Dirichlet forms and symmetric Markov processes

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Contents

1 Introduction

As is well known, the notion of Dirichlet integral of a function on an Eucledian domain had been deeply involved in the development of the mathemaical analysis from 19th century to 20th century. But the present concept of the Dirichlet form was first introduced in 1959 by A. Beurling and J. Deny[BD59] as an axiomatization of the Dirichlet integral:

A closed symmetric form on an L^2-space is called a *Dirichlet form* if the norm of $f \in L^2$ decreases whenever the variation of f gets smaller.

The study of general boundary conditions for the one-dimensional diffusions initiated by W. Feller in 1950 was successfully completed by Itô and McKean [IM65] around 1959. Thus many Japanese young probabilists in the 1960s were eager to investigate the so called *the boundary problem for more general Markov processes* looking for
firstly, analytic characterzations of possible Markovian extentions of the transition functions of a given minimal Markov processes
secondly, realization of those analytic extensions to be associated with genuine strong Markov processes with regular sample paths.

I approached to this problem starting from Dirichlet forms and revealed new significant prospects for the Dirichlet form as will be explained briefly below.

2 Dirichlet form and Markovian semigroup

2.1 Dirichlet form and Markovian semigroup

- $(S, \mathcal{B}, m)$: $\sigma-$finite measure space
  Let $(f, g) = \int_S f(x)g(x)m(dx), \quad f, g \in L^2(S; m)$
- $(\mathcal{E}, \mathcal{F})$ *symmetric form* on $L^2(S; m)$
  $\overset{\text{def}}{\Longleftrightarrow}$
  $\mathcal{F}$ is a dense linear subset of $L^2(S.m)$, $\mathcal{E}$ is a non-negative definite symmetric bilinear form: a bilinear map from $\mathcal{F} \times \mathcal{F}$ to $\mathbb{R}$, $\mathcal{E}(u, v) = \mathcal{E}(v, u)$, $\mathcal{E}(u, u) \geq 0$, $\forall u, v \in \mathcal{F}$
- a symmetric form $(\mathcal{E}, \mathcal{F})$ on $L^2(S; m)$ is *closed*
  $\overset{\text{def}}{\Longleftrightarrow}$
  $\mathcal{F}$ is a Hilbert space with inner product $\mathcal{E}_\alpha(u, v) = \mathcal{E}(u, v) + \alpha(u, v)$ for any $\alpha > 0$
- A family of symmetric linear operators $\{T_t, \ t > 0\}$ on $L^2(S; m)$ is called an L^2-*semigroup* if it is a strongly continuous contraction semigroup in the sense that

$T_tT_s = T_{t+s},\ ||T_tf||_{L^2} \le ||f||_{L^2},\ \lim_{t\downarrow 0}||T_tf - f||_{L^2} = 0, \quad \forall f \in L^2.$

- Closed symmetric forms on $L^2(S;m)$ and L^2-semigroups are in one to one correspondence by the relation

$$\mathcal{E}(f,g) = \lim_{t\downarrow 0}\frac{1}{t}(f - T_tf, g),\ \ \mathcal{F} = \{f \in L^2(S;m) : \mathcal{E}(f,f) < \infty\} \quad (2.1)$$

(a bilinear version of a generator of a semigroup)

- L^2-semigroup $\{T_t, t > 0\}$ is *Markovian*

$\overset{\text{def}}{\Longleftrightarrow}\ \ f \in L^2(S;m),\ 0 \le f \le 1\ [m]\ \Longrightarrow 0 \le T_tf \le 1\ [m], \forall t > 0$

- a real function $\varphi = \varphi(t),\ t \in \mathbb{R}$ is a *normal contraction*

$\overset{\text{def}}{\Longleftrightarrow}\ \ \varphi(0) = 0, \quad |\varphi(t) - \varphi(s)| \le |t - s|, \quad \forall t, s \in \mathbb{R}$

Theorem 2.1 (A.Beurling and J.Deny[BD59], cf. [FTake08]. [CF12]) *Let $(\mathcal{E},\mathcal{F})$ be a closed symmetric form on $L^2(S;m)$ and $\{T_t, t > 0\}$ be the associated L^2 semigroup.*

Then $\{T_t, t > 0\}$ is Markovian if and only if

(*) *every normal contraction operates on $(\mathcal{E},\mathcal{F})$*
in the sense that, for any normal contraction φ and for any $u \in \mathcal{F}$, $v = \varphi(u) \in \mathcal{F}$ and $\mathcal{E}(v,v) \le \mathcal{E}(u,u)$.

A closed symmetric form on $L^2(S;m)$ satisfying (*) is called a *Dirichlet form.*

Axiom (*) is due to Beurling [BD59], partly motivated by his study of the Douglas integral (2.4).

2.2 Examples of Dirichlet form

(a) D: connected open subset of R^n

$$\mathbf{D}(u,v) = \sum_{i=1}^{n}\int_D \frac{\partial u(x)}{\partial x_i}\frac{\partial v(x)}{\partial x_i}dx \quad (2.2)$$

with domain

$$H^1(D) = \{u \in L^2(D) : \frac{\partial u}{\partial x_i} \in L^2(D), \quad 1 \le i \le n\} \quad (2.3)$$

Sobolev space of order 1. (2.2) is *Dirichlet integral.*

(b) $D = \mathbb{D}$ unit disk with center $\mathbf{0}$. $\mathbb{T} = \partial\mathbb{D}$

$$\mathbf{C}(\varphi,\psi) = \int_{\mathbb{T}\times\mathbb{T}} (\varphi(\theta)-\varphi(\theta'))(\psi(\theta)-\psi(\theta'))U(\theta,\theta')d\theta d\theta' \qquad (2.4)$$

$$\mathcal{D}(\mathbf{C}) = \{\varphi \in L^2(\mathbb{T}) : \mathbf{C}(\varphi,\varphi) < \infty\}.$$

where $U(\theta,\theta') = [4\pi(1-\cos(\theta-\theta')]^{-1}$. Douglas integral[Dou31]

$\mathbf{C}(\varphi,\psi) = \mathbf{D}(\mathbf{H}\varphi, \mathbf{H}\psi)$;

$\mathbf{H}\varphi$ is harmonic function with boundary function $\varphi(\theta)$

studied by A. Beurling[B39] in relation to exceptional set for Lebesgue point

3 Regular Dirichlet form and symmetric Hunt process

3.1 Potential theory for regular Dirichlet form

announced in Beurling-Deny[BD59] and proved in Deny[D70]

S: locally compact separable meric space
m: positive Radon measure (Borel measure finite on compact sets) with full support

- Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(S;m)$ is *regular*

  $\overset{\text{def}}{\Longleftrightarrow}$

  $\mathcal{F}\cap C_c(S)$ is dense in $(\mathcal{F},\mathcal{E}_1)$ and in $C_c(S)$ with $||u||_\infty = \sup_{x\in S}|u(x)|$

- For open set $A \subset S$, let $\mathcal{L}_A = \{u \in \mathcal{F} : \ u \geq 1 \ m-\text{a.e.on } A\}$ and $\text{Cap}(A) = \inf_{u\in\mathcal{L}_A}\mathcal{E}_1(u.u)$, which is called the *capacity* of A.

- $B \subset S$ of zero outer capacity is called *almost polar.*

  Almost polar set is m-negligible.

  'Quasi everywhere' (q.e.) means 'except for an almost polar set'.

- A function u on S is *quasi continuous* if
  $\forall\varepsilon > 0$, $\exists A$ open with $\text{Cap}(A) < \varepsilon$, $u\big|_{S\setminus A}$ is continuous.

- If u is quasi continuous and $u \geq 0$ m-a.e., then so it is q.e.

  Under the regularity of Dirichlet form $(\mathcal{E}, \mathcal{F})$,

- any $u \in \mathcal{F}$ admits its quasi continuous m-version $\widetilde{u}$
- $u_n,\ u \in \mathcal{F},\ \ u_n$ are quasi continuous, $||u_n - u||_{\mathcal{E}_1} \to 0$

  $\Longrightarrow$

  $\exists \{n_k\}\ \ u_{n_k}$ converges q.e. to a quasi continuous version of u

3.2 Symmetric Hunt process associated with regular Dirichlet form

$S_\delta = S \cup \delta$: the one point compactification of S.

$\mathbb{X} = (\Omega, \mathcal{G}_t, X_t(\omega), \zeta(\omega), \mathbb{P}_x)$ *Markov process* on S $\quad \overset{\text{def}}{\Longleftrightarrow}$

- $X_t(\omega) \in S_\delta;\ \ X_t(\omega) \in S,\ t \in [0, \zeta(\omega)),\ X_t(\omega) = \delta,\ t \geq \zeta(\omega)$,

  $X_\cdot(\omega)$ is called *sample path.* $\quad \zeta(\omega)$ is called its *lifetime.*

  $X_t \in \mathcal{G}_t / \mathcal{B}(S_\delta)$, $\quad \{\mathcal{G}_t\}$: increasing σ-fields of subsets of Ω.

- $\mathbb{P}_x(X_0 = x) = 1, \quad x \in S_\delta$.
- $(**)\quad \mathbb{P}_x\left(X_{t+s} \in B \middle| \mathcal{G}_t\right) = \mathbb{P}_{X_t}(X_s \in B),\ \ \mathbb{P}_x$–a.s. $B \in \mathcal{B}(S_\delta)$

We put $P_t(x, B) = \mathbb{P}_x(X_t \in B)$ and call it the *transition function of* $\mathbb{X}$. If we let $P_t f(x) = \int_S P_t(x, dy) f(y)$ for $f \in \mathcal{B}_+(S)$, then $\{P_t, t > 0\}$ satisfies Markovian property: $0 \leq f \leq 1 \Rightarrow 0 \leq P_t f \leq 1$ and semigroup property: $P_t P_s = P_{t+s}$. (*transition semigroup of* $\mathbb{X}$)

A Markov process on S is called *symmetric* with respect to a measure m

$$\overset{\text{def}}{\Longleftrightarrow} \quad \int_S P_t f(x) g(x) m(dx) = \int_S f(x) P_t g(x) m(dx), \quad t > 0, \quad f, g \in \mathcal{B}_+(S).$$

A Markov process $\mathbb{X}$ on S is a *Hunt process* (introduced by [Hu57-58])

$\overset{\text{def}}{\Longleftrightarrow}$

- *strong Markov* (Markov property $(**)$ holds by replacing time t with any random time called a stopping time)

- the sample path $X_\cdot(\omega)$ is right continuous on $[0,\infty)$, has left limit on $(0,\infty)$ and quasi left continuous on $(0,\infty)$

A Hunt process whose sample path is continuous on $[0,\zeta)$ is called a *diffusion.*

$N \in \mathcal{B}(S)$ is *properly excepional* for a Hunt process $\mathbb{X}$ on S

$\overset{\text{def}}{\Longleftrightarrow}$ $m(N)=0,\ \ \mathbb{P}_x(X_t,\ X_{t-}\in S\setminus N,\ \forall t\in[0,\zeta))=1,\ \forall x\in S\setminus N,$

Namely,$S\setminus N$ is an invariant set for $\mathbb{X}$.

Theorem 3.1 *Let $(S,m,\mathcal{E},\mathcal{F})$ be a regular Dirichlet form.*

(i) *There exists an m-symmetric Hunt process $\mathbb{X}$ on S properly associated with it in the following sense:*

P_tf is a quasi continuous version of T_tf for any $f\in\mathcal{B}(S)\cap L^2(S;m)$,

(ii) *Uniqueness: If $\mathbb{X}_1,\ \mathbb{X}_2$, are properly associated, then there exists a properly excepional set N for both processes and* $\mathbb{X}_1\big|_{S\setminus N}\ \sim\ \mathbb{X}_2\big|_{S\setminus N}$

Properly exceptional set is almost polar.

Method of proof

(I) ([F71a],[F71b]) Two Dirichlet forms $(S,m,\mathcal{E},\mathcal{F})$ and $(\widetilde{S},\widetilde{m},\widetilde{\mathcal{E}},\widetilde{\mathcal{F}})$ are called to be *equivalent* if there is an algebraic homomorphism Φ between $\mathcal{F}_b$ and $\widetilde{\mathcal{F}}_b$ preserving L^∞, L^2 and $\mathcal{E}$-norms. The latter is called a *representation* of the former. The following two assertions yield a proof.

- Any Dirichlet form admits a *strongly regular representaton*: the resolvent of the associated semigroup of the latter has a nice property due to D.B. Ray[R59] and [KW67].
- If two regular Dirichlet forms are equivalent under Φ, then Φ is induced by a *quasi-homeomorphism* between S and $\widetilde{S}$.

In [F69], I could characterize all symmetric conservative Markovian extensions of the resolvent of the absorbed Brownian motion on any bounded Euclidean domain in terms of the family of Dirichlet forms on the Martin boundary dominating the Douglas integral.

So I turned to the second issue of making each extended resolvent to be associated with a strong Markov process. For this, the method due to Ray and Knight of a compactification of the underlying space was known at that time.

The strongly regular representation of any Dirichlet form is its profound intrinsic generalization.

(II) (M.L.Silverstein[S73], Fukushima[F73]) Use the Markov property of L^2-semigroup $\{T_t, t > 0\}$ and the potential theory for regular Dirichlet form directly to construct an appropriate transition function $P_t(x, B)$ from $T_t f \in \mathcal{F},\ f \in L^2(S; m)$, by removing sets of zero capacity where desired relations may fail to hold successively.

Thus, the above theorem becomes a quite general assertion independent of the boundary problem for Markov processes.

4 Road map and speed

Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form on $L^2(S; m)$ and $\mathbb{X} = (\Omega, \mathcal{G}_t, X_t(\omega), \zeta(\omega), \mathbb{P}_x)$ be the m-symmetric Hunt process properly associated with it.
How do m and $\mathcal{E}$ indicate behaviours of $\mathbb{X}$?

4.1 Invariance of $\mathcal{E}$ on $\mathcal{F}_e$ under time change

Analogously to a famous saying of W.Feller for one dimesional diffusions,
I wrote in [F71b]:
It may be asserted that the sample path $X_t(\omega)$ will run with speed indicated by the underlying measure m and along the roads indicated by the 0-order Dirichlet form $\mathcal{E}$ independent of its speed.

Concuring with this, Silverstein[S74] introduced the notion of the *extended Dirichlet space $\mathcal{F}_e$* a natural extension of the domain $\mathcal{F}$ of the 0-order Dirichlet form $\mathcal{E}$ suggesting $(\mathcal{F}_e, \mathcal{E})$ to be a right road map.

Every function in $\mathcal{F}_e$ is finite m-a.e. on S, admits its quasi continuous m-version and it holds that $\quad \mathcal{F} = \mathcal{F}_e \cap L^2(S; m)$.
A precise assertion will be stated below using the concept of *additive functional* (AF) of $\mathbb{X}$.

An extended real valued function $A_t(\omega)$ of $t \geq 0,\ \omega \in \Omega$, is an *additive functional (AF) of* $\mathbb{X}$

$\overset{\text{def}}{\Longleftrightarrow}$

- $A_t(\cdot)$ is $\mathcal{G}_t$-measurable
- $A_0 = 0,\ |A_t| < \infty, \forall t \in [0,\zeta),\ A_t$ is right continuous and has left limit $\forall t \in [0,\zeta),\ A_t = A_\zeta,\ t \geq \zeta,\ \mathbb{P}_x-$a.s.$\forall x \in S \setminus N$.
- $A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t \omega),\ \forall t, s \geq 0,\ \mathbb{P}_x-$a.s.$\forall x \in S \setminus N$,

where N is some properly exceptional subset of S.

Two AFs A_t and B_t are said to be m-equivalent if $\mathbb{P}_m(A_t \neq B_t) = 0$ for every $t \geq 0$.

Denote by $\overset{\circ}{\mathcal{S}}$ the collection of all positive Radon measures μ on S charging no almost polar set with full support.

For instance, $\mu = g \cdot m \in \overset{\circ}{\mathcal{S}}$ if $g \in \mathcal{B}_b(S)$ and $g > 0$ m-a.e. on S.

For any $\mu \in \overset{\circ}{\mathcal{S}}$, there exists a *strictly increasing positive continuous AF(PCAF)* A_t of $\mathbb{X}$ uniquely up to m-equivalence such that

$$\int_S f(x)\mu(dx) = \lim_{t \downarrow 0} \frac{1}{t} \mathbb{E}_m \left[\int_0^t f(X_s) dA_s \right],\ \forall f \in \mathcal{B}_+(E).$$

Such A_t is called the PCAF of $\mathbb{X}$ *determined by* $\mu \in \overset{\circ}{\mathcal{S}}$.

For instance, $A_t = \int_0^t g(X_s)ds$.is the PCAF determined by $\mu = g \cdot m$ as above.

Invariance of $\mathcal{E}$ on $\mathcal{F}_e$ under time change

For any $\mu \in \overset{\circ}{\mathcal{S}}$, take the PCAF A_t of $\mathbb{X}$ determined by μ. Let

$\check{\mathbb{X}} = (\check{X}_t(\omega),\ \check{\zeta}(\omega), \mathbb{P}_x)$ where $\check{X}_t(\omega) = X_{A_t(\omega)^{-1}}(\omega),\ \check{\zeta}(\omega) = A_{\zeta(\omega)-}(\omega)$,

which is called the *time changed process* of $\mathbb{X}$ by A_t.

Theorem 4.1 ([CF12]) (i) *$\check{\mathbb{X}}$ is μ-symmetric Hunt process on S.*

(ii) *Let $(\check{\mathcal{E}}, \check{\mathcal{F}})$ be the Dirichlet form on $L^2(S;\mu)$ of $\check{\mathbb{X}}$.*
Then $(\check{\mathcal{E}}, \check{\mathcal{F}})$ is regular.

(iii) *Let $(\check{\mathcal{F}}_e, \check{\mathcal{E}})$ be the extended Dirichlet space of $(\check{\mathcal{E}}, \check{\mathcal{F}})$.*
Then $(\check{\mathcal{F}}_e, \check{\mathcal{E}}) = (\mathcal{F}_e, \mathcal{E})$.

Hence, $\check{\mathcal{F}} = \mathcal{F}_e \cap L^2(S;\mu), \quad \mu \in \overset{\circ}{\mathcal{S}}$. Recall $\mathcal{F} = \mathcal{F}_e \cap L^2(S;m)$.

4.2 Probabilistic expression of $\mathcal{E}$ on $\mathcal{F}_e$ (road map)

An AF A_t of $\mathbb{X}$ is called of *finite energy* if $e(A) = \lim_{t\downarrow 0} \frac{1}{t}\mathbb{E}_m[A_t^2] < \infty$.

For $u \in \mathcal{F}_e$, $A_t^{[u]} = \widetilde{u}(X_t) - \widetilde{u}(X_0)$ is of finite energy admitting unique *Fukushima decomposition*[F79]

$$A_t^{[u]} = M_t^{[u]} + N_t^{[u]}$$

where $M^{[u]}$ is a martingale AF (mean zero and square integrable AF) of finite energy and $N^{[u]}$ is continuous AF of zero energy.

By making further orthogonal decomposition of $M^{[u]}$ as

$$M_t^{[u]} = M_t^{[u],c} + M_t^{[u].j} + M_t^{[u],k}$$

and evaluating the energy of each term, we arrive at the next theorem.

$\mathcal{E}$ is called *strongly local* if $\mathcal{E}(f,g) = 0$ whenever $f \in \mathcal{F}$ has a compact support and $g \in \mathcal{F}$ is constant on a neighborhood of the support of f.

Theorem 4.2 ([FTake08], [FOT11], [CF12]) (I) *$\mathcal{E}$ on $\mathcal{F}_e$ admits a unique expression: for $u, v \in \mathcal{F}_e$,*

$$\begin{aligned}&\mathcal{E}(u,v) = \mathcal{E}^{(c)}(u,v)\\ +\ &\frac{1}{2}\int_{S\times S\setminus d} (\widetilde{u}(x) - \widetilde{u}(y))(\widetilde{u}(x) - \widetilde{v}(y))J(dx,dy) + \int_S \widetilde{u}(x)\widetilde{v}(x)\kappa(dx),\end{aligned}$$

where $\mathcal{E}^{(c)}$ is a strongly local symmetric form, J is a symmetric Radon measure on $S \times S \setminus d$ and κ is a Radon measure on S.

(II) $J(dx,dy) = N(x,dy)\mu_H(dx), \quad \kappa(dx) = N(x,\{\partial\})\mu_H(dx)$

where $(N(x,dy),H)$ is the Lévy system of the Hunt process $\mathbb{X}$ introduced by S.Watanabe[W64] *and μ_H is the Revuz measure of the PCAF H.*

(III) *$\mathcal{E}$ is strongly local if and only if $\mathbb{X}$ is a diffusion with no killing inside S, namely, its path is continuous on $[0,\infty)$ a.s.*

5 Quasi-regularity vs regularity for Dirichlet form

Although the regularity of a Dirichlet form is a sufficient condition to associate a nice Markov process called a Hunt process, it may not be a necessary condition for its association with a nice Markov process.

Furthermore, the local compactness assumption on the underlying space S is too restrictive in dealing with infinite dimensional objects.

In this regard, an important notion of quasi-regular Dirichlet form was introduced by Albeverio-Ma[AM91], and its detailed study was presented in Ma-Roeckner[MR92] under the framework of generel non-symmetric Dirichlet forms.

Let S be a general topological Hausdorff space, m be a σ-finite measure on S with full support and $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form on $L^2(S; m)$.

For a closed set $C \subset S$, let $\mathcal{F}_C = \{u \in \mathcal{F} : u = 0 \ m\text{-a.e. on } S \setminus C\}$

An increasing sequence $\{C_k\}$ of closed sets is called an *$\mathcal{E}$-nest* if $\bigcup_{k=1}^{\infty} \mathcal{F}_{C_k}$ is $\mathcal{E}_1$-dense in $\mathcal{F}$.

A set $N \subset S$ is called *$\mathcal{E}$-polar* if $N \subset \bigcap_{k=1}^{\infty}(S \setminus C_k)$ for some $\mathcal{E}$-nest $\{C_k\}$.

$\mathcal{E}$-q.e. means 'except for an $\mathcal{E}$-polar set'.

A function u on S is called *$\mathcal{E}$-quasi continuous* if $u\big|_{C_k}$ is continuous for any $k \geq 1$ for some $\mathcal{E}$-nest $\{C_k\}$.

Let $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form on $L^2(S; m)$ for a pair (S, m) as above.

$(\mathcal{E}, \mathcal{F})$ is *quasi regular* $\overset{\text{def}}{\Longleftrightarrow}$

(i) there exists an $\mathcal{E}$-nest consisting of compact sets

(ii) there exists an $\mathcal{E}_1$-dense subset $\mathcal{F}_0$ of $\mathcal{F}$ such that each $f \in \mathcal{F}_0$ admits $\mathcal{E}$-quasi contnuous m-version

(iii) there exist functions $\{f_k, \ k \geq 1\} \subset \mathcal{F}$ and an $\mathcal{E}$-polar set N such that each f_k admits $\mathcal{E}$-quasi continuous m-version $\widetilde{f_k}$ and $\{\widetilde{f_k}\}$ separates points of $S \setminus N$.

If $(S, m, \mathcal{E}, \mathcal{F})$ is a regular Dirichlet form, then it is quasi regular.

Further $\mathcal{E}$-polarity and $\mathcal{E}$-quasi continuity are synonims of almost polarity and quasi continuity defined in §3.1 in terms of capacity.

Here are three fundamental theorems on quasi regular Dirichlet form formulated in [CF12].

Theorem 5.1 (Albeverio-Ma[AM91], Fitzsimmons[Fi01]) *Let S be a Lusin space, m be a σ-finite measure on it with full support and $\mathbb{X}$ be an m-symmetric right process on S.*

Then the Dirichlet form of $\mathbb{X}$ on $L^2(S;m)$ is quasi regular and $\mathbb{X}$ is properly associated with it.

Theorem 5.2 (Chen-Ma-Roeckner[CMR94]) *If $(S, m, \mathcal{E}, \mathcal{F})$ is quasi regular Dirichlet form, then there exist*

a regular Dirichlet form $(\widehat{S}, \widehat{m}, \widehat{\mathcal{E}}, \widehat{\mathcal{F}})$ and

a quasi homeomorphism j from S to $\widehat{S}$ such that

$\widehat{m} = m \circ j^{-1}$ and $(\widehat{\mathcal{E}}, \widehat{\mathcal{F}})$ on $L^2(\widehat{S}; \widehat{m})$ is the image Dirichlet form of $(\mathcal{E}, \mathcal{F})$ on $L^2(S;m)$ by j.

In the second theorem, the locally compact separable metric space $\widehat{S}$ is chosen as the character space of a certain closed subalgebra of $L^\infty(S;m)$ in an analogous manner to [F71a].

This theorem enables us to transfer known results for regular Dirichlet forms to quasi regular ones (*transfer method*).

In particular, Theorem in subsection 3.2 is transfered to

Theorem 5.3 *For any quasi regular Dirichlet form $(S, m, \mathcal{E}, \mathcal{F})$, there are an $\mathcal{E}$-polar Borel set $N \subset S$ and an m-symmetric special Borel standard process $\mathbb{X}$ on $S \setminus N$ that is properly associated with $(\mathcal{E}, \mathcal{F})$.*

6 Three different cases of underlying space S

6.1 Fractal sets

S: fractal set like Sierpinski gasket and Sierpinski carpet

S has a *self similar structure* but does not have a differential structure.

studied by Fukushima-Shima[FS92], S.Kusuoka, J.Kigami[Ki01], T. Kumagai, M. Hino[Hi13], et. al.

- m: Hausdorff measure on S
- *strongly local regular* Dirichlet form on $L^2(S;m)$ was constructed.

- The associated symmetric diffusion $\mathbb{X}$ on S is called the *Brownian motion* on S

- In many cases, the space of MAFs of $\mathbb{X}$ of finite energies has dimension 1, while the Hausdorff dimension of S is larger than 1 (cf. [Hi13]).

6.2 Configulation space

$$\Gamma = \{\gamma = \sum_i \delta_{x_i} : \gamma(K) < \infty, \text{ for any compact sebset } K \subset \mathbb{R}^n\}$$

equipped with vague convergence topology to be a Lusin space
studied by H. Osada, H.Tanemura[OT20] et. al.

- $S = \Gamma$
- $m = \mu$: *Gibbs measure* on Γ introduced by Osada[O13]
- $(\mathcal{E}, \mathcal{F})$: Dirichlet form on $L^2(\Gamma, \mu)$ defined in a close relation to μ
- verify that it is a *strongly local quasi regular* Dirichlet form
- The associated symmetric diffusion $\mathbb{X}$ on Γ represents a motion of *unlabeled* particles with invariant measure μ
- Similar Dirichlet forms are then used to solve infinite dimensional SDE representing a motion of *interracting infinitely many particles*

6.3 Quotient space of multiply connected planar domain

$G \subset \mathbb{C}$: domain such that either $G = \mathbb{C}$ or $\mathbb{C} \setminus G$ is continuum (closed connected, containing at least two points)

$D = G \setminus K, \quad K = \bigcup_{i=1}^N A_i$: $(N+1)$-connected domain
A_i are mutually disjoint compact continua.

$D^* = D \cup K^*, \quad K^* = \{a_i^* : 1 \le i \le N\}$: quotient topologocal space obtained from G by rendering each set A_i into singleton a_i^*

m: Lebesgue measure on D being extended to D^* by setting $m(a_i^*) = 0, \ 1 \le i \le N$

$\mathbb{X}$: *Brownian motion with darning (BMD)* $\overset{\text{def}}{\Longleftrightarrow}$

m-symmetric diffusion on D^* admitting no killing on K^* whose part process on D is identical in law with the ABM on D

(BMD is a key to extend the *SLE theory* from simply connected domains: the restriction to D of any BMD-harmonic function v admits an analytic function f on D with $\Im f = v$ up to an addtional real constant).

Existence and uniqueness of BMD $\mathbb{X}$

Proved in [CFM23] by using the following description of the Dirichlet form $(\mathcal{E}^*, \mathcal{F}^*)$ of $\mathbb{X}$ on $L^2(D^*; m)$ indicated in [CF12, Th.7.7.3 (vi)]:
$\mathcal{F}^*$ is linear subspace of $H^1(D)$ spanned by $H^1_0(D)$ and $\{u_1^{(i)}\big|_D,\ 1 \le i \le N\}$,
$\mathcal{E}^*(u, v) = \frac{1}{2}\int_D \nabla u(x) \cdot \nabla v(x) dx, \quad u, v \in \mathcal{F}^*$,
where $u_1^{(i)}$ is the 1-order hitting probability for the set A_i of the ABM $(Z_t, \mathbb{P}^G_z)$ on G: $\quad u_1^{(i)}(z) = \mathbb{E}^G_z\left[e^{-\sigma_K}; Z_{\sigma_K} \in A_i\right],\ z \in G.$

- $(\mathcal{E}^*, \mathcal{F}^*)$ is a *strongly local regular* Dirichlet form on $L^2(D^*; m)$ and each a_i^* is of positive capacity $\quad\Longrightarrow\quad$ associated symmetric diffusion $\mathbb{X}$ can be refined to be a BMD starting at every point of D^*

- $\mathbb{X}$ is BMD on $S = D^* \Longrightarrow$

  the associated Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(D^*; m)$ is *quasi-regular* and the *transfer method* works in getting $(\mathcal{E}, \mathcal{F}) = (\mathcal{E}^*, \mathcal{F}^*)$.

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