i$ in place of $\underline{u}_i$ whenever r_i is regular.
Hene the next theorem can be proved in a similar way to the proof of the preceding theorem by using (4.16) and the table (4.3):

Theorem 4.3 *$u \in \mathcal{D}(\mathcal{G}^r)$ if and only if*

$$\begin{cases} u \in C_b(I^*), \quad \dfrac{dD_s u - u dk}{dm} \in C_b(I^*), \quad \text{and} \\ D_s u(r_i) = 0 \quad \text{if } r_i \text{ is regular}, \quad u(r_i) = 0 \text{ if } \quad r_i \text{ is exit.} \end{cases} \tag{4.17}$$

In this case,

$$\mathcal{G}^r u = \frac{dD_s u - u dk}{dm}, \quad u \in \mathcal{D}(\mathcal{G}^r). \tag{4.18}$$

5 Proper symmetric diffusion extensions of X^0

5.1 Symmetric diffusion extensions with no sojourn nor killing

X^0: minimal diffusion on $I = (r_1, r_2)$ with attached triplet (s, m, k)
S: a closed set into which I is embedded as a dense open subset
m is extended to S by setting $m(S \setminus I) = 0$.
Suppose X^S is an m-symmetric diffusion Hunt process on S whose part process on I coincides with X^0.
Then the Dirichlet form $(\mathcal{E}^S, \mathcal{F}^S)$ of X^S on $L^2(S; m) = L^2(I; m)$ is quasi-regular (cf. [CF]) and satisfies (cf. [FOT])

$$\mathcal{F}^0 \subset \mathcal{F}^S, \quad \mathcal{E}^S(u, v) = \mathcal{E}^0(u, v), \quad \text{for any } u, v \in \mathcal{F}^0. \tag{5.1}$$

For two closed sets $S_1,\ S_2$ as above, we write $S_1 \sim S_2$ if they are quasi-homeomorphic.
X^F is called a **proper symmetric diffusion extension** of X^0 to S **with no sojourn nor killing** if
(a) X^S it is an m-symmetric diffusion Hunt process on S,
(b) X^S admits no killing on $S \setminus I$,
(c) the part process of X^S on I coincides with X^0, and
(d) $\mathcal{F}^0$ is a proper subspace of $\mathcal{F}^S$.

Theorem 5.1 (i) X^0 *admits a proper symmetric diffusion extension* X^S *with no sojourn nor killing if and only if*

$$\text{either } r_1 \text{ or } r_2 \text{ is a regular boundary of } I. \tag{5.2}$$

(ii) *If* r_1 *(resp.* r_2*) is regular and* r_2 *(resp.* r_1*) is non-regular, then* $S \sim I^*$ *and* $X^S = X^r$.
(iii) *If both* r_1 *and* r_2 *are regular, then four cases can occur:*

1. $S \sim [r_1, r_2]$, $\quad X^S = X^r$,

2. $S \sim [r_1, r_2)$, $\quad X^S = X^r$ being killed upon hitting r_2,

3. $S \sim (r_1, r_2]$, $\quad X^S = X^r$ being killed upon hitting r_1,

4. $S \sim \dot{I}$, $\quad X^S =$ the one-point extension of X^0 from I to $\dot{I}$.

Here $\dot{I}$ *denotes the one-point compactification of* I.

Proof. By quasi-homeomorphism and Theorem 3.3.8 of [CF],
$\mathcal{F}^0$ can be identified with a subspace $\mathcal{F}^{S,0} = \{u \in \mathcal{F}^S : u = 0 \text{ q.e. on } S\}$ of $\mathcal{F}^S$.
In particlular, $\mathcal{F}^0$ is an ideal of $\mathcal{F}^S$ and we have by Theorem 6.6.9 of [CF], $\mathcal{F}^S \subset \mathcal{F}^r$ and $\mathcal{E}^S(u,u) \geq \mathcal{E}^r(u,u), \ u \in \mathcal{F}^S$.
This combined with (5.1) and property (b) of X^S leads us to

$$\mathcal{F}^0 \subset \mathcal{F}^S \subset \mathcal{F}^r, \quad \mathcal{E}^S(u,v) = \mathcal{E}^{(s),k}(u,v), \ u,v \in \mathcal{F}^S. \tag{5.3}$$

On the other hand, $\mathcal{E}^r_\alpha$-orthogonal complement $\mathcal{H}_\alpha$ of $\mathcal{F}^0$ in $\mathcal{F}^r$ consists of α-hamonic functions. The integration by parts gives, $r_1 < a < b < r_2$,

$$\begin{aligned}\int_a^b (D_s u_i(x))^2 ds(x) \quad &+ \quad \int_a^b u_i(x)^2 dk(x) + \alpha \int_a^b u_i(x)^2 dm(x) \\ &= \quad u_i(b) D_s u_i(b) - u_i(a) D_s u_i(a).\end{aligned}$$

On account of the table (4.3), we thus conclude that $u_i \in \mathcal{H}_\alpha$ if and only if r_i is regular. Consequently

$$\mathcal{H}_\alpha = \{c_1 u_i + c_2 u_2 : \ c_i = 0, \quad \text{unless } r_i \text{ is regular}\}. \tag{5.4}$$

Theorem 4.4 follows readily from (5.3) and (5.4).

The C_b-generator of X^S of Theorem 4.4 can be described as Theorem 4.3 by replacing the boundary condition in (4.17) according to the cases of S as follows:

case (ii). $D_s u(r_1) = 0$ (resp. $D_s u(r_2) = 0$),

$u(r_2) = 0$ (resp. $u(r_1) = 0$), if r_2 (resp. r_1) is exit

case (iii), 1. $D_s u(r_1) = 0, \quad D_s u(r_2) = 0$

case (iii), 2. $D_s u(r_1) = 0, \quad u(r_2) = 0$

case (iii), 3. $u(r_1) = 0, \quad D_s u(r_2) = 0$

case (iii), 4. $u(r_1) = u(r_2), \quad D_s u(r_1) = D_s u(r_2)$.

5.2 Symmetric diffusion extensions with sojourn and killing

Given a minimal diffusion X^0 on $I = (r_1, r_2)$ with attached triplet (s, m, k), the most general proper symmetric diffusion extension X^S of X^0 with no sojourn nor killing on $S \setminus I$ has been studied in the preceding section.
We can admit sojourn and killing for X^S that amounts to extending m and k to S by allowing them to have positive point masses on $S \setminus I$ and considering a proper symmetric diffusion extension of X^0 associated with the resulting Dirichlet form.
We consider the typical case where the left boundary r_1 of I is regular but the right boundary r_2 is non-regular. Let m^* and k^* be extensions of m and k from I to $I^* = [r_1, r_2)$, respectively allowing point masses at r_1 so that

$$m^*(r_1) =: m^*(\{r_1\}) \geq 0, \quad k^*(r_1) =: k^*(\{r_1\}) \geq 0. \tag{5.5}$$

Define the Dirichlet form $(\mathcal{E}^*, \mathcal{F}^*)$ on $L^2(I^*; m^*)$ by

$$\begin{cases} \mathcal{F}^* = \mathcal{F}^{(s)} \cap L^2(I^*; k^*) \cap L^2(I^*; m^*) \\ \mathcal{E}^*(u, v) = \mathcal{E}^{(s),k}(u, v) + u(r_1)v(r_1)k^*(r_1), \quad u, v \in \mathcal{F}^*. \end{cases} \tag{5.6}$$

$(\mathcal{E}^*, \mathcal{F}^*)$ is then a regular, local irreducible Dirichlet form on $L^2(I^*; m^*)$ and it admits an associated m^*-symmetric diffusion process $X^* = (X^*_t, \mathbf{P}^*_x)$ on I^*.
$\{R^*_\alpha; \alpha > 0\}$ denotes the resolvent kernel of X^*.
Just as the case of the reflecting extension X^r, if we define the space $C_b(I^*)$ by

$$C_b(I^*) = \{u \in C_b(I) : u(r_1) = \lim_{x \to r_1,\, x \in I} u(x)\}, \tag{5.7}$$

then $R^*_\alpha(C_b(I)) \subset C_b(I^*)$ and the C_b-generator $\mathcal{G}^*$ of X^* is well defined by

$$\begin{cases} \mathcal{D}(\mathcal{G}^*) = R^*_\alpha(C_b(I^*)), \\ (\mathcal{G}^* u)(x) = \alpha u(x) - f(x), \quad x \in I^*, \quad \text{for } u = R^*_\alpha f,\ f \in C_b(I^*). \end{cases}$$

Furthermore, for $\alpha > 0$, the Hilbert space $(\mathcal{F}^*, \mathcal{E}^*_\alpha)$ admits a reproducing kernel $g^*_\alpha(x,y),\ x,y \in I^*$: for each $y \in I^*$,

$$g^*_\alpha(\cdot,y) \in \mathcal{F}^*, \quad \mathcal{E}^*_\alpha(g^*_\alpha(\cdot,y),v) = v(y), \quad \text{for any } v \in \mathcal{F}^*.$$

The following is the counterpart of Lemma 4.1 for X^*.:

Lemma 5.2 (i) *$g^*_\alpha(x,y)$ admits an expression*

$$g^*_\alpha(x,y) = \begin{cases} W(u_1,u_2)^{-1}u_1(x)u_2(y) & \text{if } x \le y, \quad x,y \in I^*, \\ W(u_1,u_2)^{-1}u_2(x)u_1(y) & \text{if } x \ge y \quad x,y \in I^*. \end{cases} \tag{5.8}$$

where

$$u_1 = \begin{cases} \overline{u}_1 & if \quad k^*(r_1) + m^*(r_1) = 0, \\ \overline{u}_1 + \frac{\overline{u}_1(r_1)}{D_s\underline{u}_1(r_1)}(k^*(r_1) + \alpha m^*(r_1))\underline{u}_1 & if \quad k^*(r_1) + m^*(r_1) > 0. \end{cases} \tag{5.9}$$

(ii) *For $f \in C_b(I^*)$ and $x \in I^*$, $R^*_\alpha f(x)$ admits an expression*

$$R^*_\alpha f(x) = \int_I g^*_\alpha(x,y)f(y)m(dy) + w(x), \tag{5.10}$$

where

$$w(x) = \begin{cases} 0 & if \quad m^*(r_1) = 0, \\ \frac{f(r_1)m^*(r_1)}{-D_s u_2(r_1) + k^*(r_1) + \alpha m^*(r_1)} u_2 & if \quad m^*(r_1) > 0. \end{cases} \tag{5.11}$$

The following is the counterpart of Theorem 4.3 for X^*:

Theorem 5.3 *$u \in \mathcal{D}(\mathcal{G}^*)$ if and only if*

$$u \in C_b(I^*), \quad \frac{dD_s u - u dk}{dm} \in C_b(I^*), \tag{5.12}$$

and

$$\begin{cases} D_s u(r_1) - u(r_1)k^*(r_1) = \mathcal{G}^* u(r_1) m^*(r_1), \\ u(r_2) = 0, \quad \text{if } r_2 \text{ is exit}, \end{cases} \tag{5.13}$$

where $\mathcal{G}^*u(r_1)$ denotes the value of the function $\frac{dD_s u - udk}{dm}(\in C_b(I^*))$ at r_1. In this case, for $u \in \mathcal{D}(\mathcal{G}^*)$,

$$\begin{cases} \mathcal{G}^* u(x) = \dfrac{dD_s u - udk}{dm}(x), \quad x \in I, \\ \mathcal{G}^* u(r_1) = (D_s u(r_1) - u(r_1)k^*(r_1))/m^*(r_1), \quad \text{if } m^*(r_1) > 0. \end{cases} \tag{5.14}$$

6 All possible diffusion extensions of X^0

X^0: a minimal diffusion on $I = (r_1, r_2)$ with an attached triplet (s, m, k)
We have considered a family of symmetric diffusion extensions of X^0 to a closed sets obtained by adding to I regular boundaries or their identification.
As was verified in the preceding section, this family exhausts
all possible proper symmetric diffusion extensions of X^0
All of them have been constructed by using Dirichlet form extensions of $(\mathcal{E}^0, \mathcal{F}^0)$.

$\mathbf{Ext_{DF}(X^0)}$: the collection of X^0 and all of its proper symmetric diffusion extensions as above.
By convention, we exclude from $\mathrm{Ext}_{\mathrm{DF}}(X^0)$ the one point extension $\dot{X}$ of X^0
to the one-point compactification $\dot{I}$ consider in Theorem 5.1, (iii), 4.
The C_b-generator of any $X \in \mathrm{Ext}_{\mathrm{DF}}(X^0)$ has been characterized in terms of boundary conditions.
In particular, the most general boundary condition at a regular boundary r_i is

$$D_s u(r_i) - u(r_i)k^*(r_i) = \mathcal{G}^* u(r_i) m^*(r_i), \tag{6.1}$$

where $\mathcal{G}^* u(r_i)$ denotes the value of the function $\frac{dD_s u - udk}{dm} (\in C_b([r_1, r_2]))$ at r_i.

$\mathbf{Ext_{IM}(X^0)}$: the collection of all diffusion extensions of X^0 to $[r_1, r_2]$ studied in §4.4 and §4.7 of Itô-McKean's book [IM2].
$\mathrm{Ext}_{\mathrm{IM}}(X^0)$ consists all (not necessarily symmetric) diffusion extensions of X^0 to $[r_1, r_2]$
except that the processes starting at entrance boundaries and remaining there until life time are excluded from $\mathrm{Ext}_{\mathrm{IM}}(X^0)$.
The C_b-generator of any $X \in \mathrm{Ext}_{\mathrm{IM}}(X^0)$ was characterized in terms of the boundary conditions imposed at both r_1 and r_2.
In particular, the most general boundary condition at a regular boundary r_i can be seen to be identical with (6.1).
$\mathrm{Ext}_{\mathrm{IM}}(X^0)$ contains an extension X of X^0 with a trivial boundary condition

$$\mathcal{G}u(r_i) + \kappa u(r_i) = 0, \ 0 \le \kappa < \infty,$$

at an non-entrace boundary r_i,
which means that X starting at r_i remains there until its life time.

We modify such X by
(a) killing it at time σ_{r_1} whenever it is finite and
(b) discarding r_i from its state space.

The resulting modified family is designated as $\mathbf{Ext'_{IM}(X^0)}$.

On the other hand, when I has entrance boundaries, they can be added to the state space of any $X \in \mathrm{Ext_{DF}}(X^0)$ to produce a improper symmetric extension $\widetilde{X}$ of X.
For simplicity, we explain this procedure only for $X = X^0 \in \mathrm{Ext_{DF}}(X^0)$.
When r_1 is entrance, there exists a diffusion $\widetilde{X}^0 = (\widetilde{X}^0_t, \widetilde{\mathbf{P}}^0_x)$
on the extended state space $[r_1, r_2)$ such that

$$\widetilde{X}^0\big|_I = X^0 \quad \text{and} \quad \widetilde{\mathbf{P}}^0_{r_1}(\widetilde{X}^0_t \in I \text{ for any } t \in (0, \widetilde{\zeta}^0)) = 1. \tag{6.2}$$

In particular,
the part process of $\widetilde{X}^0$ on I equals X^0 and
the one-point set $\{r_1\}$ is polar for $\widetilde{X}^0$
so that $\widetilde{X}^0$ can be viewed as an m-symmetric diffusion extension of X^0 that is improper, in the sense that,
$\widetilde{X}^0$ has the same Dirichlet form $(\mathcal{E}^0, \mathcal{F}^0)$ on $L^2(I; m)$ as X^0.
In fact, since r_1 is entrance, we have

$$\lim_{\epsilon \downarrow 0} \mathbf{E}^0_{r_1+}[e^{-\sigma_{\mathcal{R}_1+\epsilon}}] = 1, \quad \lim_{\epsilon \downarrow 0} \mathbf{P}^0_{r_1+\epsilon}(\sigma_{r_1+} < \infty) = 1.$$

Using these properties of X^0, the above mentioned extension $\widetilde{X}^0$ of X^0 can be constructed as in Problem 3.6.3 of Itô-Mckean's book
by defining $(\widetilde{X}^0_t, \widetilde{\mathbf{P}}^0_{r_1})$ to be a kind of limit of $(X^0_t, \mathbf{P}^0_{r_1+\frac{1}{n}})$ as $n \to \infty$
using the direct product $\prod_{n=1}^{\infty} \mathbf{P}^0_{r_1+\frac{1}{n}}$.
The C_b-generator of $\widetilde{X}^0$ can be readily identified as follows.
The property (6.2) implies $\widetilde{\mathbf{E}}^0_{r_1}[e^{-\sigma_{r_1+}}] = 1$ and consequently
the resolvent $\{\widetilde{R}^0_\alpha; \alpha > 0\}$ of $\widetilde{X}^0$ satisfies $\widetilde{R}^0_\alpha(\mathcal{B}_b(I)) \subset C_b([r_1, r_2))$.
We introduce the C_b-generator $\widetilde{\mathcal{G}}^0$ of $\widetilde{X}^0$ by

$$\begin{cases} \mathcal{D}(\widetilde{\mathcal{G}}^0) = \widetilde{R}^0_\alpha(C_b([r_1, r_2)), \\ (\widetilde{\mathcal{G}}^0 u)(x) = \alpha u(x) - f(x), \text{ for } u = \widetilde{R}^0_\alpha f, \ f \in C_b([r_1, r_2)), \ x \in [r_1, r_2). \end{cases}$$

Then we see just as in the proof of Theorem 4.2 that $u \in \mathcal{D}(\widetilde{\mathcal{G}}^0)$
if and only if u satisfies the condition (4.11) with $C_b([r_1, r_2))$ in place of $C_b(I)$.

If both r_1 and r_2 are entrance, we can replace the above $\widetilde{X}^0$ by its further improper m-symmetric extension to $[r_1, r_2]$ so that the resulting diffusion $\widetilde{X}^0$ has the same Dirichlet form $(\mathcal{E}^0, \mathcal{F}^0)$ as X^0 and its C_b-generator is characterized as (4.11) but with $C_b([r_1, r_2])$ in place of $C_b(I)$.

$\widetilde{\mathbf{Ext}}_{\mathbf{DF}}(\mathbf{X^0})$: the collection of all $X \in \mathrm{Ext}_{\mathrm{DF}}(X^0)$ but being modified to be $\widetilde{X}$ as above
by adding entrance boundaries whenever they are present.
We can then readily verify that

$$\mathbf{Ext}'_{\mathbf{IM}}(\mathbf{X^0}) = \widetilde{\mathbf{Ext}}_{\mathbf{DF}}(\mathbf{X^0}). \tag{6.3}$$

Thus every element X of $\mathrm{Ext}'_{\mathrm{IM}}(X^0)$ is symmetric with respect to m or its extension m^* to regular boundaries.
Furthermore we can verify that the transition function P_t of $X \in \mathrm{Ext}'_{\mathrm{IM}}(X^0)$ determines a Feller semigroup on the space $C_\infty(\widehat{I})$.
Here $\widehat{I}$ denotes the interval obtained from I by adding the boundaries r_i to it only in the following two cases:
(I) r_i is regular and X is not absorbed at r_i,
(II) r_i is entrance.
$C_\infty(\widehat{I})$ denotes the space of all continuous functions on $\widehat{I}$ vanishing at infinity of $\widehat{I}$.
Indeed, combining general expressions in Lemma 5.2 of the resolvent R_α of X
with Theorem 5.14.1 in Itô's book [I] and the table (4.3),
we can see that R_α makes invariant the space of bounded continuous functions on I vanishing at a natural boundary.
Therefore, on account of the observations we have made on the C_b-generator of X,
we can conclude that $R_\alpha(C_\infty(\widehat{I})) \subset C_\infty(\widehat{I})$.
Moreover $\lim_{\alpha\to\infty} \alpha R_\alpha f(x) = f(x),\ x \in \widehat{I},\ f \in C_\infty(\widehat{I})$, by the path continuity of X.
Hence $\{R_\alpha; \alpha > 0\}$ becomes a strongly continuous contraction resolvent on $C_\infty(\widehat{I})$ and consequently $\{P_t; t > 0\}$ is a strongly continuous contraction semigroup on $C_\infty(\widehat{I})$ with generator

$$\widehat{\mathcal{G}} = \alpha I - R_\alpha^{-1},\ \mathcal{D}(\widehat{\mathcal{G}}) = R_\alpha(C_\infty(\widehat{I})).$$

Proposition 6.1 *The transition function $\{P_t; t > 0\}$ of $X \in \mathrm{Ext}'_{\mathrm{IM}}(X^0)$ determines a strongly continuous contraction semigroup on $C_\infty(\widehat{I})$.*

Let $\widehat{\mathcal{G}}$ be its infinitesimal generator. $u \in \mathcal{D}(\widehat{\mathcal{G}})$ if and only if

$$u \in C_\infty(\widehat{I}), \qquad \frac{dD_s u - u dk}{dm} \in C_\infty(\widehat{I}), \tag{6.4}$$

and

$$D_s u(r_i) - u(r_i)k^*(r_i) = \widehat{\mathcal{G}}u(r_i)m^*(r_i), \quad \text{if } r_i \text{ is regular and } r_i \in \widehat{I}, \tag{6.5}$$

where $\widehat{\mathcal{G}}u(r_i)$ denotes the value of the function $\frac{dD_s u - udk}{dm} (\in C_\infty(\widehat{I}))$ at r_i, and $m^*(r_i)$, $k^*(r_i)$ are non-negative parameters.
In this case, it holds for $u \in \mathcal{D}(\widehat{\mathcal{G}})$ that

$$\begin{cases} \widehat{\mathcal{G}}u(x) = \dfrac{dD_s u - udk}{dm}(x), \quad x \in I, \\ \widehat{\mathcal{G}}u(r_i) = (D_s u(r_i) - u(r_i)k^*(r_i))/m^*(r_i), \quad \text{if } m^*(r_i) > 0. \end{cases} \tag{6.6}$$

Conversely, given a linear operator $\widehat{\mathcal{G}}$ on $C_\infty(\widehat{I})$ satisfying (6.4), (6.5) and (6.6), we can solve the equation $(\alpha - \widehat{\mathcal{G}})u = f$ in the space $C_\infty(\widehat{I})$ using the functions $g^*(x, y)$ and $w(x)$ defined in Lemma 5.2.

But it is not easy to verify that $\mathcal{D}(\widehat{\mathcal{G}})$ is dense in $C_\infty(\widehat{I})$ unless the associated Dirichlet form is utilized.

The Dirichlet form method gives us a direct and quickest way to construct the diffusion in $\mathrm{Ext}'_{\mathrm{IM}}(X^0)$,
firstly by constructing $X \in \mathrm{Ext}_{\mathrm{DF}}(X^0)$ by a regular Dirichlet form and secondly by considering the improper symmetric diffusion extension $\widetilde{X}$ of X to entrance boundaries.

The constructed diffusion in $\mathrm{Ext}'_{\mathrm{IM}}(X^0)$ has a Feller transition function by the above proposition.

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