On extended Dirichlet spaces and the space of BL functions *

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Abstract

We first formulate the notions of the Dirichlet form and the extended Dirichlet space without assuming any topology on the underlying space. We next study the space of BL functions on an Euclidean domain in relation to the extended Dirichlet space of the Sobolev space of order 1 as well as an extension of the time changed transient Brownian motion by reflection at infinity.

The (symmetric) Dirichlet space and the Dirichlet form were introduced by A. Beurling and J. Deny [1] in 1959 and then the extended Dirichlet space was defined by M.L. Silverstein [15] in 1974. In both cases, the underlying space was assumed to be a locally compact Hausdorff topological space. We first formulate them without assuming any topology on the underlying space using an idea of B. Schmuland [13].

The notion of the extended Dirichlet space serves as a useful tool to give criteria for the transience and recurrence of the associated Markovian semigroup. Besides, it plays the role of an invariant in describing the time changes of the process or equivalently, the exchanges of the underlying measure, and it enables us to formulate its maximal extension called the reflected Dirichlet space in transient case.

The space of BL (Beppo Levi) functions on an Euclidean domain was introduced and studied in J. Deny and J.L. Lions [8] in 1953-4 that preceded [1]. The second aim of the present paper is to examine this space in relation to the above mentioned concepts for the Brownian motion, producing a new extension of it at infinity.

1 Extended Dirichlet spaces

Let $(E, \mathcal{B}(E))$ be a measurable space and m be a σ-finite measure on it. Numerical functions $f,\ g$ on E are said to be m-equivalent ($f = g\ [m]$ in notation) if $f = g$ m-a.e.

Definition 1.1 For $1 \le p \le \infty$,, a linear operator L on $L^p(E; m)$ is called *Markovian* if

$$0 \le f \le 1\ [m],\ f \in \mathcal{D}(L) \quad \Longrightarrow \quad 0 \le Lf \le 1\ [m].$$

A real function φ, namely, a mapping from $\mathbb{R}$ to $\mathbb{R}$, is said to be a *normal contraction* if

$$\varphi(0) = 0, \quad |\varphi(s) - \varphi(t)| \le |s - t|, \quad \forall s, t \in \mathbb{R}.$$

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A function defined by $\varphi(t) = (0 \vee t) \wedge 1,\ t \in \mathbb{R}$, is a normal contraction which is called the *unit contraction*. For any $\epsilon > 0$, a real function φ_ϵ satisfying the next condition is a normal contraction:

$$\begin{aligned} &\varphi_\epsilon(t) = t,\ \forall t \in [0,1];\ -\epsilon \le \varphi_\epsilon(t) \le 1+\epsilon,\ \forall t \in \mathbb{R}, \\ &s < t\ \rightarrow\ 0 \le \varphi_\epsilon(t) - \varphi_\epsilon(s) \le t - s,\ \forall s,t \in \mathbb{R}. \end{aligned} \tag{1.1}$$

Definition 1.2 A symmetric form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(E;m)$ is called *Markovian* if, for any $\epsilon > 0$, there exists a real function φ_ϵ satisfying (1.1) and

$$f \in \mathcal{D}(\mathcal{E}) \quad \Longrightarrow \quad g = \varphi_\epsilon \circ f \in \mathcal{D}(\mathcal{E}), \quad \mathcal{E}(g,g) \le \mathcal{E}(f,f). \tag{1.2}$$

A closed symmetric form $(\mathcal{E}, \mathcal{F})$ on $L^2(E;m)$ is called a *Dirichlet form* if it is Markovian. In this case, the domain $\mathcal{F}(= \mathcal{D}(\mathcal{E}))$ is said to be a *Dirichlet space.*

Theorem 1.3 *Let $(\mathcal{E}, \mathcal{F})$ be a closed symmetric form on $L^2(E;m)$ and $\{T_t\}_{t>0}$, $\{G_\alpha\}_{\alpha>0}$ be the strongly continuous contraction semigroup and resolvent on $L^2(E;m)$ generated by $(\mathcal{E}, \mathcal{F})$, respectively. Then the next conditions are mutually equivalent:*

(a) *T_t is Markovian for each $t > 0$.*

(b) *αG_α is Markovian for each $\alpha > 0$.*

(c) *$(\mathcal{E}, \mathcal{F})$ is a Dirichlet form on $L^2(E;m)$.*

(d) *The unit contraction operates on $(\mathcal{E}, \mathcal{F})$:*

$$f \in \mathcal{F} \implies g = (0 \vee f) \wedge 1 \in \mathcal{F}, \quad \mathcal{E}(g,g) \le \mathcal{E}(f,f).$$

(e) *Every normal contraction operates on $(\mathcal{E}, \mathcal{F})$: for any normal contraction φ*

$$f \in \mathcal{F} \implies g = \varphi \circ f \in \mathcal{F}, \quad \mathcal{E}(g,g) \le \mathcal{E}(f,f).$$

This theorem can be shown in exactly the same way as the proof of [9, Th.1.4.1] except for the implication (**a**) $\Rightarrow$ (**e**), which follow however from the more general theorem formulated below. In what follows, we occasionally use for a Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(E;m)$ the notations $||f||_{\mathcal{E}} = \sqrt{\mathcal{E}(f,f)},\ f \in \mathcal{F}$.

Definition 1.4 Let $(\mathcal{E}, \mathcal{F})$ be a closed symmetric form on $L^2(E;m)$. Denote by $\mathcal{F}_e$ the totality of m-equivalence classes of all m-measurable functions f on E satisfying the next condition:

$$|f| < \infty\ [m],\ \exists \{f_n\} \subset \mathcal{F},\ \lim_{n,n' \to \infty} \|f_n - f_{n'}\|_{\mathcal{E}} = 0,\ \lim_{n \to \infty} f_n = f\ [m]. \tag{1.3}$$

$\{f_n\} \subset \mathcal{F}$ in the above is called an *approximating sequence of $f \in \mathcal{F}_e$*. We call the space $\mathcal{F}_e$ the *extended space* attached to $(\mathcal{E}, \mathcal{F})$. When the latter is a Dirichlet form on $L^2(E;m)$, the space $\mathcal{F}_e$ will be called its *extended Dirichlet space.*

Theorem 1.5 *Let $(\mathcal{E}, \mathcal{F})$ be a closed symmetric form on $L^2(E;m)$ and $\mathcal{F}_e$ be the extended space attached to it. If the semigroup $\{T_t; t > 0\}$ generated by $(\mathcal{E}, \mathcal{D}(\mathcal{F}))$ is Markovian, then the followings are true:*

(i) *For any* $f \in \mathcal{F}_e$ *and for any approximating sequence* $\{f_n\} \subset \mathcal{F}$ *of* f*, the limit* $\mathcal{E}(f,f) = \lim_{n\to\infty} \mathcal{E}(f_n, f_n)$ *exists independently of the choice of an approximating sequence* $\{f_n\}$ *of* f.
(ii) *Every normal contraction operates on* $(\mathcal{F}_e, \mathcal{E})$*: for any normal contration* φ

$$f \in \mathcal{F}_e \implies g = \varphi \circ f \in \mathcal{F}_e, \quad \mathcal{E}(g,g) \le \mathcal{E}(f,f).$$

(iii) $\mathcal{F} = \mathcal{F}_e \cap L^2(E;m)$. *In particular,* $(\mathcal{E}, \mathcal{F})$ *is a Dirichlet form on* $L^2(E;m)$.

Assertions (ii) and (iii) of this theorem imply the implication (**a**) $\Rightarrow$ (**e**) in Theorem 1.3. Before giving the proof of Theorem 1.5, we shall fix a Markovian contractive symmetric linear operator T on $L^2(E;m)$ and make some preparatory observations.

By the linearity and the Markovian property of T on $L^2(E;m) \cap L^\infty(E;m)$,

$$f_1,\ f_2 \in L^2 \cap L^\infty,\ 0 \le f_1 \le f_2\ [m] \Longrightarrow 0 \le Tf_1 \le Tf_2 \le \|f_2\|_\infty\ [m].$$

Due to the σ-finitness of m, we can construct a Borel function $\eta \in L^1(E;m)$ which is strictly positive on E. If we put $\eta_n(x) = (n\eta(x)) \wedge 1$, then $0 < \eta_n \le 1,\ \eta_n \uparrow 1,\ n \to \infty$. Hence we can define an extension of T from $L^2(E;m) \cap L^\infty(E;m)$ to $L^\infty(E;m)$ by

$$\begin{cases} Tf(x) &= \lim_{n\to\infty} T(f\cdot\eta_n)(x),\ m\text{-a.e. } x \in E,\ f \in L^\infty_+(E;m), \\ Tf &= Tf^+ - Tf^-, \quad f \in L^\infty(E;m), \quad f = f^+ - f^-. \end{cases} \tag{1.4}$$

By the symmetry of T, $(g, T(f\cdot\eta_n)) = (Tg, f\cdot\eta_n),\ g \in bL^1(E;m)$, and we let $n \to \infty$ to see that the function $Tf,\ f \in L^\infty(E;m)$, defined by (1.4) satisfies the identity

$$\langle g, Tf\rangle = \langle Tg, f\rangle, \quad \forall g \in bL^1(E;m), \tag{1.5}$$

where $\langle g, f\rangle$ denotes the integral $\int_E gf dm$ for $g \in L^1(E;m),\ f \in L^\infty(E;m)$. Consequently Tf is uniquely determined up to the m-equivalence for $f \in L^\infty(E;m)$. T becomes a Markovian linear operator on $L^\infty(E;m)$ and satisfies

$$f_n,\ f \in L^\infty_+(E;m),\ f_n \uparrow f\ [m],\ n \to \infty \Longrightarrow \lim_{n\to\infty} Tf_n = Tf\ [m]. \tag{1.6}$$

Further, if a sequence $\{f_n\} \subset L^\infty(E;m)$ is uniformly bounded and converges to f m-a.e. as $n \to \infty$, then

$$\lim_{n\to\infty} \langle g, Tf_n\rangle = \langle g, Tf\rangle, \quad \forall g \in bL^1(E;m). \tag{1.7}$$

Lemma 1.6 (i) *For any* $g \in L^\infty(E;m)$,

$$T(g^2) - 2gTg + g^2T1 \ge 0 \quad [m].$$

(ii) *For any* $g \in L^\infty(E;m)$*, define*

$$\mathcal{A}_T(g) = \frac{1}{2}\int_E [T(g^2) - 2gTg + g^2T1]dm + \int_E g^2(1 - T1)dm. \tag{1.8}$$

It holds for $g \in L^2(E;m) \cap L^\infty(E;m)$ *that*

$$\mathcal{A}_T(g) = (g - Tg, g). \tag{1.9}$$

(iii) *For any* $g \in L^\infty(E;m)$ *and for any normal contraction* φ,

$$\mathcal{A}_T(\varphi \circ g) \le \mathcal{A}_T(g). \tag{1.10}$$

Proof. Since $T(g^2) - 2sTg + s^2T1 \geq 0 \quad [m]$ for $g \in L^\infty$ and a simple function s, we get (i) by letting $s \to g$. (ii) can be shown first for $\mathcal{A}_T^k(g) = \langle T(g^2) - 2gTg + g^2T1, \eta_k\rangle + \langle g^2(1 - T1), \eta_k\rangle$ by approximating g with simple functions and using (1.7). We then let $k \to \infty$ to obtain (ii). □

Let φ^ℓ be a specific sequence of normal contractions defined by

$$\varphi^\ell(t) = [(-\ell) \vee t] \wedge \ell, \quad t \in \mathbb{R}. \tag{1.11}$$

For any m-measurable function g on E with $|g| < \infty$ $[m]$, $\mathcal{A}_T(\varphi^\ell \circ g)$ is increasing as ℓ increases, as is clear from $\varphi^\ell \circ (\varphi^{\ell+1} \circ g) = \varphi^\ell \circ g$ and Lemma 1.6 (iii). We can then extend the definition of $\mathcal{A}_T(g)$ to g by letting

$$\mathcal{A}_T(g) = \lim_{\ell\to\infty} \mathcal{A}_T(\varphi^\ell \circ g) \ (\leq \infty). \tag{1.12}$$

Lemma 1.7 (i) *For $g \in L^2(E; m)$, $\mathcal{A}_T(g) = (g - Tg, g)$.*
(ii) **(Fatou's property)** *For any m-measurable functions g_n, g on E with $|g_n| < \infty$, $|g| < \infty[m]$, $\lim_{n\to\infty} g_n = g$ $[m]$,*

$$\mathcal{A}_T(g) \leq \liminf_{n\to\infty} \mathcal{A}_T(g_n). \tag{1.13}$$

(iii) *For any m-measurable function g on E with $|g| < \infty$ $[m]$ and for any normal contraction φ, $\mathcal{A}_T(\varphi \circ g) \leq \mathcal{A}_T(g)$.*

Proof. (i) follows from Lemma 1.6 (ii) and the contraction property of T on $L^2(E; m)$. Assertion (ii) for uniformly bounded $\{g_n\}$ can be shown in the same manner as the proof of [13, Prop.1] with a help of property (1.6). It then suffices to use the contraction φ^ℓ. For the proof of (iii), we put $f = \varphi \circ g$, $f_\ell = \varphi \circ \varphi_\ell \circ g$. Then $f_\ell \to f$, $\ell \to \infty$, and it suffices to combine (ii) with (1.10). □

More details of the proof of Lemma 1.6 and Lemma 1.7 are being given in [10] and [6].

Proof of Theorem 1.5 (i) For any $f \in \mathcal{F}_e$, take its approximating sequence $\{f_n\} \subset \mathcal{F}$. f_n being $\mathcal{E}$-Cauchy, the triangular inequality guarantees the existence of the limit $\mathcal{E}(f, f) = \lim_{n\to\infty} \mathcal{E}(f_n, f_n)$. Let us prove that

$$\frac{1}{t}\mathcal{A}_{T_t}(f) \ \uparrow \ \mathcal{E}(f, f) \quad t \downarrow 0, \tag{1.14}$$

which in particular implies that $\mathcal{E}(f, f)$ does not depend the choice of the approximating sequence.

Since $f - f_\ell \in \mathcal{F}_e$ for each ℓ and $\{f_n - f_\ell\} \subset \mathcal{F}$ is its approximating sequence, we have from Lemma 1.7

$$\frac{1}{t}\mathcal{A}_{T_t}(f - f_\ell) \leq \liminf_{n\to\infty} \frac{1}{t}\mathcal{A}_{T_t}(f_n - f_\ell) \leq \lim_{n\to\infty} \|f_n - f_\ell\|_{\mathcal{E}}^2.$$

Therefore $\lim\limits_{\ell\to\infty} \mathcal{A}_{T_t}(f - f_\ell) = 0$, and by the triangular inequality
$\mathcal{A}_{T_t}(f) = \lim_{\ell\to\infty} \mathcal{A}_{T_t}(f_\ell)$, which particularly implies the monotonicity of the left

hand side of (1.14) in t. Since $\lim_{t\downarrow 0}\frac{1}{t}\mathcal{A}_{T_t}(f_\ell) = \|f_\ell\|_{\mathcal{E}}^2$, we can get from the triangular inequality and the inequality obtained above that

$$\left|\lim_{t\downarrow 0}\sqrt{\frac{1}{t}\mathcal{A}_{T_t}(f)} - \|f_\ell\|_{\mathcal{E}}\right| \le \lim_{t\downarrow 0}\sqrt{\frac{1}{t}\mathcal{A}_{T_t}(f-f_\ell)} \le \lim_{n\to\infty}\|f_n - f_\ell\|_{\mathcal{E}}.$$

The last term tends to 0 as $\ell\to\infty$, yielding (1.14).
(ii) By Lemma 1.7 and (1.14), we have

$$\frac{1}{t}\mathcal{A}_{T_t}(\varphi\circ f) \le \frac{1}{t}\mathcal{A}_{T_t}(f) \le \mathcal{E}(f,f), \quad \forall t>0,$$

and consequently, it is enough to prove that $\varphi\circ f\in\mathcal{F}_e$. But this is an easy consequence of a Babach-Saks type theorem. □

Remark 1.8 Theorem 1.3 has been formulated in Bouleau-Hirsch [2] and in Ma-Röckner [12] without any topology on E, while Theorem 1.5 was shown in Silverstein [15] and Fukushima-Oshima-Takeda [9] under a certain topological assumption on E.

Let $(\mathcal{E},\mathcal{F})$ be a Dirichlet form on $L^2(E;m)$, $(\mathcal{F}_e,\mathcal{E})$ be its extended Dirichlet space and $\{T_t; t>0\}$ be the associated strongly continuous contraction semigroup of Markovian symmetric operators on $L^2(E;m)$. We put

$$S_t f = \int_0^t T_s f ds, \quad t>0, \quad f\in L^2(E;m), \tag{1.15}$$

the integral being taken in Bochner's sense. S_t is a linear operator on $L^2(E;m)$ and satisfies the boundedness $||S_tf||_2\le t||f||_2,\ f\in L^2(E;m)$.

Take $f\in L^2(E;m)\cap L^1(E;m)$. Choose $B_n\in \underline{(}E)$ with $m(B_n)<\infty,\ B_n\uparrow E$. By the symmetry and the Markov property of T_t,

$$\int_{B_n}|T_tf(x)|m(dx) \le (T_t|f|, 1_{B_n}) = (|f|, T_t1_{B_n}) \le \int_E |f(x)|m(dx).$$

Letting $n\to\infty$, we get $||T_tf||_1\le||f||_1$. Similarly, we get $||S_tf||_1\le t||f||_1$ and hence both $\{T_t\}$, $\{S_t\}$ can be extended to linear operators on $L^1(E;m)$ satisfying

$$T_sT_tf = T_{s+t}f,\ ||T_tf||_1\le||f||_1,\ ||S_tf||_1\le t||f||_1,\ f\in L^1(E;m).$$

Further T_t and $\frac{1}{t}S_t$ are Markovian.

Since S_t so extended satisfies for $f\in L^1_+(E;m)$ the positivity and the monotonicity $0\le S_sf\le S_tf,\ [m],\ 0<s<t$, we can define a function $Gf(x)(\le\infty)$ by

$$Gf(x) = \lim_{N\to\infty} S_Nf(x)\ [m],\ f\in L^1_+(E;m), \tag{1.16}$$

uniquely up to the m-equivalence.

Definition 1.9 (i) $\{T_t; t>0\}$ is called *transient* if
$Gg(x)<\infty\ [m]$, for some $g\in L^1_+(E;m)$ with $g>0\ [m]$.
(ii) $\{T_t;t>0\}$ is called *recurrent* if, for any $f\in L^1_+(E;m)$,
$Gf(x)$ is either ∞ or $0\ [m]$, namely, $m\{x\in E: 0<Gf(x)<\infty\}=0$.

It is known that the transience is equivalent to the condition that $Gf(x) < \infty$ $[m]$ for all $f \in L^1_+(E;m)$, while the recurrence is equivalent to $Gf(x) = \infty$ $[m]$ for all $f \in L^1(E;m)$ with $f > 0$ $[m]$. We take the following theorem from [9] (cf. [15]).

Theorem 1.10 (i) *The transience of* $\{T_t; t > 0\}$ *is equivalent to one of the following conditions:*

$$u \in \mathcal{F}_e, \quad \mathcal{E}(u,u) = 0 \Longrightarrow u = 0. \tag{1.17}$$

$$(\mathcal{F}_e, \mathcal{E}) \text{ is a real Hilbert space.} \tag{1.18}$$

(ii) $\{T_t; t > 0\}$ *is recurrent if and only if*

$$1 \in \mathcal{F}_e, \qquad \mathcal{E}(1,1) = 0. \tag{1.19}$$

2 The space of BL functions and Brownian motion

For a non-empty domain D of the Euclidean n-space $\mathbb{R}^n$, we let

$$\mathbf{D}(u,v) = \sum_{i=1}^{n} \int_D \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx, \tag{2.1}$$

$$H^1(D) = \{u \in L^2(D) : \frac{\partial u}{\partial x_i} \in L^2(D),\ 1 \le i \le n.\} \tag{2.2}$$

Here the derivatives $\dfrac{\partial u}{\partial x_i}$, $1 \le i \le n$, are taken in Schwartz distribution sense and $L^2(D)$ denotes the L^2-space on D based on the Lebesgue measure dx.

We forcus our attention on a related space

$$\mathrm{BL}(D) = \left\{ T : \frac{\partial T}{\partial x_i} \in L^2(D),\ 1 \le i \le n \right\} \tag{2.3}$$

of Schwartz distributions T. It is known that any distribution $T \in \mathrm{BL}(D)$ can be identified with a function in $L^2_{loc}(D)$ (cf. L. Schwartz [14], J. Deny and J.L. Lions [8]) so that

$$\mathrm{BL}(D) = \{u \in L^2_{loc}(D) : \frac{\partial u}{\partial x_i} \in L^2(D),\ 1 \le i \le n,\} \tag{2.4}$$

where again the derivatives are taken in Schwartz distribution sense. Members in $\mathrm{BL}(D)$ are called BL (Beppo-Levi) *functions* on D. The space $\mathrm{BL}(D)$ is known to enjoy the following properties (cf. [8]):

(BL.1) The quotient space $\dot{\mathrm{BL}}(D)$ of $\mathrm{BL}(D)$ by the subspace of constant functions is a Hilbert space with inner product $\mathbf{D}$. Any $\mathbf{D}$-Cauchy sequence $u_n \in \mathrm{BL}(D)$ admits $u \in \mathrm{BL}(D)$ and constants c_n such that u_n is $\mathbf{D}$-convergent to u and $u_n + c_n$ is L^2_{loc}-convergent to u.

(BL.2) A function u on D is in $\mathrm{BL}(D)$ if and only if, for each i $(1 \le i \le n)$, there is a version $u^{(i)}$ of u such that it is absolutely continuous on almost all straight lines parallel to x_i-axis and the derivative $\partial u^{(i)}/\partial x_i$ in the ordinary sense (which exists a.e. on D) is in $L^2(D)$. In this case, the ordinary derivatives coincide with the distribution derivatives of u.

Notice that the version $u^{(i)}$ in the above for $u \in \mathrm{BL}(D)$ depends on the choice of the coordinate x_i. However each $u \in \mathrm{BL}(D)$ admits a quasi continuous version $\widetilde{u}$ called a BLD (Beppo Levi-Deny) *function*, which enjoys the absolute continuity property (BL.2) no matter how the coordinates are choosen (cf. [8]).

The space $H^1(D) = \mathrm{BL}(D) \cap L^2(D)$ is called the *Sobolev space* of order 1 on D.

$$(\mathcal{E}, \mathcal{F}) = (\frac{1}{2}\mathbf{D}, H^1(D)) \tag{2.5}$$

is a closed symmetric form on $L^2(D)$. To see this, suppose $\{u_n\} \subset \mathcal{F}$ is $\mathcal{E}_1$-Cauchy. Then $\frac{\partial u_n}{\partial x_i}$ is $L^2(D)$-convergent to some $v_i \in L^2(D)$ for each $1 \le i \le n$ and u_n is $L^2(D)$-convergent to some $u \in L^2(D)$. Then, for any $f \in C_0^\infty$,

$$(v_i, f) = \lim_{n\to\infty} (\frac{\partial u_n}{\partial x_i}, f) = -\lim_{n\to\infty} (u_n, \frac{\partial f}{\partial x_i}) = -(u, \frac{\partial f}{\partial x_i})$$

and hence $v_i = \dfrac{\partial u}{\partial x_i}$, $1 \le i \le n$. It is a Dirichlet form on $L^2(D)$ because its Markov property (1.2) can be verified by a direct use of the property (BL.2).

Let us denote by $H_e^1(D)$ the extended Dirichlet space of the Dirichlet form (2.5). When $D = \mathbb{R}^n$, the Dirichlet form (2.5) on $L^2(\mathbb{R}^n)$ is associated with the transition density $g_t(x) = (2\pi t)^{-n/2} \exp(-|x|^2/(2t))$ of the n-dimensional standard Brownian motion. Since $\displaystyle\int_0^\infty g_t(x)dt$, $x \ne 0$, is divergent when $n = 1,\ 2$, but convergent and equal to the Newtonian kernel $\mathcal{N}(x)$ when $n \ge 3$, the corresponding L^2-semigroup is recurrent in the former case and transient in the latter case.

When $n \ge 3$, the extended Dirichlet space $(H_e^1(\mathbb{R}^n), \mathcal{E})$ of (2.5) for $D = \mathbb{R}^n$ is a real Hilbert space and

$$u \in H_e^1(\mathbb{R}^n),\ \mathcal{E}(u,u) = 0 \implies u = 0, \tag{2.6}$$

in view of Theorem 1.10.

Theorem 2.1 *Assume that* $n \ge 3$. $\mathrm{BL}(\mathbb{R}^n)$ *is the linear space spanned by* $H_e^1(\mathbb{R}^n)$ *and constant functions. The space* $(H_e^1(\mathbb{R}^n), \mathcal{E})$ *is isometric with the space* $(\dot{\mathrm{BL}}(\mathbb{R}^n), \frac{1}{2}\mathbf{D})$ *by the canonical map* $\mathrm{BL}(\mathbb{R}^n) \mapsto \dot{\mathrm{BL}}(\mathbb{R}^n)$.

Proof. For $u \in H_e^1(\mathbb{R}^n)$, there is a sequence $\{u_n\} \subset H^1(\mathbb{R}^n)$ which is $\mathbf{D}$-Cauchy and convergent to u a.e. $\mathbf{D}(u_n, u_n)$ then converges to $\mathcal{E}(u,u)$. By (BL.1), there exist $v \in \mathrm{BL}(\mathbb{R}^n)$ and constants c_n such that $\{u_n\}$ is $\mathbf{D}$-convergent to v and the sequence $\{u_n + c_n\}$ is convergent to v in $L^2_{loc}(\mathbb{R}^n)$. By choosing a subsequence if necessary, we may assume that the latter sequence converges to v a.e. Then $\lim_{n\to\infty} c_n = c$ exists, $u = v - c$ and consequently, $u \in \mathrm{BL}(\mathbb{R}^n)$ and $\mathcal{E}(u,u) = \mathbf{D}(u,u)$. Further we see from (2.6) that the Hilbert space $(H_e^1(\mathbb{R}^n), \mathcal{E})$ is isometrically imbedded into a closed subspace of $(\dot{\mathrm{BL}}(\mathbb{R}^n), \frac{1}{2}\mathbf{D})$ by the canonical map $\mathrm{BL}(\mathbb{R}^n) \mapsto \dot{\mathrm{BL}}(\mathbb{R}^n)$. We denote by $\dot{u}$ the equivalence class represented by $u \in \mathrm{BL}(\mathbb{R}^n)$.

If $\dot{u} \in \dot{\mathrm{BL}}(\mathbb{R}^n)$ is $\mathbf{D}$-orthognal to this closed subspace, then, since $C_0^\infty(\mathbb{R}^n) \subset H^1(\mathbb{R}^n)$, we have

$$(\Delta u, f) = -\mathbf{D}(u, f) = 0, \quad \forall f \in C_0^\infty(\mathbb{R}^n),$$

which implies that $\Delta u = 0$, namely, (a version of) u is harmonic on $\mathbb{R}^n$. Since the ordinary derivatives $\dfrac{\partial u}{\partial x_i}$, $1 \le i \le n$, are also harmonic on $\mathbb{R}^n$, we get from the mean-value theorem the estimate

$$\left|\frac{\partial u}{\partial x_i}(x)\right| \le \frac{1}{|B_r(x)|}\int_{B_r(x)}\left|\frac{\partial u}{\partial x_i}\right| dx \le \left(\frac{1}{|B_r(x)|}\mathbf{D}(u,u)\right)^{1/2}, \ x \in \mathbb{R}^n,$$

where $B_r(x)$ is the ball of rarius r centered at x and $|B_r(x)|$ denotes it volume. By letting $r \to \infty$, we see that all derivatives of u vanish and hence u is constant, and consequently $\dot{u}$ is the 0 element of $\dot{\mathrm{BL}}(\mathbb{R}^n)$. □

Remark 2.2 Example 1.5.3 and Example 6.2.1 of Fukushima-Oshima-Takeda [9] contained incorrect statements that, when $n \ge 3$, the space $H^1_e(\mathbb{R}^n)$ is obtained from $\mathrm{BL}(\mathbb{R}^n)$ by removing non-zero constant functions. Taking this opportunity, we would like mention that they should be corrected as the statement of Theorem 2.1 of the present paper.

We next consider, on a general domain $D \subset \mathbb{R}^n$, a measure $m(dx) = m(x)dx$ with a density function $m(x)$ satisfying

$$m(x) > 0 \quad \forall x \in D, \quad m \in bC(D) \cap L^1(D) \tag{2.7}$$

and an associated form

$$(\mathcal{E}, \mathcal{F}) = \left(\frac{1}{2}\mathbf{D}, \mathrm{BL}(D) \cap L^2(D; m)\right), \tag{2.8}$$

which is obtained just by replacing $L^2(D)$ with $L^1(D; m)$ in (2.5). Since the convergence in $L^2(D; m)$ implies the convergence in $L^2_{loc}(D)$, (2.8) can be readily seen to be a Dirichlet form on $L^2(D; m)$.

Theorem 2.3 *The Dirichlet form* (2.8) *on* $L^2(D; m)$ *is recurrent. Its extended Dirichlet space* $(\mathcal{F}_e, \mathcal{E})$ *coincides with the space* $(\mathrm{BL}(D), \frac{1}{2}\mathbf{D})$.

Proof. Since m is assumed to be a finite measure on D, the present Dirichlet form enjoys the recurrence condition (1.19). We further have

$$u \in \mathcal{F}_e, \ \mathcal{E}(u,u) = 0 \implies u \text{ is constant } a.e. \tag{2.9}$$

To see this, suppose $u \in \mathcal{F}_e$, $\mathcal{E}(u,u) = 0$ and put $u_\ell = \varphi^\ell \circ u$ by the contraction φ^ℓ of (1.11). Then $u_\ell \in L^2(E; m) \cap \mathcal{F}_e = \mathcal{F}$ and hence $\frac{1}{2}\mathbf{D}(u_\ell, u_\ell) = \mathcal{E}(u_\ell, u_\ell) = 0$. Therefore u_ℓ is a constant and we get (2.9) by letting $\ell \to \infty$.

Denote by $\dot{\mathcal{F}}_e$ the quotient space of $\mathcal{F}_e$ by the subspace of constant functions. Just as in the proof of the preceding theorem but using (2.9) in place of (2.6), we conclude that the space $(\dot{\mathcal{F}}_e, \mathcal{E})$ is isometrically embedded into the space $(\dot{\mathrm{BL}}(D), \frac{1}{2}\mathbf{D})$.

Take any $u \in \mathrm{BL}(D)$ and put $u_\ell = \varphi^\ell \circ u$ as above. By (BL.2), $u_\ell \in \mathrm{BL}(D)$ and

$$\mathbf{D}(u - u_\ell, u - u_\ell) = \int_{\{x:|u(x)|>\ell\}} |\nabla u|^2 dx \ \to \ 0, \quad \ell \to \infty.$$

Since $u_\ell \in \mathcal{F}$ and u_ℓ converges to u pointwise, u must be an element of $\mathcal{F}_e$. Hence the above isometric embedding is an onto map and $\mathcal{F}_e = \mathrm{BL}(D)$. □

When the Lebesgue measure of the domain D is finite, then we can take $m(dx)$ to be the Lebesgue measure in (2.8) in reducing $\mathcal{F}$ to $H^1(D)$. Hence

Corollary 2.4 *If the domain D is of finite Lebesgue measure, then*

$$H^1_e(D) = \mathrm{BL}(D). \tag{2.10}$$

Finally, when $D = \mathbb{R}^n$ and m is a finite measure on $\mathbb{R}^n$ with density satisfying (2.7) on $\mathbb{R}^n$, we designate the Dirichlet form (2.8) on $L^2(\mathbb{R}^n; m)$ as

$$(\mathcal{E}^*, \mathcal{F}^*) = \left(\frac{1}{2}\mathbf{D}, \mathrm{BL}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n; m)\right). \tag{2.11}$$

In parallel to this, let us consider the symmetric form on $L^2(\mathbb{R}^n, m)$ defined by

$$(\mathcal{E}^{(0)}, \mathcal{F}^{(0)}) = \left(\frac{1}{2}\mathbf{D}, H^1_e(\mathbb{R}^n) \cap L^2(\mathbb{R}^n; m)\right), \tag{2.12}$$

which is known to be the Dirichlet form associated with the n-dimensional Brownian motion being time changed by its additive functional $A_t = \int_0^t m(X_s)ds$ (cf.[9, Example 6.2.1]).

Assume that $n \geq 3$. Then they are different; (2.11) is recurrent while (2.12) is transient. Further we see from Theorem 2.1 that

$$\mathcal{F}^* = \{u = u_0 + c : u_0 \in \mathcal{F}^{(0)},\ c \text{ is constant}\}, \quad \mathcal{E}^*(u, u) = \mathcal{E}^{(0)}(u_0, u_0). \tag{2.13}$$

Let $\mathbb{R}^n_* = \mathbb{R}^n \cup \{\infty\}$ be the one-point compactification of $\mathbb{R}^n$ and m^* be the extension of m from $\mathbb{R}^n$ to $\mathbb{R}^n_*$ with $m^*(\{\infty\}) = 0$. By identifying $L^2(\mathbb{R}^n; m)$ with $L^2(\mathbb{R}^n_*; m^*)$, we can regard $(\mathcal{E}^*, \mathcal{F}^*)$ as a Dirichlet form on $L^2(\mathbb{R}^n_*; m^*)$.

Theorem 2.5 *When $n \geq 3$, $(\mathcal{E}^*, \mathcal{F}^*)$ is a strongly local regular Dirichlet form on $L^2(\mathbb{R}^n_*, m^*)$.*

Indeed, if we let $\mathcal{C} = \{u + c : u \in C_0^\infty(\mathbb{R}^n),\ c \text{ is constant}\}$, then $\mathcal{C} \subset C(\mathbb{R}^n_*)$ and $\mathcal{C}$ is readily seen to be a core of the Dirichlet form $(\mathcal{E}^*, \mathcal{F}^*)$. The strong locality of $(\mathcal{E}^*, \mathcal{F}^*)$ can be proved in the same way as in the proof of Theorem 3.2 of [11], where a Dirichlet form quite similar to (2.13) was studied in a rather general context.

The m^*-symmetric diffusion process X^* on $\mathbb{R}^n_*$ associated with the Dirichlet form of Theorem 2.5 is an extension of the time changed Brownian motion X^0 on $\mathbb{R}^n$ associated with $(\mathcal{E}^{(0)}, \mathcal{F}^{(0)})$. X^0 approaches to ∞ at a finite life time ζ^0 with mean $\mathbf{E}_x[\zeta^0]$ being equal to the Newtonian potential $\mathcal{N} * m(x)$ of the function m. X is obtained by prolonging the path of X^0 after ζ^0 with a specific reflection at ∞ by piecing together the excursions of X^0 around ∞. Such a probabilistic construction has been studied in [11] and [4]. The relationship between $(\mathcal{E}^*, \mathcal{F}^*)$ and $(\mathcal{E}^{(0)}, \mathcal{F}^{(0)})$ will be studied in terms of the reflected Dirichlet space in [5].

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References

[1] A. Beurling and J. Deny, Dirichlet spaces, Proc. Nat. Acad. Sci. U.S.A. 45(1959),208-215

[2] N. Bouleau and F. Hirsch, *Dirichlet forms and analysis on Wiener space,* Walter de Gruyter, 1991

[3] Z.-Q. Chen, On reflected Dirichlet spaces, *Probab. Theory Relat. Fields* **94** (1992), 135-162.

[4] Z.-Q. Chen and M. Fukushima, One-point extensions of symmetric Markov processes by darning, *Probab. Theory Relat. Fields* **141**(2008), 61-112

[5] Z.-Q. Chen and M. Fukushima, On a unique extension of time changed reflecting Brownian motions, Preprint

[6] Z.-Q. Chen and M. Fukushima, *Symmetric Markov processes, time changes and boundary theory*, Princeton University Press, in preparation

[7] J. Deny, Méthods Hilberitiennes en théorie du potentiel, Potential Theory, Centro Internazionale Matematico Estivo, Edizioni Cremonese, Roma, pp.121-201, 1970

[8] J. Deny and J.L. Lions, Les espaces du type de Beppo Levi, *Annales Inst. Fourier* **5**(1953-54), 305-370

[9] M. Fukushima, Y. Oshima and M. Takeda, *Dirichlet forms and symmetric Markov processes*, Walter de Gruyter, 1994

[10] M. Fukushima and M. Takeda, *Markov processes* (in Japanese), Baifukan, 2008

[11] M. Fukushima and H. Tanaka, Poisson point processes attached to symmetric diffusions, *Ann. Inst. Henri Poincaré Probab. Statist.* **41**(2005), 419-459

[12] Z.M. Ma and M. Röckner, *Introduction to the theory of* (*non-symmetric*) *Dirichlet forms,* Springer, 1992.

[13] B. Schmuland, Positivity preserving forms have the Fatou property, *Potential Analysis,* **10**(1999), 373-378

[14] L. Schwartz, *Théorie des distributions* I, II, Hermann, Paris, 1950, 1951

[15] M. L. Silverstein, *Symmetric Markov processes.* Lecture Notes in Math. **426**. Springer-Verlag, 1974.