On boundary problems for symmetric Markov processes

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Contents

1 Symmetric Markovian extensions of ABM: An analytic characterization

Based on [F69]

D bounded domain of $\mathbb{R}^n$

$G_\alpha(x,y),\ \alpha>0, x,y\in D,\ x\neq y$ *resolvent density* on $D \overset{\text{def}}{\Longleftrightarrow}$
$G_\alpha(x,y)\geq 0,\ \alpha\int_D G_\alpha(x,z)dz\leq 1.$
$G_\alpha(x,y)-G_\beta(x,y)+(\alpha-\beta)\int_D G_\alpha(x,z)G_\beta(z,y)dz=0,\ \alpha,\beta>0.$

It is called *symmetric and conservative* if
$G_\alpha(x,y)=G_\alpha(y,x),\quad \alpha\int_D G_\alpha(x.z)dz=1$

Let $G^0_\alpha(x,y)$ be the resolvent density of the *absorbed Brownian motion* (ABM) $\mathbb{X}^0$ on D

A function u on D is called *α-harmonic* if $\quad (\alpha - \frac{1}{2}\Delta)u = 0$ on D

Let $\mathbb{G}$ be the family of all symmetric conservative resolvent densities $G_\alpha(x,y)$ on D satisfying

($\mathbb{G}.a$) $\quad G_\alpha(x,y) = G^0_\alpha(x,y) + R_\alpha(x,y)$ where $R_\alpha(x,y)$ is non-negative and α-harmonic in $x \in D$,

($\mathbb{G}.b$) $\quad \sup_{x\in K, y\in D} R_\alpha(x,y) < \infty$ for any compact set $K \subset D$

The first assertion of the next propsition is a conterpart of [F67, Lemma 2.1].

Proposition 1.1 (i) *Any non-negative α-harmonic function u on D admits the expression*

$$(*) \qquad u(x) = \mathbb{E}_x\left[e^{-\alpha\tau_B}u(X_{\tau_B})\right], \quad x \in B,$$

for any ball B with $\overline{B} \subset D$ in terms of the n-dimensional Brownian motion $(X_t, \mathbf{P}_x)$.

(ii) *For any $G_\alpha(x,y) \in \mathbb{G}$, $G_\alpha f(x) = \int_D G_\alpha(x,y)f(y)dy$, $x \in D$, is not only a symmetric contraction resolvent on $L^2(D)$ but also a strongly continuous resolvent.*

Proof (i) The retriction to B of an α-harmonic function u on D is the unique slution of $(u - \frac{1}{2}\Delta)u = 0$ on B with continuous boundary function $u\big|_{\partial B}$ (cf. [GT77, Cor.6.0]).

On the other hand, we can see just as in the proof of [CF12, Lemma 3,1] that the function on the ball B defined by the right hand side of (*) also has this property by noting that the restriction to B of any α-excessive function on D is α-excesive relative to the ABM on B according to [Dy65, Th 12.9, Th.12.9].

(ii) Since G^0_α has the stated properties, it is enough to show that $\lim_{\alpha\to\infty} \alpha||R_\alpha f||_{L^2(D)} = 0$ for any $f \in L^2(D)$.For any $\varepsilon > 0$, take $g \in C_c(D)$ with $||f - g||_{L^2} < \varepsilon$. We then readily get $\alpha||R_\alpha f||_{L^2} \leq \varepsilon + \alpha||R_\alpha g||_{L^2}$

By using the α-harmonicity of $R_\alpha(x,y)$ in $x \in D$ and its expression (*), we see as in the proof of [F67, Lemma 2.9] that $\lim_{\alpha\to\infty} \alpha|R_\alpha g(x)| = 0$ for any $x \in D$. Since $\alpha||R_\alpha g||_\infty \leq ||g||_\infty$, the second term of the right hand side of the preceding inequality tends to 0 as $\alpha \to \infty$ by the bounded convergence theorem.

Remark 1.2 *Assume that $n \geq 3$. and let $g(x,y)$ be the Green function for the domain $D \subset \mathbb{R}^n$. Denote $G^0_{0+}(x,y)$ by $G^0(x,y)$. Then $g(x,y) = \frac{q}{2}G^0(x,y) \quad x,y \in D$, for $q = (n-2)\sigma_n$ where σ_n is the unit ball boundary area: σ_n equals $\pi^{\frac{n}{2}}n/(\frac{n}{2})!$ when n is even and $2^{\frac{n+1}{2}}\pi^{\frac{n-1}{2}}/\{1\cdot 3\cdot 5\cdots(n-2)\}$ when n is odd.*

According to K.Itô [I60, §3.5], $G^0(x,y) = \frac{2}{C}g(x,y)$ for $C = 4\pi^{\frac{n}{2}}/\Gamma(\frac{n}{2}-1)$, and we readily get $C = q$.

M: Martin boundary of D. μ: harmonic measure on M relative to a reference point $x_0 \in D$
$K(x,\xi),\ x \in D,\ \xi \in M$: Martin kernel:
$K(x,\xi) = \lim_{y\to\xi} \frac{g(x,y)}{g(x_0,y)} = \lim_{y\to\xi} \frac{G^0(x,y)}{G^0(x_0,y)}$.
α-order Martin kernel is then defined by
$K_\alpha(x,\xi) = K(x,\xi) - \alpha\int_D G^0_\alpha(x,z)K(z,\xi)dz$

For a function φ on M, let
$\mathbf{H}\varphi(x) = \int_M K(x,\xi)\varphi(\xi)\mu(d\xi),\ \mathbf{H}_\alpha\varphi(x) = \int_M K_\alpha(x,\xi)\varphi(\xi)\mu(d\xi),\ x \in D$

α-order Feller kernel is defined by

$$U_\alpha(\xi,\eta) = \alpha\int_D K(z,\xi)K_\alpha(z.\eta)dz, \quad \xi,\eta \in M,$$

which is increasing in α. *Feller kernel* is defined by

$$U(\xi,\eta) = \lim_{\alpha\to\infty} U_\alpha(\xi,\eta), \quad \xi,\eta \in M$$

For a function φ in M, the *Douglas integral* is defined by

$$\mathbf{C}(\varphi,\varphi) = \frac{1}{2}\int_{M\times M}(\varphi(\xi)-\varphi(\eta))^2 U(\xi,\eta)\mu(d\xi)\mu(d\eta)$$

Define $\mathbb{H}_M = \{\varphi \in L^2(M;\mu_0) : \mathbb{C}(\varphi,\varphi) < \infty\}$ where $\mu_0(d\xi) = U_1 1(\xi)\mu(d\xi)$.

Proposition 1.3 *If a function u on D is harmonic with finite Dirichlet integral $\mathbf{D}_D(u,u)$, then u admits fine limit $\varphi(\xi) = \gamma u(\xi)$ for μ-a.e. $\xi \in M$, $u = \mathbf{H}\varphi$ and*

$$\frac{1}{2}\mathbf{D}_D(u,u) = \mathbf{C}(\varphi,\varphi).$$

This was established by J.L.Doob [Do62, (7.7)] with $\frac{q}{2}\Theta(\xi,\eta)$ for the *Naim kernel*
$\Theta(\xi,\eta) = \lim_{x\to\xi,y\to\eta} \frac{g(x,y)}{g(x_0,x)g(x_0,y)} = \frac{2}{q}\lim_{x\to\xi,y\to\eta} \frac{K(x,y)}{G^0(x_0,x)}$
in place of the Feller kernel U.

[F64] then proved that $\frac{q}{2}\Theta = U$.

The notion U was introduced by W.Feller [Fe57] in his seminal study of boundary condition for a Markov process on a denumerable state space with finite number of boundary points.

Theorem 1.4 *There is one-to-one correspondence between the family* $\mathbb{G}$ *of resolvent deisities on* D *and the family of Dirichlet forms* $(\mathcal{E}_M, \mathcal{F}_M)$ *on* $L^2(M;\mu_0)$ *such that*
$\mathcal{F}_M \subset \mathbb{H}_M, \quad \mathcal{E}_M(\varphi,\varphi) \geq \mathbf{C}(\varphi,\varphi),\ \forall\varphi\in\mathcal{F}_M, \quad \mathcal{E}_M(1,1) = 0.$

For each $G_\alpha(x,y) \in \mathbb{G}$, the corresponding $(\mathcal{E}_M.\mathcal{F}_M)$ is defined as follows:

Let $(\mathcal{E}_D, \mathcal{F}_D)$ be its associated Dirichlet form on $L^2(D)$ which is well defined due to Prposition 1.1. Define
$\mathcal{H}_\alpha = \{u \in \mathcal{F}_D : u \text{ is } \alpha\text{−harmonic}\}$.
$u \in \mathcal{H}_\alpha$ admits the fine boundary function $\gamma u \in L^2(M;\mu_0)$ and $u = \mathbf{H}_\alpha \gamma u$.

Let
$\mathcal{F}_M = \{\gamma u : u \in \mathcal{H}_\alpha\}$,
$\mathcal{E}_M(\varphi,\varphi) = \mathcal{E}_\alpha(\mathbf{H}_\alpha\varphi, \mathbf{H}_\alpha\varphi) - U_\alpha(\varphi,\varphi),\ \varphi\in\mathcal{F}_M$.

Then $(\mathcal{E}_M, \mathcal{F}_M)$ is independent of $\alpha > 0$ and satisfies the stated properties.

Maximum and minimum ones among $(\mathcal{F}_M, \mathcal{E}_M)$ **in a semi-order** (cf. [CF12, Def 6.6.8])

(**I**) *Maximum one*: $\mathcal{F}_M = \mathbb{H}_M, \quad \mathcal{E}_M = \mathbf{C}$

Let $G_\alpha(x,y) \in \mathbb{G}$ be the corresponding resolvent dencity and $(\mathcal{E}_D, \mathcal{F}_D)$ be its associated Dirichlet form on $L^2(D)$. Then

$\mathcal{F}_D = H^1(D) = \{u \in L^2(D) : |\nabla u| \in L^2(D)\}$
$\mathcal{E}_D(u,u) = \frac{1}{2}\mathbf{D}_D(u,u)$.

Hence $(\mathcal{F}_D, \mathcal{E}_D)$ is the *active reflected Dirichlet space* of the Dirichlet form $(\mathcal{E}^0_D, \mathcal{F}^0_D) = (\frac{1}{2}\mathbf{D}_D, H^1_0(D))$ of the *ABM* on D (cf. [CF12, §6.5(4°)])

and furthermore, for $u \in \mathcal{F}_D$ and $u_0 = u - \mathbf{H}_\alpha u \in \mathcal{F}_D^0$,
$\mathcal{E}_{D,\alpha}(u,u) = \mathcal{E}_{D,\alpha}^0(u_0.u_0) + \mathbf{C}(\gamma u, \gamma u) + U_\alpha(\gamma u, \gamma u)$.

(**II**) *Minimum one*: $\mathcal{F}_M = \{\text{constant functions}\}, \quad \mathcal{E}_M = \mathbf{C}$
The associated objects:
$G_\alpha(x,y) = G_\alpha^0(x,y) + \frac{\mathbf{H}_\alpha 1_M(x)\mathbf{H}_\alpha 1_M(y)}{\alpha(1_D, \mathbf{H}_\alpha 1_M)_{L^2(D)}} \in \mathbb{G}$,
$\mathcal{F}_D = H_0^1(D) \oplus \text{constants}, \quad \mathcal{E}_D(u,u) = \frac{1}{2}\mathbf{D}_D(u,u)$

2 Active reflected Dirichlet space for a part process $\mathbb{X}^0$

Based on [CF12, Th.7.1.8]

E: locally compact separable metric space
m: positive Radon measure on E with full support
$(\mathcal{E}, \mathcal{F})$: regular Dirichlet form on $L^2(E;m)$
$\mathbb{X}$: associated m-symmetric Hunt process on E
(N,H): the Lévy system of $\mathbb{X}$
$J(dx,dy) = N(x,dy)\mu_H(dx), \quad \kappa(x,\{\delta\})\mu_H(dx)$ jumping and killing measures

F: a neaerly Borel measurable finely closed subset of E, $\quad E_0 = E \setminus F$

Assume that F is non-$\mathcal{E}$-polar and that $\mathbb{X}$ admits no jump from E_0 to F: $J(E_0 \times F) = 0$

$\mathbb{X}^0 = (X_t^0, \zeta^0, \mathbb{P}^0)$: the *part process* of $\mathbb{X}$ on E_0
$(\mathcal{E}^0, \mathcal{F}^0)$: the Dirichlet form of $\mathbb{X}^0$ on $L^2(E_0, m_0)$, $\;m_0 = m\big|_{E_0}$

$(\mathcal{F}_a^{0,\mathrm{ref}}, \mathcal{E}^{0,\mathrm{ref}})$: the *active reflected Dirichlet space* of $(\mathcal{E}^0, \mathcal{F}^0)$

For a function φ on F, let $\quad \mathbf{H}\varphi(x) = \mathbb{E}_x\left[\varphi(X_{\sigma_F}); \sigma_F < \infty\right]$, and $\mathbf{H}_\alpha\varphi(x) = \mathbb{E}_x\left[e^{-\alpha\sigma_F}\varphi(X_{\sigma_F}); \sigma_F < \infty\right], \; \alpha > 0$.

Define for $\varphi, \psi \in \mathcal{B}_+(F)$,
$U_\alpha(\varphi, \psi) = \alpha \int_{E^0} \mathbf{H}\varphi(x)\mathbf{H}_\alpha\psi(x) m_0(dx)$ which increases in α.

Let $U(\varphi,\psi) = \lim_{\alpha\to\infty} U_\alpha(\varphi,\psi)$.

U_α and U are bimesures on F called *Feller measures*.

$U(\varphi,\psi)$ coincides with the *energy functional* $L^0(\mathbf{H}\varphi, \mathbf{H}\psi)$ of the $\mathbb{X}^0$-excessive functions $\mathbf{H}\varphi, \mathbf{H}\psi$ in Meyer's sense.

Further the *supplementary Feller measure* V is defined by
$V(\varphi) = \lim_{\alpha\to\infty} \alpha \int_{E_0} \mathbf{H}_\alpha\varphi(x)(1-\mathbf{H}1(x))m_0(dx),\ \varphi \in \mathcal{B}_+(F)$.

According to [CF12, Th.5.7.6], U is *rate of excursion* of $\mathbb{X}$ from F to F and V is *rate of no returning excursion* of $\mathbb{X}$ from F.

Finally the *Douglas integral* of a function φ on F is defined by
$\mathbf{C}(\varphi,\varphi) = \frac{1}{2}\int_{F\times F}(\varphi(\xi)-\varphi(\eta))^2 U(d\xi,d\eta) + \int_F \varphi(\xi)^2 V(d\xi) (\le \infty)$.

Theorem 2.1 *For any* $u \in \mathcal{F}$, $u\big|_{E_0} \in \mathcal{F}_a^{0,\mathrm{ref}}$ *and*

$$(\mathbf{a})\ \mathcal{E}_\alpha^{0,\mathrm{ref}}(u\big|_{E_0}, u\big|_{E_0}) = \mathcal{E}^0(u_0,u_0) + \mathbf{C}(u\big|_F, u\big|_F) + U_\alpha(u\big|_F, u\big|_F),$$

where $u_0 = u - \mathbf{H}_\alpha u.$ *Furthermore*

$$\begin{aligned}(\mathbf{b})\ \mathcal{E}_\alpha(u,u) &= \mathcal{E}_\alpha^{0.\mathrm{ref}}(u\big|_{E_0}, u\big|_{E_0}) \\ &+ \frac{1}{2}\mu^c_{\langle \mathbf{H}u\rangle}(F) + \frac{1}{2}\int_{F\times F}(u(\xi)-u(\eta))^2 J(d\xi,d\eta) \\ &+ \int_F u(\xi)^2\kappa(d\xi) + \alpha\int_F u(\xi)^2 m(d\xi)\end{aligned}$$

Every function in $\mathcal{F}$ is represented by its quasi-continuous version.

Since $u \in \mathcal{F} \subset \mathcal{F}_e$, $\mathbf{H}u \in \mathcal{F}_e$ and $\mu^c_{\langle \mathbf{H}u\rangle>}$ is well defined.

3 Unique extension of $\mathbb{X}^0$ when F is a finite set

E: locally compact separable metric space
m: positive Radon measure on E with full support
$F = \{a_1, a_2, \cdots, a_N\}$: finite subset of E, $E_0 = E\setminus F$, $m_0 = m\big|_{E_0}$

$\mathbb{X}^0 = (X_t^0, \zeta^0, \mathbb{P}_x^0)$: m_0-symmetric Borel standard process on E_0 admitting no killing inside E_0 and

$$(**)\quad \mathbb{P}_x^0(\zeta^0 < \infty, X^0_{\zeta^0-} = a_i) > 0,\ \forall x \in E_0,\ 1 \le i \le N$$

$(\mathcal{E}^0, \mathcal{F}^0)$: Dirichlet form of $\mathbb{X}^0$ on $L^2(E_0, m_0)$
$u_\alpha^{(i)}(x) = \mathbb{E}_x^0\left[e^{-\alpha\zeta^0}; X^0_{\zeta^0-} = a_i\right],\quad x \in E_0,\ 1 \le i \le N.$
$\varphi^{(i)}(x) = \mathbb{P}_x^0(\zeta^0 < \infty, X^0_{\zeta^0-} = a_i),\quad x \in E_0,\ 1 \le i \le N.$

A right process $\mathbb{X} = (X_t, \zeta, \mathbb{P}_x)$ on E is *an N-points reflection of* $\mathbb{X}^0$

$\overset{\text{def}}{\Longleftrightarrow}$

$\mathbb{X}$ is m-symmetric, admits no killing on F, admits no jump from F to F and $\mathbb{X}^0$ is the part of $\mathbb{X}$ on E_0

Based on [CF12, Th. 7.7.3]

Let $(\mathcal{F}^{0,\text{ref}}, \mathcal{E}^{0,\text{ref}})$ be the *reflected Dirichlet space of* $(\mathcal{E}^0, \mathcal{F}^0)$ and $(\mathcal{F}_a^{0,\text{ref}}, \mathcal{E}^{0,\text{ref}})$ be the *active reflected Dirichlet space of* $(\mathcal{E}^0, \mathcal{F}^0)$

Theorem 3.1 *An N-points reflection $\mathbb{X}$ of $\mathbb{X}^0$ is unique.*

Let $(\mathcal{E}, \mathcal{F})$ be the Dirichlet form of $\mathbb{X}$ on $L^2(E; m)$. Then

(i) $\mathcal{F} \subset \mathcal{F}_a^{0,\text{ref}}$. $\mathcal{F}_e \subset \mathcal{F}^{0,\text{ref}}$. $\mathcal{E}(u,u) = \mathcal{E}^{0,\text{ref}}(u,u), \quad \forall u \in \mathcal{F}_e$.

(ii) $\mathcal{F}$ *is a linear subspace of* $\mathcal{F}_a^{0.\text{ref}}$ *spanned by* $\mathcal{F}^0$ *and* $u_\alpha^{(i)}$, $1 \le i \le N$.

(iii) $\mathcal{F}_e$ *is a linear subspace of* $\mathcal{F}^{0.\text{ref}}$ *spanned by* $\mathcal{F}_e^0$ *and* $\varphi^{(i)}$, $1 \le i \le N$.

(iv) *If* $\mathbb{X}^0$ *is a diffusion, then so is* $\mathbb{X}$.

It is remarked that $\mathbb{X}$ then admits no jump from E_0 to F.

$(\mathcal{E}, \mathcal{F})$ is quasi regular. By the transfer method, we may assume that it is a regular Dirichlet form on $L^2(E; m)$ and $\mathbb{X}$ is the associated m-symmetric Hunt process on E.

(i) then follows from **(b)** of preceding theorem by noticing that $\mu^c_{\langle \mathbf{H}u \rangle}$ charges no level set of $\mathbf{H}u$,

while **(a)** implies that boundary value of resolvent of $\mathbb{X}$ is uniquely determined by $\mathbf{C}$ and U_α, yielding uniqueness of $\mathbb{X}$.

Fix i for $1 \le i \le N$. (ii) and (iii) follow from the existence of $v \in \mathcal{F}$ with $v(a_j) = \delta_{ij}$, $1 \le j \le N$.

Take an open set $U \subset E$ with $a_i \in U$, $a_j \notin U$, $j \ne i$, and an m-integrable strictly positive bounded continuous function f on E. Let
$w(x) = \mathbb{E}_x \left[\int_0^\infty e^{-\alpha t} f(X_t) dt \right], \quad x \in E.$

$v(x) = w(x)/w(a_i)$ is then such a function and
$u_\alpha^{(i)}(x) = \mathbf{H}_\alpha v(x) \in \mathcal{F}, \quad \varphi^{(i)}(x) = \mathbf{H} v(x) \in \mathcal{F}_e$

Construction of N-points reflection of $\mathbb{X}^0$

One-point reflection [CF12, Th.7.5.6] and [FTa05]

Under conditions (A.1), (A.2) and (A.3) on $\mathbb{X}^0$ in [CF12, §7.5], piecing together excursions around a_1 by Poisson point process in Itõ's sense whose characteristic measure is determined by the entrance law ν_t from a_1 to E_0 defined by

$\int_0^\infty \nu_t dt = \mathbb{P}^0(\zeta^0 < \infty, X^0_{\zeta^0-} = a_1)m_0$

N-points reflection [CF12, Th.7.7.4]

Under conditions (M.1), (M.2) and (M.3) on $\mathbb{X}^0$ in [CF12, §7.7], repeat the above procedure inductively.

4 Several applications

4.1 Walsh's Brownian motion

based on [CF15]

$E = \mathbb{R}^2, E_0 = \mathbb{R}^2 \setminus \{\mathbf{0}\}$.

J.B.Walsh [W78] heuristically described the motion $\mathbb{X}$ starting at $x \in E_0$ as the 1-dimensional BM on a ray connecting x and $\mathbf{0}$, upon hitting $\mathbf{0}$, it reflects in a random direction θ with a given distribution η.

$E_0 = \{(r, \theta) : r \in (0, \infty) \quad \theta \in [0, 2\pi)\}$. $m = \lambda \times \eta$ for the Lebesgue measure λ on $(0, \infty)$. m is extended to E by setting $m(\mathbf{0}) = \mathbf{0}$
p_t^0: the transition function of the ABM on $(0, \infty)$

The transition function P_t^0 of $\mathbb{X}^0$ is then given by $(P_t^0 f)(r, \theta) = (p_t^0 f_\theta)(r)$ for $f_\theta(r) = f(r, \theta)$, whch is m-symmetric because p_t^0 is λ-symmetric.

Define *Walsh's Brownian motion* $\mathbb{X}$ on E to be the one-point reflection of the m-symmetric diffusion $\mathbb{X}^0$. It can be constructed from $\mathbb{X}^0$ by means of the $\mathbb{X}^0$-entrance law $\nu_t(dx) = \frac{1}{(2\pi t^3)^{1/2}} r e^{-t^2/(2t)} dr \cdot \eta(d\theta)$
because $\mathbb{X}^0$ satisfies (A.1),(A.2),(A.3).

Barlow,Pitman and Yor [BPY89] constructed a Feller semi-group on E for an extension of $\mathbb{X}^0$, which can be verified to be m-symmetric and consequently corresponds to Walsh's BM.

4.2 Brownian motion with darning(BMD)

based on [CFM23]

$G \subset \mathbb{C}$: domain such that either $G = \mathbb{C}$ or $\mathbb{C} \setminus G$ is continuum (closed connected, containing at least two points)
$D = G \setminus K, \quad K = \bigcup_{i=1}^N A_i$: $(N+1)$-connected domain
A_i are mutually disjoint compact continua.

$D^* = D \cup K^*, \quad K^* = \{a_i^* : 1 \le i \le N\}$: quotient topologocal space obtained from G by rendering each set A_i into singleton a_i^*
m: Lebesgue measure on D being extended to D^* by setting $m(a_i^*) = 0,\ 1 \le i \le N$

$\mathbb{X}$: *Brownian motion with darning (BMD)* $\overset{\text{def}}{\Longleftrightarrow}$
m-symmetric diffusion on D^* admitting no killing on K^* whose part process on D is identical in law with the ABM on D

ABM $\mathbb{X}^0$ on D satisfies condition (**) due to the specific properties of harmonic and α-harmonic functions on $\mathbb{C}$ ([CFM23, Lemmas 1.1.3 and 1.3.1]).

Hence BMD $\mathbb{X}$ can be regarded as the unique N-points reflection of $\mathbb{X}^0$ from D to D^*

By the above theorem, the Dirichlet form $(\mathcal{E}^*, \mathcal{F}^*)$ of BMD on $L^2(D^*; m)$ equals the linear subspace of $H^1(D)$ spanned by $H_0^1(D)$ and $\{u_\alpha^{(i)},\ 1 \le i \le N\}$.

$(\mathcal{E}^*, \mathcal{F}^*)$ can be verified to be strongly local and regular.
Further the associated capacity of each point a_i^* is positive.

So, by refining the associated diffusion, we can construct BMD on D^* starting at every point of D^*.

BMD is a key to extend the *SLE theory* from simply connected domains: because the restriction to D of any BMD-harmonic function v admits an analytic function f on D with $\Im f = v$ up to an addtional real constant.

4.3 Reflections at infinity of a time changed RBM

based on [CF18]

For a domain $D \subset \mathbb{R}^d$ with $d \ge 3$, consider the Dirichlet form

$(***) \quad (\mathcal{E}, \mathcal{F}) = (\frac{1}{2}\mathbf{D}_D, H^1(D))$ on $L^2(D)$.
$H_e^1(D)$ and $\mathrm{BL}(D) = \{u \in L^2_{\mathrm{loc}}(D) : |\nabla| \in L^2(D)\}$

are the extended Dirichlet space and the reflected Dirichlet space of (***), respectively, $\mathbf{D}_D$ extends to both spaces.

Let $\mathcal{H}^* = \{u \in \mathrm{BL}(D) : \ \mathbf{D}_D(u,v) = 0 \ \forall v \in H^1_e(D)\}$

D is called a *Liouville domain* if (***) is transient and $\dim(\mathcal{H}^*) = 1$.

An example of Liouville domain is the truncated infinite cone defined by

$C_{A,a} = \{(r,\omega) : r > a, \quad \omega \in A\} \subset \mathbb{R}^d$
for $a > 0$ and a connected open set $A \subset S^{d-1}$ with Lipschitz boundary.

Fix a domain $D \subset \mathbb{R}^d$ for $d \geq 3$ with Lipschitz boundary satisfying

$$D \setminus \overline{B_r(\mathbf{0})} \ = \bigcup_{j=1}^{N} C_j$$

for some $r > 0$ where $C_1, \cdots C_N$ are Liouville domains with Lipschitz boundaries such that $\overline{C_1}, \cdots, \overline{C_N}$ are mutually disjoint.

Owing to [FTo96], there exists a strong Feller conservative diffusion process $\mathbb{Z} = (Z_t, \mathbb{Q}_x)$ on $\overline{D}$ which is a refined version of the RBM associated with the regular Dirichlet form (***) on $L^2(\overline{D})$.

As $D \supset C_1$, the Dirichlet form (***) for D is transient.
Hence it follows from [CF12, Th.3.5.2] that

$\mathbb{Q}_x(\lim_{t\to\infty} Z_t = \partial) = 1, \quad \forall x \in \overline{D}$.

Define

∂_j: point at infinity of $\overline{C_j}, \quad 1 \leq j \leq N$
$F = \{\partial_1, \cdots, \partial_N\}, \quad \overline{D}^* = \overline{D} \cup F$ compact Hausdorff space

Let $\varphi_j(x) = \mathbb{Q}_x\left(\lim_{t\to\infty} Z_t = \partial_j\right) \ x \in \overline{D}, \ 1 \leq j \leq N.$, Then
$\varphi_j(x) > 0, \ 1 \leq j \leq N, \quad \sum_{j=1}^N \varphi_j(x) = 1, \ \ \forall x \in \overline{D}$.

Take a strictly positive bounded integrable function f on $\overline{D}$ and define
$A_t = \int_0^t f(Z_s)ds, \quad t \geq 0,$, which is a PCAF of $\mathbb{Z}$ and
$\mathbb{Q}_x(A_\infty < \infty) = 1, \quad \forall x \in \overline{D}$.

Let $\mathbb{X} = (X_t, \zeta, \mathbb{P}_x)$ be the *time changed process of* $\mathbb{Z}$ *by* A_t. Then

$\mathbb{P}_x(\zeta < \infty) = \mathbb{Q}_x(A_\infty < \infty) = 1, \quad \mathbb{P}_x(\zeta < \infty, \ X_{\zeta-} = \partial_j) = \varphi_j(x) > 0,$
for any $x \in \overline{D}, \ 1 \leq j \leq N$.

$\mathbb{X}$ is symmetric with respect to $m(dx) = f(x)dx$.
The Dirichlet form $(\mathcal{E}^{\mathbb{X}}, \mathcal{F}^{\mathbb{X}})$ of $\mathbb{X}$ on $L^2(\overline{D}; m)$ is given by

$$\mathcal{E}^{\mathbb{X}} = \frac{1}{2}\mathbf{D}_D. \quad \mathcal{F}^{\mathbb{X}} = H^1_e(D) \cap L^2(\overline{D}; m).$$

Extend m from $\overline{D}$ to $\overline{D}^*$ by setting $m(\{\partial_1, \cdots, \partial_N\}) = 0$.

N-points reflection $\mathbb{X}^*$ of $\mathbb{X}$ from $\overline{D}$ to $\overline{D}^*$ uniquely exists because $\mathbb{X}$ satisfies (M.1), (M.2), (M.3).

$\mathbb{X}^*$ is conservatve.

Let $(\mathcal{E}^*, \mathcal{F}^*)$ and $\mathcal{F}^*_e$ be the Dirichlet form of $\mathbb{X}^*$ on $L^2(\overline{D}^*, m)$ and its extended Dirichlet space. Then
$\mathcal{F}^*_e = H^1_0(D) \ \oplus \ \{\sum_{j=1}^N c_j\varphi_j : \ c_j \in \mathbb{R}\} \ \subset \ \mathrm{BL}(D)$,
$\mathcal{E}^*(u, u) = \frac{1}{2}\mathbf{D}_D(u, u) \quad u \in \mathcal{F}^*_e$.

4.4 All possible symmetric conservative diffusion extensions of the time changed RBM $\mathbb{X}$ on $\overline{D}$

A map Π from the boundary set $F = \{\partial_1, \cdots, \partial_N\}$ onto a finite set $\hat{F} = \{\hat{\partial}_1. \cdots, \hat{\partial}_\ell\}$ with $\ell \leq N$ is called a *partition of* F. We let $\overline{D}^{\Pi,*} = \overline{D} \cup \hat{F}$. Π is extended from F to $\overline{D}^*$ by setting $\Pi x = x, \ x \in \overline{D}$, and $\overline{D}^{\Pi,*}$ is equipped with the quotient topology by Π.

$\overline{D}^{\Pi.*}$ is a compact Hausdorff space and may be called an *ℓ-point compactification of* $\overline{D}$ obtained from $\overline{D}^*$ by identifying the points in the set $\Pi^{-1}\partial_i \subset F$ as a single point $\hat{\partial}_i$ for each $1 \leq i \leq \ell$.

The approaching probabilities of the RBM $\mathbb{Z} = (Z_t, \mathbb{Q}_x)$ on $\overline{D}$ to $\hat{\partial}_i \in \hat{F}$ are defined by $\quad \hat{\varphi}_i(x) = \sum_{j \in \Pi^{-1}\hat{\partial}_i} \varphi_j(x), \quad x \in \overline{D}, \ 1 \leq i \leq \ell$.

The time changed process $\mathbb{X} = (X_t, \zeta, \mathbb{P}_x)$ of the RBM $\mathbb{Z}$ on $\overline{D}$ is defined as above.

The measure m on $\overline{D}$ is extended to $\overline{D}^{\Pi,*}$ by seeting $m(\hat{F}) = 0$.

Just as in the above, there exists then an m-symmetric conservative diffusion extension $\mathbb{X}^{\Pi,*}$ of $\mathbb{X}$ from $\overline{D}$ to $\overline{D}^{\Pi,*}$ with the following Dirichlet form $(\mathcal{E}^{\Pi,*}, \mathcal{F}^{\Pi,*})$ on $L^2(\overline{D}^{\Pi,*}; m)(= L^2(D; m))$.

Let $(\mathcal{E}^*, \mathcal{F}^*)$ and $\mathcal{F}_e^*$ be the Dirichlet form of $\mathbb{X}^*$ on $L^2(\overline{D}^*, m)$ and its extended Dirichlet space. Then

$\mathcal{F}_e^{\Pi,*} = H_0^1(D) \oplus \{\sum_{i=1}^{\ell} c_j \hat{\varphi}_i : \ c_i \in \mathbb{R}\} \subset \mathrm{BL}(D),$

$\mathcal{E}^{\Pi,*}(u,u) = \frac{1}{2}\mathbf{D}_D(u,u) \quad u \in \mathcal{F}_e^{\Pi,*}.$

Theorem 4.1 [CF18, Th.5.1] $\{\mathbb{X}^{\Pi,*} : \Pi$ is a partition of $F\}$ *exhausts all possible m-symmetric conservative diffusion extensions of the time changed RBM $\mathbb{X}$ on $\overline{D}$.*

The extended Dirichlet space of $\mathbb{X}^{\Pi,*}$ does not depend on the measure $m(dx) = f(x)dx$ taking part in the time change of the RBM $\mathbb{Z}$ on $\overline{D}$.

An analogous theorem holds for the reflecting diffusion process constructed by [FTo96].

5 About symmetry

(I) One-dimensional difusions [F14]

$\mathbb{X}^0$: One dimensional minimal diffusion on $I = (r_1, r_2)$ with canonical (speed) measure m. Then $\mathbb{X}^0$ is m-symmetric. The general boundary conditions for it are very well formulated in terms of Dirichlet forms including much quicker construction of associated diffusions.

(II) Duality preserving extensions [CF07]

$\mathbb{X}$ and $\hat{\mathbb{X}}$ are in weak duality with respect to a measure m (in m-duality)
$\overset{\text{def}}{\Longleftrightarrow} \quad \int G_\alpha f \cdot g \; dm = \int f \cdot \hat{G}_\alpha g \; dm, \quad f, g \geq 0.$

E locally compact separable metric space, $F = \{a_1, \cdots, a_N\} \subset E$, $E_0 = E \setminus F$

$\mathbb{X}^0$, $\hat{\mathbb{X}}^0$: standard processes on E_0 in weak duality w.r.to a measure m_0 on E_0, both being approachable to each a_i.
Extend m_0 to m on E by setting $m(F) = 0$.

Look for Markovian extensions $\mathbb{X}$, $\hat{\mathbb{X}}$ of $\mathbb{X}^0$, $\hat{\mathbb{X}}^0$ to E which are in m-duality.

Let $\mathbb{X}$, $\hat{\mathbb{X}}$ be standard processes on E whose parts on E_0 are $\mathbb{X}^0$, $\hat{\mathbb{X}}^0$, admitting no jump from F to F, nor from E_0 to F, but admitting killing on F with killing measure κ_i, $\hat{\kappa}_i$ at a_i, $1 \leq i \leq N$.

It is possible to construct such $\mathbb{X}$ and $\hat{\mathbb{X}}$ under some conditions on $\mathbb{X}^0$, $\hat{\mathbb{X}}^0$.

$\mathbb{X}$ and $\hat{\mathbb{X}}$ are in m-duality if and only if

$$\sum_{k\neq i} U_{ik} + V_i + \kappa_i = \sum_{k\neq i} U_{ki} + \hat{V}_i + \hat{\kappa}_i, \quad 1 \leq i \leq N,$$

where $U_{ij},\ V_i$ (resp. $\hat{V}_i$) are Feller measures of $\mathbb{X}^0$ (resp. $\hat{\mathbb{X}}^0$).

When $\mathbb{X}^0$ is m-symmetric, $U_{ik} = U_{ki},\ V_i = \hat{V}_i,\ \kappa_i = \hat{\kappa}_i$ so that the above identity holds with $\kappa_i = 0$.

When $\mathbb{X}^0$ is non-symmetric, one needs to allow suitable killings on the boundary in order to preserve the m-duality in the extention.

References

[BPY89] M.T.Barlow, J.W.Pitman and M.Yor, On Walsh's Brownian motion, in Seminaire de Probabilités, 23, Lecture Notes in Math. vol.1372, Springer, 1989, pp 25-293

[CF07] Z.-Q. Chen and M. Fukushima, On Feller's boundary prolem for Markov processes in weak duality, *J. Func. Anal.* **252**(2007), 289-316

[CF09] Z.-Q. Chen and M. Fukushima, On unique extension of time changed reflecting Brownian motions, *Ann.Inst.Henri Poincare, Probab. statist.,* **45**(2009), 861-875 f

[CF12] Z.-Q. Chen and M. Fukushima, *Symmeric Markov Processes, Time Change, and Boundary Theory*, Princeton University Press, Princeton and Oxford, 2012

[CF15] Z.-Q. Chen and M. Fukushima, One point reflections, *Stochastic Processes Appl.* **125**(2015), 1368-1393

[CF18] Z.-Q. Chen and M. Fukushima, Reflection at infinity of time changed RBMs on a domain with Liouville branches, *J. Math. Soc. Japan* **70**(2018), 833-852

[CFM23] Z.-Q.Chen, M.Fukushima and T.Murayama, *Stochastic Komatu-Loewner Evolutions*, World Scientific, 2023

[Do62] J.L.Doob, Boundary properties of functions with finite Dirichlet integrals, *Ann.Inst.Fourier,* **12**(1962), 573-621

[Dy65] E,B, Dynkin, *Markov Processes*, vol.1, 3, Springer, 1965

[Fe57] W.Feller, On boundaries and lateral conditions for the Kolmogorov differential equations, *Ann.Math.***55**(1957), 527-570

[F64] M. Fukushima, On Feller's kernel and the Dirichlet norm, *Nagoya Math.J.,*,**24**(1964), 167-175

[F67] M. Fukushima, A construction of reflecting barrier Brownian motions for bounded domains, *Osaka J. Math.***4**(1967), 183-215

[F69] M. Fukushima, On boundary conditions for multi-dimensional Brownian motions with symmetric resolvent densities, *J.Math.Soc.Japan* **21**(1969), 58-93

[F80] M. Fukushima, *Dirichlet Forms and Markov Processes*, North Holland, Amsterdom-New York/ Kodansha, Tokyo, 1980

[F10] M. Fukushima, From one-dimensional diffusions to symmetric Markov processes, *Stochastic Process Appl.***120**(2010), 590-604

[F14] M. Fukushima, On general boundary conditions for one-dimensional diffusions with symmetry, *J.Math.Soc.Japan* **66**(2014), 289-316

[F18] M.Fukushima, Liouville property of harmonic functions with finite energy for Dirichlet forms, in: *Stochastic Partial Differential Equations and Relaed Fields* A.Eberle et al. (eds.), Springer Proceedings in Mathematics and Statistics, Vol 229, 2018, pp25-42

[F20] M. Fukushima, Komatu-Loewner differential equations, *SUGAKU Expositions,* **33**(2020), AMS, 239-260

[FOT11] M. Fukushima, Y. Oshima and M.Takeda, *Dirichlet Forms and Symmetric Markov Processes*, de Gruyter, Berlin,1994
Second revised and extended editions, de Gruyter, Berlin, 2011

[FTa05] M. Fukushima and H. Tanaka, Poisson point processes attached to symmetric diffusions, *Ann.Inst. Henri-Poincaré Probab. Stat.,* **41**(2005), 419-459

[FTo96] , M.Fukushima and M.Tomisaki, Construction and decomposition of reflecting diffusions on Lipschitz domains with Hölder cusps, *Probab.Theory Relat.Fields* **106**(1996), 521-557

[GT77] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer 1977

[I60] K. Itô, *Lecture on Stochastic Processes*, Tata Instiute of Fundamental Research, Bombay, 1965

[IM66] K. Itô and H.P. McKean,Jr., *Diffusion Processes and their Sample Paths*, Springer, 1965, Springer's Classics in Mathmatics Series, 1996

[S74] M.L. Silverstein, *Symmetric Markov Processes*, Lecture Notes in Math. **426**, Springer, Berlin-Heidelberg-New York, 1974

[W78] J.B.Walsh, A diffusion with discontinous local time, *Asterisque* **52-53**(1978), 37-45