On Villat's kernels and BMD Schwarz kernels in Komatu-Loewner equations

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Dedicated to Professor Terry Lyons on the occasion of his 60th birthday

Abstract

The classical Loewner differential equation for simply connected domains is attracting new attention since Oded Schramm launched in 2000 the stochastic Loewner evolution (SLE) based on it. The Loewner equation itself has been extended to various canonical domains of multiple connectivity after the works by Y. Komatu in 1943 and 1950, but the Komatu-Loewner (K-L) equations have been derived rigorously only in the left derivative sense. In a recent work, Z.-Q. Chen, M. Fukushima and S.Rhode prove that the K-L equation for the standard slit domain is a genuine ODE by using a probabilistic method together with a SDE method, and that the right hand side of the equation admits an expression in terms of the complex Poisson kernel of the Brownian motion with darning (BMD).

In the present paper, K-L equations for the annulus and circularly slit annili are investigated. For the annulus, we establish a K-L equation as a genuine ODE possessing a normalized Villat's kernel on its right hand side by using a variant of the Carathéodory convergence theorem for annuli indicated by Komatu. This method is also used to obtain the same K-L equation in the right derivative sense on annulus for a more general family of growing hulls that satisfies a specific right continuity condition usually adopted in the SLE theory. Villat's kernel is then identified with a BMD Schwarz kernel for the annulus. Finally we derive K-L equations for circularly slit annuli in terms of their normalized BMD Schwarz kernels, but only in the left derivative sense when at least one circular slit is present.

1 Introduction

The celebrated Loewner differential equation for the planar unit disk has been extended to various canonical domains of multiple connectivity, first by Y. Komatu [K1] to the annulus, then by Y. Komatu [K3] to the circularly slit annulus, much later by R.O. Bauer and R.M. Friedrich [BF1] to the circularly slit disk, and further by R.O. Bauer and R.M. Friedrich [BF2] to the circularly slit annulus as well as to the standard slit domain, namely, a domain obtained from the upper half plane by removing a finite number of disjoint line segments parallel to the x-axis. However, the Komatu-Loewner differential equation has been derived only in the left derivative sense. Recall that, even in the case of the classical Loewner equation for a disk, its derivation in the right derivative sense is harder (cf.[A, §6-2]).

In a recent paper by Z.-Q. Chen, M. Fukushima and S. Rhode [CFR], the Komatu-Loewner equation (the K-L equation in abbreviation) for the standard slit domain is established to be a genuine differential equation with the kernel appearing on its right hand side being the complex Poisson kernel of the Brownian motion with darning (BMD in abbreviation) on the standard slit domain. In order to obtain the right differentiability in t of the family of conformal mappings $g_t(z)$ involved in the equation, a probabilistic representation of $\Im g_t(z)$ in terms of the BMD as well as a Lipschitz continuity of the BMD complex Poisson kernel under the perturbation of the standard slit domains are utilized.

The purpose of the present paper is to investigate the counterparts of K-L equations for the annulus and circularly slit annuli.

In §3, we consider an annulus whose outer boundary component is the unit circle and establish the K-L equation for it as the genuine differential equation (3.10) with a normalized Villat's kernel on its right hand. The right differentiability of $g_\cdot(z)$ will be shown by using a variant of Carathéodory kernel convergence theorem for annuli formulated in Appendix. In Komatu [K1], K-L equations for the annulus were obtained in terms of the Weierstrass zeta function and Jacobi's elliptic function instead of Villat's function. The stated variant of Carathéodory theorem for annuli was also presented in [K1] without proof to ensure the continuity of the modulus of the domain with respect to the parameter of the Jordan arc being removed. But the proof of the stated differentiability was not as rigorous as in the present paper. Villat's kernel was adopted 8 years later by G.M. Goluzin [G1] to derive a K-L equation in a different setting (for annuli located outside the unit disk).

In §4, we consider a general family of growing hulls in annulus that satisfies a specific right continuity condition usually adopted in the SLE theory (cf. [L]) and in SKLE as well (cf. [CF2]). We show that the same method as in §3 works to derive the associated K-L equation (3.10) in the right derivative sense. D. Zhan presented in [Z, Proposition 2.1] a variant of Corollary 4.2 without proof for his study of an annulus SLE that was defined based on the unnormalized Villat's kernel. One may formulate an annulus SLE based directly on the K-L equation (3.10) or its reparametrization (3.21) driven by the Brownian motion (with constant drifts) on the outer circle of the annulus.

The Brownian motion with darning (BMD) for an $(N+1)$-connected planar domain is defined as follows, A closed connected subset of $\mathbb{C}$ containing at least two points is called a *continuum*. Let E be a domain in $\mathbb{C}$ such that $\mathbb{C} \setminus E$ is an unbounded continuum and $\{A_1, \cdots, A_N\}$ be a collection of mutually disjoint compact continua contained in E. We write $E_0 = E \setminus \bigcup_{j=1}^N A_j$ and consider the topological space $E^* = E_0 \cup \{a_1^*, \cdots, a_N^*\}$ obtained from E by rendering each 'hole' A_j of E into a single point a_j^*. Extend the Lebesgue measure m on E_0 to E^* by setting $m(a_j^*) = 0,\ 1 \le j \le N$. There exists then a unique m-symmetric diffusion process Z^* on E^* admitting no killing at $a_1^*, \cdots, a_N^*$ whose part (killed) process Z^0 on E_0 is just the absorbing Brownian motion on E_0 (cf. [CF1, §7.7]). We call Z^* the *BMD for* E_0. Informally we may say that Z^* is the diffusion process on E^* obtained from the absorbing Brownian motion on E by rendering each hole A_j into a single point a_j^* (darning).

A simple way to conceive the BMD Z^* is to consider the Dirichlet form $(\mathcal{E}^*, \mathcal{F}^*)$ defined by

$$\mathcal{F}^* = \{u \in H_0^1(E) : \widetilde{u} \quad \text{is constant q.e. on each } A_j\}$$

$$\mathcal{E}^*(u, v) = \frac{1}{2} \int_E \nabla u(x) \cdot \nabla v(x) dx,$$

where $\widetilde{u}$ denotes a quasi-continuous version of u. Then $(\mathcal{E}^*, \mathcal{F}^*)$ turns out to be a regular Dirichlet form on $L^2(E^*, m)$ and the associated diffusion process on E^* is nothing but the BMD for E_0 (cf. [CFR]).

The notion of a BMD-harmonic function for E_0 is well defined to be a function on E^* satisfying a usual probabilistic averaging property with respect to the BMD Z^* (cf. [CFR]). Thus a BMD-harmonic function is harmonic on E_0 in the classical sense but it has an additional important property that its period around each hole A_j vanishes, and accordingly it admits a unique harmonic conjugate on E_0 up to the addition of a constant. If ∂E is smooth, every bounded BMD-harmonic function u on E^* with continuous boundary value on ∂E admits an expression $u(z) = \int_{\partial E} K^*(z, \zeta) u(\zeta) ds(\zeta),\ z \in E^*$, in terms of the uniquely determined kernel $K^*(z, \zeta),\ z \in E^*,\ \zeta \in \partial E$, called the *BMD-Poisson kernel*.

Since $K^*(z, \zeta)$ is BMD-harmonic in z for each $\zeta \in \partial E$, it admits an analytic function $\Psi(z, \zeta),\ z \in E_0$, with $\Im \Psi(z, \zeta) = K^*(z, \zeta)$ uniquely up to the addition of a real constant. $\Psi(z,\zeta)$ with the normalization $\lim_{z\to\infty}\Psi(z,\zeta)=0$ is called a *BMD complex Poisson kernel for* E_0 and it appears on the right hand side of the K-L equation for the standard slit domain (cf. [CFR]).

There exists also a function $\mathcal{S}(z,\zeta),\ z\in E_0,\ \zeta\in\partial E$, analytic in z with $\Re\mathcal{S}(z,\zeta)=K^*(z,\zeta)$ uniquely up to the addition of an imaginary constant. We call $\mathcal{S}(z,\zeta)$ a *BMD Schwarz kernel for* E_0 because its counterpart for the unit disk is the classical Schwarz kernel $\frac{1}{2\pi}\frac{\zeta+z}{\zeta-z}$. We may expect that the BMD Schwarz kernel would play important roles in the K-L equations for the annulus and circularly slit annuli.

Indeed we shall show in §4 that, in the case of the annulus $\mathbb{A}_q=\{z\in\mathbb{C}: q<|z|<1\},\ 1<q<1$, ($E=\mathbb{D},\ A=\{z\in\mathbb{C}:|z|\le q\}$ and $E_0=\mathbb{A}_q$ in the preceding notation), Villat's kernel for $\mathbb{A}_q$ coincides with a BMD Schwarz kernel for $\mathbb{A}_q$ up to a constant factor.

In §5, we shall consider more generally a circlularly slit annulus and derive a K-L differential equation possessing a normalized BMD Schwarz kernel on its right hand side by making computations similar to [CFR]. Such a representation of the equation in terms of a BMD Schwarz kernel was obtained neither in [K3] nor in [BF2]. But, when at least one circular slit is present, the equation will be shown to hold only in the sense of left derivative and the problem to make it a genuine ODE is left open.

In this connection, we mention a recent work by C. Boehm and W. Lauf [BL] where a K-L equation for a circularly slit disk is obtained as a genuine ODE by using an extended version of the Carathéodory convergence theorem.

2 Villat's kernel representing analytic functions on annulus

Define an annulus by $\mathbb{A}_q=\{z\in\mathbb{C}: q<|z|<1\}$ for $q\in(0,1)$. Sometimes $\mathbb{A}_q$ is written as $\mathbb{A}$ by omitting q. Define *Villat's function* by

$$\begin{aligned}\mathcal{K}_q(z) &= \lim_{N\to\infty}\sum_{n=-N}^{N}\frac{1+q^{2n}z}{1-q^{2n}z}\\ &= \frac{1+z}{1-z}+\lim_{N\to\infty}\sum_{n=1}^{N}\left(\frac{1+q^{2n}z}{1-q^{2n}z}+\frac{1+q^{-2n}z}{1-q^{-2n}z}\right),\qquad z\in\mathbb{A}_q.\end{aligned}\tag{2.1}$$

It holds that

$$\mathcal{K}_q(z) = \frac{1+z}{1-z} + 2\sum_{n=1}^{\infty} \frac{q^{2n}}{q^{2n}-z} + 2\sum_{n=1}^{\infty} \frac{q^{2n}z}{1-q^{2n}z}, \quad z \in \mathbb{A}_q, \tag{2.2}$$

both sums on the righthand side being convergent. This is because

$$\frac{q^{2n}}{q^{2n}-z} + \frac{q^{2n}z}{1-q^{2n}z} = \frac{1}{1-q^{2n}z} + \frac{z}{q^{2n}-z}, \quad n \geq 1.$$

For $z \in \mathbb{A}_q$ and $\zeta \in \partial\mathbb{A}_q$, define *Villat's kernel* by

$$\mathcal{K}_q(z,\zeta) = \mathcal{K}_q(z/\zeta) = \frac{\zeta+z}{\zeta-z} + 2\sum_{n=1}^{\infty}\left(\frac{q^{2n}\zeta}{q^{2n}\zeta - z} + \frac{q^{2n}z}{\zeta - q^{2n}z}\right). \tag{2.3}$$

The following representation by Villat's kernel of any analytic function on $\mathbb{A}$ that is continuous on $\overline{\mathbb{A}}$ has been known:

Theorem 2.1 *If f is analytic on $\mathbb{A}$ and $f \in C(\overline{\mathbb{A}}, \mathbb{C})$, then it holds that*

$$f(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{A}} \Re f(\zeta)\mathcal{K}_q(z,\zeta)\frac{d\zeta}{\zeta} - \frac{1}{2\pi i}\int_{|\zeta|=q} \Re f(\zeta)\frac{d\zeta}{\zeta} + ic, \quad z \in \mathbb{A}, \tag{2.4}$$

where

$$c = \frac{1}{2\pi i}\int_{|\zeta|=q} \Im f(\zeta)\frac{d\zeta}{\zeta}.$$

Furthermore

$$\frac{1}{2\pi i}\int_{\partial\mathbb{A}} \Re f(\zeta)\frac{d\zeta}{\zeta} = 0, \text{ namely, } \int_0^{2\pi} \Re f(e^{i\theta})d\theta = \int_0^{2\pi} \Re f(qe^{i\theta})d\theta. \tag{2.5}$$

This theorem is taken from PhD thesis by T. Vaitsiakhovich [Vai] that is quoted in a paper [CD-MG] of M.D. Contreras, S. Diaz-Madrigal and P. Gumenyuk. Denote by $\mathcal{L}(z,\zeta)$ the infinite sum in (2.3). For $z \in \mathbb{A}$, $\mathcal{L}(z,\zeta)$ and $\mathcal{L}(1/z,\zeta)$ are both analytic in $\zeta \in \mathbb{A}$ and continuous on $\overline{\mathbb{A}}$, and the expression (2.4) is an easy consequence of the Cauchy theorem and the Cauchy integral formula. Using expression (2.4), we get

$$\lim_{r\uparrow 1} \Re f(re^{i\theta}) = \Re f(e^{i\theta}), \ \lim_{r\downarrow q} \Re f(re^{i\theta}) = \Re f(qe^{i\theta}) + \frac{1}{2\pi i}\int_{\partial\mathbb{A}} \Re f(\zeta)\frac{d\zeta}{\zeta}, \tag{2.6}$$

which yields (2.5)

This theorem goes back to H. Villat [V2]. In page 12-20 of this book, the expression like (2.4) was obtained in terms of the kernel (2.3) by matching the coefficients in the Laurent expansion of f and in Fourier expansion of $\phi\big|_{\partial\mathbb{A}}$. In fact, (2.3) for $|\zeta|=1$ coincides with $1+2S$ for the kernel S in [V2]. (2.3) for $|\zeta|=q$ is also related to the kernel T in [V2]. The expressions of S and T were then rewritten in [V2] to derive the celebrated Villat's formula to represent an analytic function f on $\mathbb{A}$ in terms of the Weierstrass zeta functions. Apparently it was in G.M. Goluzin [G1] where the sum (2.2) was first rewritten as a sum (2.1) in the principal value sense.

The next proposition will be utilized in §3 and §5. We adopt the notations $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$, $\mathbb{D}_q=\{z\in\mathbb{C}:|z|<q\}$.

Proposition 2.2 (i) *Suppose that f is analytic on $\mathbb{A}$, $f\in C(\overline{\mathbb{A}},\mathbb{C})$ and*

$$\Re f \text{ is equal to a real constant } A \text{ on } \partial\mathbb{D}_q. \tag{2.7}$$

Then

$$\frac{1}{2\pi}\int_0^{2\pi}\Re f(e^{i\theta})d\theta=A, \tag{2.8}$$

and moreover f can be expressed as

$$f(z)=\frac{1}{2\pi}\int_0^{2\pi}\Re f(e^{i\theta})\,\mathcal{K}_q(z,e^{i\theta})d\theta+ic,\quad z\in\mathbb{A}, \tag{2.9}$$

for some real constant c.
(ii) *Conversely, for any $\phi\in C(\partial\mathbb{D},\mathbb{R})$ and $c\in\mathbb{R}$, define $f(z)$, $z\in\mathbb{A}$, by* (2.9) *and A by* (2.8) *with ϕ in place of $\Re f$, respectively. Then*

$$\lim_{r\downarrow q}\Re f(re^{i\eta})=A \text{ for any } \eta\in[0,2\pi),\ \lim_{r\uparrow 1}\Re f(re^{i\theta})=\phi(e^{i\theta}),\ \theta\in[0,2\pi). \tag{2.10}$$

Proof. (i) Condition (2.7) implies (2.8) by Theorem 2.1. Under the condition (2.7), the contribution of the integral on the inner circle $|\zeta|=q$ to the righthand side of (2.4) is $-\frac{A}{2\pi i}\int_{|\zeta|=q}\mathcal{K}_q(z,\zeta)\frac{d\zeta}{\zeta}-A$, which vanishes because $\frac{1}{2\pi i}\int_{|\zeta|=q}\mathcal{K}_q(z.\zeta)\frac{d\zeta}{\zeta}=-1$ on account of (2.3) and
$\mathrm{Res}_{\{\zeta=0\}}\frac{\zeta+z}{\zeta-z}\cdot\frac{1}{\zeta}=-1$, $\int_{|\zeta|=q}\frac{d\zeta}{\zeta-q^{-2n}z}=0$, $\mathrm{Res}_{\{\zeta=0\}}\frac{q^{2n}z}{\zeta-q^{2n}z}\cdot\frac{1}{\zeta}=-1$,
$\mathrm{Res}_{\{\zeta=q^{2n}z\}}\frac{q^{2n}z}{\zeta-q^{2n}z}\cdot\frac{1}{\zeta}=1$.
(ii) By (2.3), we readily have $\lim_{r\downarrow q}\Re\mathcal{K}_q(re^{i\eta},e^{i\theta})=1$ boundedly, yielding the first identity of (2.10). Then f admits the expression (2.4) by the observation made in (i) and so the second identity of (2.10) is nothing but the first one in (2.6). □

The following extension of Proposition 2.2 (i) will be utilized in §4.

Proposition 2.3 *Suppose that f is analytic and bounded on $\mathbb{A}$, and*

$$\Re f \text{ admits a constant limit } A \text{ at each point of } \partial\mathbb{D}_q. \tag{2.11}$$

Then the limit

$$\phi(e^{i\theta}) = \lim_{r\uparrow 1} \Re f(re^{i\theta}) \tag{2.12}$$

exists for a.e. $\theta \in [0, 2\pi)$ *and*

$$\frac{1}{2\pi}\int_0^{2\pi} \phi(e^{i\theta})d\theta = A. \tag{2.13}$$

Furthermore f can be expressed as

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} \phi(e^{i\theta})\ \mathcal{K}_q(z, e^{i\theta})d\theta + ic, \quad z \in \mathbb{A}, \tag{2.14}$$

for some real constant c.

Proof. Since $\Re f$ is a bounded harmonic function on $\mathbb{A} = \mathbb{A}_q$, the Fatou theorem (cf.[GM]) yields its boundary limit (2.12) on $\partial\mathbb{D}$. On account of the assumption (2.11), f can be extended to be an analytic function on $\{z : q^2 < |z| < 1\}$ denoted by f again across $\partial\mathbb{D}_q$ by the mirror reflection. For any $Q \in (q, 1)$, the function $f_Q(z) = f(Qz)$ is analytic on $\mathbb{A}_q$ continuous on $\overline{\mathbb{A}}_q$ so that (2.4) and (2.5) hold for f_Q. By letting $Q \uparrow 1$, we get (2.13) and also (2.4) with $\Re f\big|_{\partial\mathbb{D}} = \phi$ and $\Re f\big|_{\partial\mathbb{D}_q} = A$, which is reduced to (2.14) as in the proof of Proposition 2.2. $\square$

3 Komatu-Loewner equation on annulus in terms of Villat's kernel

Fix an annulus $\mathbb{A}_Q$ for $0 < Q < 1$, and a Jordan arc $\gamma = \{\gamma(t) : 0 \le t \le t_\gamma\}$ satisfying $\gamma(0) \in \partial\mathbb{D}$, $\gamma(0, t_\gamma] \subset \mathbb{A}_Q$.

According to [G2, Chap.V, §1], there exists then a strictly increasing function $\alpha : [0, t_\gamma] \mapsto [Q, Q_\gamma]$ $(\alpha(t_\gamma) = Q_\gamma < 1)$ with the following property: if $\alpha(t) = q$, then there is a unique conformal map g_q from $\mathbb{A}_Q \setminus \gamma[0, t]$ onto $\mathbb{A}_q$ with the normalization condition

$$g_q(Q) = q. \tag{3.1}$$

We shall prove the continuity of α eventually, but we do not assume it presently. Nevertheless we can reparametrize the curve γ as $\{\widetilde{\gamma}(q): \ q \in \mathrm{dom}(\widetilde{\gamma})\}$ by setting $\widetilde{\gamma}(q) = \gamma(\alpha^{-1}(q))$ where $\mathrm{dom}(\widetilde{\gamma}) = \alpha[0, t_\gamma] \subset [Q, Q_\gamma]$.

Take $0 \leq t^* < t \leq t_\gamma$ and put $q = \alpha(t),\ q^* = \alpha(t^*)$, then $Q \leq q^* < q \leq Q_\gamma$. Define

$$g_{q^*q} = g_{q^*} \circ g_q^{-1}, \qquad S_{q^*q} = g_{q^*}\gamma[t^*t]. \tag{3.2}$$

g_{q^*q} is a conformal map from $\mathbb{A}_q$ onto $B_{q^*q} = \mathbb{A}_{q^*} \setminus S_{q^*q}$ such that

$$g_{q^*q}(q) = q^*. \tag{3.3}$$

Let

$$\lambda(q) = g_q(\widetilde{\gamma}(q)) \tag{3.4}$$

be the image of the tip of the curve $\gamma[0, t]$ under g_q, which is a unique point on the outer circle of $\mathbb{A}_q$. The pre-image $\delta_{q^*q} = g_{q^*q}^{-1}(S_{q^*q})$ is a subarc $\{e^{i\theta} : \beta_1(t^*, t) < \theta < \beta_2(t^*, t)\}$ of the outer circle of $\mathbb{A}_q$ containing the point $\lambda(q)$.

We consider the function

$$\Phi(w) = \log\frac{g_{q^*q}(w)}{w},\ w \in \mathbb{A}_q, \quad \Phi(q) = \log\frac{q^*}{q}, \tag{3.5}$$

which is a well defined analytic function on $\mathbb{A}_q$, continuously extendable to $\overline{\mathbb{A}}_q$ with

$$\Re\Phi(w) = \log\frac{q^*}{q} \quad \text{for any} \quad w \in \partial\mathbb{D}_q. \tag{3.6}$$

Since $\Re\Phi(e^{i\theta}) = \log|g_{q^*q}(e^{i\theta})|$, we have by Proposition 2.2 (i),

$$\frac{1}{2\pi}\int_0^{2\pi} \log|g_{q^*q}(e^{i\theta})| d\theta = \log\frac{q^*}{q}. \tag{3.7}$$

and, for some real constant c,

$$\log\frac{g_{q^*q}(w)}{w} = \frac{1}{2\pi}\int_0^{2\pi} \log|g_{q^*q}(e^{i\theta})|\mathcal{K}_q(w, e^{i\theta}) d\theta + ic. \tag{3.8}$$

We now substitute $w = g_q(z),\ z \in \mathbb{A}_Q \setminus \gamma[0, t]$ in (3.8) to get

$$\log\frac{g_{q^*}(z)}{g_q(z)} = \frac{1}{2\pi}\int_0^{2\pi} \log|g_{q^*q}(e^{i\theta})|\mathcal{K}_q(g_q(z), e^{i\theta}) d\theta + ic.$$

We next put $z = Q$ and obtain from the normalization condition (3.5) that

$$\log\frac{q^*}{q} = \frac{1}{2\pi}\int_0^{2\pi} \log|g_{q^*q}(e^{i\theta})|\mathcal{K}_q(q, e^{i\theta}) d\theta + ic,$$

and consequently

$$c = -\frac{1}{2\pi}\int_0^{2\pi} \log|g_{q^*q}(e^{i\theta})| \Im \mathcal{K}_q(q, e^{i\theta}) d\theta.$$

Thus we arrive at

$$\log\frac{g_{q^*}(z)}{g_q(z)} = \frac{1}{2\pi}\int_0^{2\pi} \log|g_{q^*q}(e^{i\theta})| \left[\mathcal{K}_q(g_q(z), e^{i\theta}) - i\Im\mathcal{K}_q(q, e^{i\theta})\right] d\theta. \quad (3.9)$$

Theorem 3.1 *$q = \alpha(t)$ is a strictly increasing continuous function from $[0, t_\gamma]$ onto $[Q, Q_\gamma]$.*
$g_q(z)$, $z \in \mathbb{A}_Q \setminus \gamma[0,t]$, is continuously differentiable in $q \in [Q, Q_\gamma]$ and satisfies the differential equation

$$\frac{\partial \log g_q(z)}{\partial \log q} = \mathcal{K}_q(g_q(z), \lambda(q)) - i\Im\mathcal{K}_q(q, \lambda(q)), \ Q \le q \le Q_\gamma, \ g_Q(z) = z. \quad (3.10)$$

Proof. **(I)** We first prove that $\alpha(t)$, $t \in [0, t_\gamma)$, is left continuous in t, $g_q(z)$ is left-differentiable in q and the equation (3.11) holds in the left-derivative sense.

We maintain the notations in the above. Every point on the outer circle of $\mathbb{A}_q$ off the set δ_{q^*q} is sent by g_{q^*q} to a point on the outer circle of $\mathbb{A}_{q^*}$. Accordingly the domain $[0, 2\pi]$ of the integration in both equations (3.7) and (3.9) can be replaced by a smaller interval $[\beta_1(t^*, t), \beta_2(t^*, t)]$.

We fix t and let $t^* \uparrow t$. Denote by $\gamma^+(t^*)$, $\gamma^-(t^*)$ the points of 'both sides of the Jordan arc γ corresponding to $\gamma(t^*)$. Then as $t^* \uparrow t$, $\gamma^+(t^*) \to \gamma(t) = \widetilde{\gamma}(q)$, $\gamma^-(t^*) \to \gamma(t) = \widetilde{\gamma}(q)$ so that

$$\begin{cases} \beta_1(t^*, t) = g_q(\gamma^-(t^*)) \uparrow g_q(\widetilde{\gamma}(q)) = \lambda(q), \\ \beta_2(t^*, t) = g_q(\gamma^+(t^*)) \downarrow g_q(\widetilde{\gamma}(q)) = \lambda(q). \end{cases} \quad (3.11)$$

Since the integrand in the left hand side of (3.7) is bounded, we have $q^* \uparrow q$ the left continuity of α. We divide the both hand sides of the equation (3.9) by the both hand sides of (3.7) and let $t^* \uparrow t$ to obtain the left-differentiablility of $g_q(z)$ in q together with the equation (3.10) holding in the left-derivative sense.

(II) We use the following notations: for $r > 0$, $0 < s < t < \infty$,

$$\mathbb{D}(z, r) = \{w \in \mathbb{C} : |w - z| < r\}, \quad \mathbb{A}_{s,t} = \{w \in \mathbb{C} : s < |w| < t\}.$$

The mirror reflection with respect to the circle $\partial\mathbb{D}(\mathbf{0}, r)$ will be denoted by Π_r.

For $0 \leq t^* < t \leq t_\gamma$, $q^* = \alpha(t^*)$, $q = \alpha(t)$ as before, we consider the inverse conformal map

$$h_{q^*q} = g_{q^*q}^{-1} = g_q \circ g_{q^*}^{-1} : \ \mathbb{A}_{q^*} \setminus S_{q^*q} \mapsto \mathbb{A}_q.$$

h_{q^*q} satisfies $h_{q^*q}(q^*) = q$ and it sends the inner circle $\partial\mathbb{D}(\mathbf{0}, q^*)$ of $\mathbb{A}_{q^*}$ onto the inner circle $\partial\mathbb{D}(\mathbf{0}, q)$ of $\mathbb{A}_q$. It further sends $\partial\mathbb{D} \setminus \{\lambda(q^*)\}$ onto $\partial\mathbb{D} \setminus \delta_{q^*q}$. Hence we can extend h_{q^*q} by the mirror reflection Π_{q^*} to a univalent function (denoted by h_{q^*q} again) on

$$\mathbb{A}_{q^{*2}} \setminus (S_{q^*q} \cup \Pi_{q^*} S_{q^*q}) \ (\supset \mathbb{A}_{q^*} \setminus S_{q^*q}).$$

Furthermore, by means of the mirror reflection Π_1, we can extend h_{q^*q} to a univalent function (denoted by h_{q^*q} again) on

$$\mathbb{A}_{q^{*2},(q^*)^{-2}} \setminus (\overline{S}_{q^*q} \cup \Pi_{q^*} S_{q^*q}) \setminus \Pi_1(S_{q^*q} \cup \Pi_{q^*} S_{q^*q}). \tag{3.12}$$

By fixing t^*, we claim that

$$\lim_{t \downarrow t^*} q = q^*, \ \text{ namely, } \ \alpha \text{ is right continuous}, \tag{3.13}$$

$$\lim_{t \downarrow t^*} h_{q^*q}(z) = z \quad \text{locally uniformly on } \mathbb{A}_{q^{*2},(q^*)^{-2}} \setminus \{\lambda(q^*)\}. \tag{3.14}$$

As $t \downarrow t^*$, the domain of definition of the univalent function h_{q^*q} increases to $\mathbb{A}_{q^{*2},(q^*)^{-2}} \setminus \{\lambda(q^*)\}$. Obviously $\{h_{q^*q} : t \in (t^*, t_\gamma]\}$ is a uniformly bounded family of univalent functions. Take any sequence $\{t_n\}$ decreasing to t^* and write $h_n = h_{q^*q_n}$, $q_n = \alpha(t_n)$. By taking a subsequence if necessary, h_n converges to a function h locally uniformly on $\mathbb{A}_{q^{*2},(q^*)^{-2}} \setminus \{\lambda(q^*)\}$.

To prove the claims (3.13) and (3.14), Let us consider the restriction of h_n to E_n for $E_n = \mathbb{A}_{q^*} \setminus S_{q^*q_n}$, which is denoted by h_n again. Then $\{h_n\}$ satisfies all the conditions (i) $\sim$ (iv) of Corollary 7.2, yielding (3.13) and also (3.14) holding on $\mathbb{A}_{q^*}$. Obviously (3.14) then holds on $\mathbb{A}_{q^{*2},(q^*)^{-2}} \setminus \{\lambda(q^*)\}$ as well.

We note that, since $h_{q^*q}(g_{q^*}(z)) = g_q(z)$, (3.14) implies

$$\lim_{t \downarrow t^*} g_q(z) = g_{q^*}(z), \qquad z \in \mathbb{A}_Q \setminus \gamma[0, t^* + \delta], \ \delta > 0. \tag{3.15}$$

(III) The continuity of α has been established by **(I)** and (3.13). Keeping the notations in **(I)**, we shall prove that

$$\lim_{t \downarrow t^*} \beta_1(t^*, t) = \lambda(q^*), \ \lim_{t \downarrow t^*} \beta_2(t^*, t) = \lambda(q^*), \ \lim_{t \downarrow t^*} \lambda(q) = \lambda(q^*). \tag{3.16}$$

Once (3.16) is established, then we can combine it with (3.15) and the continuity of the Villat's kernel $\mathcal{K}_q$ in q to prove the following readily from (3.7) and (3.9) with the domain of the integration being $[\beta_1(t^*,t), \beta_2(t^*,t)]$ in place of $[0, 2\pi]$:
$g_q(z)$ is right differentiable in $q \in [Q, Q_\gamma)$, the equation (3.10) holds in the right-derivative sense and the right hand side of (3.10) is right continuous.

Just as in [Dur], (3.16) can be obtained from (3.14) in the following way. For any $\epsilon > 0$ with $\epsilon < 1 - q^*$, choose $\delta > 0$ such that

$$\overline{S}_{q^*q} \cup \Pi_1 S_{q^*q} \subset \mathbb{D}(\lambda(q^*), \epsilon) \text{ for any } t \in (t^*, t^* + \delta). \tag{3.17}$$

Let $C = \partial\mathbb{D}(\lambda(q^*), \epsilon)$ and $\chi = h_{q^*q}(C)$. Then $\delta_{q^*q} \subset$ ins χ. By virtue of (3.14), we have for a sufficiently small $\delta > 0$

$$|h_{q^*q}(z) - z| < \epsilon, \quad \text{for any } z \in C \text{ and } t \in (t^*, t^* + \delta), \tag{3.18}$$

which particularly means that diam $\chi < 3\epsilon$. By taking any $z \in C$, we then get for any $t \in (t^*, t^* + \delta)$

$$|\lambda(q^*) - \lambda(q)| \leq |\lambda(q^*) - z| + |z - h_{q^*q}(z)| + |h_{q^*q} - \lambda(q)| < 5\epsilon,$$

$$|\lambda(q^*) - \beta_i(t^*, t)| \leq |\lambda(q^*) - z| + |z - h_{q^*q}(z)| + |h_{q^*q} - \beta_i(t^*, t)| < 5\epsilon,$$

for $i = 1, 2$.

(VI) We finally show that $\lambda(q)$ is left continuous:

$$\lim_{q^* \uparrow q} \lambda(q^*) = \lambda(q), \tag{3.19}$$

which implies the left continuity of the right hand side of the equation (3.10) completing the proof of Theorem 3.1.

It follows from (3.9) that, for $z \in \mathbb{A}_q$,

$$\log \frac{g_{q^*q}(z)}{z} = \frac{1}{2\pi} \int_{\beta_1(t^*,t)}^{\beta_2(t^*,t)} \log |g_{q^*q}(e^{i\theta})| \left[\mathcal{K}_q(z, e^{i\theta}) - i\Im\mathcal{K}_q(q, e^{i\theta})\right] d\theta.$$

For any $\epsilon > 0$, we can choose $\delta > 0$ such that $\{e^{i\theta} : \beta_1(t^*,t) < \theta < \beta_2(t^*,t)\} \subset \mathbb{D}(\lambda(q), \epsilon)$ for $t^* \in (t - \delta, t)$ by (3.12). For such t^*, we can therefore see from the expression (2.3) of the Villat's kernel $\mathcal{K}_q(z, \zeta)$ that the integrand in the right hand side of the above identity is bounded uniformly in $z \in \overline{\mathbb{A}}_q \setminus \mathbb{D}(\lambda(q), \epsilon)$ and in $q^* = \alpha(t^*)$. Thus we deduce from (3.11)

$$\lim_{q^* \uparrow q} g_{q^*q}(z) = z, \quad \text{locally uniformly in } z \in \overline{\mathbb{A}}_q \setminus \{\lambda(q)\}. \tag{3.20}$$

By the mirror reflection Π_1, we further extend g_{q^*q} to $\mathbb{A}_{q,q^{-1}} \setminus \delta_{t^*t}$ across $\partial\mathbb{D}(\mathbf{0},1)$. Then (3.20) is still valid locally uniformly in $z \in \mathbb{A}_{q,q^{-1}} \setminus \{\lambda(q)\}$ and we can repeat the same argument as in **(III)** for g_{q^*q} in place of h_{q^*q} to obtain (3.19). □

Remark 3.2 For the function $g_q \circ g_{Q_\gamma}^{-1}$ in place of g_q in the above, Komatu [K1, K2] derived the equation (3.10) in terms of the Weierstrass zeta function as well as Jacobi's elliptic function in place of the present Villat's function. A variant of the Carathéodory kernel convergence theorem for annuli as Theorem 7.1 of the present paper was also stated there without proof, that implicitly implied the continuity of the correspondence $\alpha : t \mapsto q$ (as is shown in step **(II)** in the above proof). But the proof of the right differentiability of $g_q \circ g_{Q_\gamma}^{-1}$ in q was not given as rigorously as in steps **(II), (III)** of the present one. Goluzin[G1] obtained a counterpart of Theorem 3.1 in terms of Villat's kernel under a different setting for annuli located outside the unit disk $\mathbb{D}$. □

Remark 3.3 Since α is shown to be continuous, the Jordan arc γ can be reparametrized in terms of q as $\{\gamma(q) : Q \le q \le Q_\gamma\}$ by redefining $\gamma(\alpha^{-1}(q))$ as $\gamma(q)$ so that g_q is a conformal map from $\mathbb{A}_Q \setminus \gamma[0,q]$ onto $\mathbb{A}_q$ with the normalization (3.1). $g_q(z)$ satisfies the ODE (3.10) for $z \in \mathbb{A}_Q \setminus \gamma[0,q]$.

It is sometimes convenient to reparametrize the curve γ further in terms of the modulus p of the annulus $\mathbb{A}_q$: $p = -\log q,\ q = e^{-p}$. Denote by $P,\ P_\gamma$ the modulus of $\mathbb{A}_Q,\ \mathbb{A}_{Q_\gamma}$, respectively. Villat's kernel is denoted in terms of p as $S_p(z,\zeta) = \mathcal{K}_{e^{-p}}(z,\zeta)$. We further change the parameter q to s in a way that $q = e^{-P}e^s,\ 0 \le s \le s_\gamma = P_\gamma - P$. Since the module of $\mathbb{A}_q$ equals $P - s$, (3.10) reads for $z \in \mathbb{A}_q \setminus \gamma[0,s]$ and $s \in [0, s_\gamma]$

$$\frac{\partial \log g_s(z)}{\partial s} = S_{P-s}(g_s(z), \lambda(s)) - i\Im S_{P-s}(e^{s-P}, \lambda(s)), \quad g_0(z) = z, \quad (3.21)$$

for the conformal mapping g_s from $\mathbb{A}_Q \setminus \gamma[0,s]$ onto $\mathbb{A}_{Qe^s}$ with $g_s(Q) = Qe^s$. Here $\lambda(s) = g_s(\gamma(s))$. D. Zhan defined in [Z] an annulus SLE based on the equation (3.21) with the second normalization term of its right hand side being dropped however. One may formulate an annulus SLE based directly on (3.10) or (3.21) driven by the Brownian motion (with constant drifts) on the outer circle of $\mathbb{A}_Q$ by making analogous considerations to the case of standard slit domains in [CF2]. □

4 K-L equation on annulus for right continuous growing hulls

We consider an annulus $\mathbb{A}_Q$ for a fixed $Q \in (0,1)$. A closed subset F of $\mathbb{A}_Q$ is called a *hull* in $\mathbb{A}_Q$ if the set $\mathbb{A}_Q \backslash F$ is doubly connected possessing $\partial \mathbb{D}_Q$ as one of its boundary components. A strictly increasing family $\{F_t : 0 < t \leq T\}$ of hulls in $\mathbb{A}_Q$ is said to be a *family of growing hulls* in $\mathbb{A}_Q$. A typical example of a family of growing hulls in $\mathbb{A}_Q$ is $\{F_t = \gamma(0,t]; t \in (0,t_\gamma]\}$ for a Jordan arc γ considered in the preceding section.

Let $\{F_t; 0 < t \leq T\}$ be a family of growing hulls in $\mathbb{A}_Q$. We define $F_0 = \emptyset$ by convention. According to [G2, Chap.V, §1] again, there exists then a strictly increasing function $\alpha : [0,T] \mapsto [Q, Q_T](\beta(T) = Q_T < 1)$ with the following property: if $\alpha(t) = q$, then there is a unique conformal map g_q from $\mathbb{A}_Q \setminus F_t$ onto $\mathbb{A}_q$ with the normalization condition

$$g_q(Q) = q. \tag{4.1}$$

Needless to say, the function α is determined depending on $\{F_t\}$ and it is different in general from α in the preceding section.

Take $0 \leq t^* < t \leq t_\gamma$ and put $q = \alpha(t),\ q^* = \alpha(t^*)$, then $Q \leq q^* < q \leq Q_\gamma$. Define

$$g_{q^*q} = g_{q^*} \circ g_q^{-1}, \qquad S_{q^*q} = g_{q^*}(F_t \setminus F_{t^*}). \tag{4.2}$$

g_{q^*q} is a conformal map from $\mathbb{A}_q$ onto $\mathbb{A}_{q^*} \setminus S_{q^*q}$ such that

$$g_{q^*q}(q) = q^*. \tag{4.3}$$

We also consider the inverse map $h_{q^*q} = g_{q^*q}^{-1}(= g_q \circ g_{q^*}^{-1})$. h_{q^*q} is a conformal map from $\mathbb{A}_{q^*} \setminus S_{q^*q}$ onto A_q sending the inner circle of $\mathbb{A}_{q^*}$ onto the inner circle of $\mathbb{A}_q$ homeomorphically.

Denote by $\delta_{q^*q} (\subset \mathbb{C})$ the set of accumulation points of $h_{q^*q}(z)$ as z approaches to S_{q^*q}. δ_{q^*q} is then a closed subset of the outer circle of $\mathbb{A}_q$ so that we can write $\delta_{q^*q} = \{e^{i\theta} : \theta \in \ell_{q^*q}\}$ for a closed subset ℓ_{q^*q} of $[0, 2\pi)$. Observe that any point on the outer circle of $\mathbb{A}_{q^*}$ off the closure of S_{q^*q} is a simple boundary point of $\mathbb{A}_{q^*} \setminus S_{q^*q}$ in the sense of [Co]. In view of [Co, Theorem 15.3.6], the map h_{q^*q} extends to a continuous one-to-one map (denoted by h_{q^*q} again) from $\overline{\mathbb{A}}_{q^*} \setminus \overline{S}_{q^*q}$ into $\overline{\mathbb{A}}_q$.

We show that

$$h_{q^*q}(\overline{\mathbb{A}}_{q^*} \setminus K) = \overline{\mathbb{A}}_q \setminus \delta_{q^*q} \quad \text{for} \quad K = \overline{S}_{q^*q}. \tag{4.4}$$

Denote the outer circle of $\mathbb{A}_{q^*}$ (resp. $\mathbb{A}_q$) by C^* (resp. C). For any $z \in C^* \backslash K$, take a crosscut γ of $\mathbb{A}_{q^*}$ separating z and K. Then $h_{q^*q}(\gamma)$ separates $h_{q^*q}(z) \in$

C from δ_{q^*q} so that we have the inclusion $\subset$ in (4.4). Next, take any sequence $w_n \in \mathbb{A}_q$ converging to $w \in C \backslash \delta_{q^*q}$. Then $z_n = g_{q*q}(w_n) \in \mathbb{A}_{q^*} \backslash S_{q^*q}$ converges to a point $z \in C^* \cup K$ by taking a suitable subsequence if necessary. If $z \in K$, then $w_n = h_{q^*q}(z_n)$ accumulates to δ_{q^*q} that is absurd. Hence $z \in C^* \setminus K$. Since z is simple, w_n converges to a point $w' \in C$ that must equal w by the assumption, yielding (4.4).

Analogously to §3, we consider the function

$$\Phi(w) = \log \frac{g_{q^*q}(w)}{w}, \ w \in \mathbb{A}_q, \quad \Phi(q) = \log \frac{q^*}{q}, \tag{4.5}$$

which is a well defined bounded analytic function on $\mathbb{A}_q$ with

$$\Re\Phi(w) = \log \frac{q^*}{q} \quad \text{for any} \quad w \in \partial\mathbb{D}_q. \tag{4.6}$$

Hence, by virtue of Proposition 2.3, the limit

$$\phi(e^{i\theta}) = \lim_{r\uparrow 1} \log |g_{q^*q}(re^{i\theta})| \tag{4.7}$$

exists for a.e. $\theta \in [0, 2\pi)$, and the identities (2.13) with $A = \log \frac{q^*}{q}$ and (2.14) hold true. But, by the observation made above, $\lim_{r\uparrow 1} |g_{q^*q}(re^{i\theta})| = 1$ for any $\theta \in [0, 2\pi) \setminus \ell_{q^*q}$ so that the domain of integration $[0, 2\pi)$ in those identities can be replaced by ℓ_{q^*q}, yielding as in §3

$$\frac{1}{2\pi} \int_{\ell_{q^*q}} \phi(e^{i\theta}) d\theta = \log \frac{q^*}{q}. \tag{4.8}$$

$$\log \frac{g_{q^*}(z)}{g_q(z)} = \frac{1}{2\pi} \int_{\ell_{q^*q}} \phi(e^{i\theta}) \left[\mathcal{K}_q(g_q(z), e^{i\theta}) - i\Im\mathcal{K}_q(q, e^{i\theta}) \right] d\theta. \tag{4.9}$$

We now state a specific right continuity condition on a family of growing hulls. Let $\{F_t; t \in (0, T]\}$ be a family of growing hulls in the annulus $\mathbb{A}_Q$. We keep the related notations introduced above. Let $q^* = \alpha(t^*)$ for $t^* \in [0, T)$. The family is called *right continuous at* t^* *with limit* $\lambda(q^*)$ if there exists a point $\lambda(q^*)$ on the outer boundary of $\mathbb{A}_{q^*}$ such that

$$\bigcap_{t > t^*} \overline{S}_{q^*q} = \lambda(q^*), \tag{4.10}$$

for S_{q^*q} defined by (4.2). This condition is obviously satisfied when the hulls are generated by a Jordan arc γ, in which case $\lambda(q^*) = g_{q^*}(\gamma(t^*))$. But such a condition is also satisfied by more general families of growing hulls arising in SLE (cf. [L]) and in SKLE (cf. [CF2]) as well.

Theorem 4.1 *Let $\{F_t; t \in (0,T]\}$ be a family of growing hulls in the annulus $\mathbb{A}_Q$ that is right continuous at $t^* \in [0,T)$ with limit $\lambda(q^*)$. Then $q = \alpha(t)$ is right continuous at $t = t^*$, $g_q(z)$ is right differentiable at $q = q^*$ and*

$$\left.\frac{\partial^+ \log g_q(z)}{\partial \log q}\right|_{q=q^*} = \mathcal{K}_{q^*}(g_{q^*}(z), \lambda(q^*)) - i\Im\mathcal{K}_{q^*}(q^*, \lambda(q^*)), \tag{4.11}$$

for $z \in \mathbb{A}_Q \setminus F_{t^*+\delta}$, $\delta > 0$, where the left hand side denotes the right derivative.

Proof. It suffices to repeat the steps (II) and (III) in the proof of Theorem 3.1 almost word for word.

Indeed we have verified in the above that the conformal map h_{q^*q} extends to a continuous one-to-one map from $\overline{\mathbb{A}}_{q^*} \setminus \overline{S}_{q^*q}$ onto $\overline{\mathbb{A}}_q \setminus \delta_{q^*q}$. Accordingly, using the mirror reflections Π_{q^*} and Π_1, it can be further extended to a conformal map from the region specfied by (3.12) that increases to $\mathbb{A}_{q^{*2},(q^*)^{-2}} \setminus \{\lambda(q^*)\}$ as $t \downarrow t^*$ owing to the current condition (4.10). The functions h_n and regions E_n defined in the paragraph above (3.15) satisfy all the conditions (i) $\sim$ (iv) of Corollary 7.2 again owing to condition (4.10). Hence we get the right continuity (3.13) of α and a local uniform convergence (3.14) of h_{q^*q} together with the right continuity (3.15) of $g.(z)$.

For any $\epsilon > 0$ with $\epsilon < 1 - q^*$, we can choose $\delta > 0$ such that (3.17) is valid due to condition (4.10). Let $C = \partial\mathbb{D}(\lambda(q^*), \epsilon)$ and $\chi = h_{q^*q}(C)$. By virtue of (3.14), we have for a sufficiently small $\delta > 0$ the property (3.18) which particularly means that diam $\chi < 3\epsilon$. Since $\delta_{q^*q} \subset$ ins χ, we get for every $\zeta \in \delta_{q^*q}$

$$|\lambda(q^*) - \zeta| < 5\epsilon, \quad \text{for any } t \in (t^*, t^* + \delta). \tag{4.12}$$

By taking the continuity of Villat's kernel $\mathcal{K}_q(z,\zeta)$ and (3.15) into account, we can now deduce the desired conclusion of Theorem 4.1 from (4.8), (4.9) and (4.12). $\square$

Corollary 4.2 *Let $\{F_t; t \in [0,T]\}$, $F_0 = \emptyset$ be a family of growing hulls in the annulus A_Q satisfying the following conditions:*

(1) *α is continuous on $[0,T]$ so that $\alpha[0,T] = [Q, Q_T]$.*

(2) *There exists a continuous map λ from $[Q, Q_T]$ to $\partial\mathbb{D}$ and F_t is right continuous at each $t \in [0,T]$ with limit $\lambda(q)$ for $q = \alpha(t)$.*

(3) *$g_q(z)$ is continuous in $q \in [Q, Q_T]$ for each $z \in \mathbb{A}_Q \setminus F_T$.*

Then $g_q(z),\ z \in \mathbb{A}_Q \setminus F_T$, *is continuously differentiable in* $q \in [Q, Q_T]$ *and satisfies the differential equation*

$$\frac{\partial \log g_q(z)}{\partial \log q} = \mathcal{K}_q(g_q(z), \lambda(q)) - i\Im\mathcal{K}_q(q, \lambda(q)), \quad g_Q(z) = z. \tag{4.13}$$

In fact, under the stated conditions, (4.13) holds in the right derivative sense by virtue of Theorem 4.1. As the right hand side of (4.13) is continuous in q, it becomes a genuine ODE.

5 Villat's kernel is a BMD Schwarz kernel

The Schwarz kernel on a planar domain is by definition an analytic function with its real part being the Poisson kernel to represent harmonic functions by their values on the boundary. But we need to specify which class of harmonic functions and which part of the boundary are involved. We consider a BMD Schwarz kernel $\mathcal{S}(z, \zeta)$ defined in Introduction. $\Re\mathcal{S}(z, \zeta),\ z \in \mathbb{A}_q,\ \zeta \in \partial\mathbb{D}$, for the annulus $\mathbb{A}_q$ thus represents BMD harmonic functions for $\mathbb{A}_q$ by their boundary values on $\partial\mathbb{D}$. We now deduce from Proposition 2.2 (ii) that the Villat's kernel $\mathcal{K}_q(z, \zeta)$ for $z \in \mathbb{A}_q,\ \zeta \in \partial\mathbb{D}$, is equal to a BMD Schwarz kernel $\mathcal{S}(z, \zeta)$ for $\mathbb{A}_q$ up to a constant factor.

The BMD on $\mathbb{A}_q$ is the diffusion process on $\mathbb{A}_q \cup \{a^*\}$ obtained from the absorbing Brownian motion on $\mathbb{D}$ by rendering the inner concentric disk $\mathbb{D}_q = \{z : |z| < q\}$ into a single point a^*. The BMD-Poisson kernel $K^*(z, e^{i\theta}),\ z \in \mathbb{A}_q,\ 0 \le \theta < 2\pi$, to represent BMD-harmonic functions by their values on $\partial\mathbb{D}$ admits the same expression as (5.2) of [CFR]:

$$K^*(z, e^{i\theta}) = -\frac{1}{2}\frac{d}{dr}G^0(z, re^{i\theta})\big|_{r=1} - \varphi(z)p^{-1}\frac{d}{dr}\varphi(re^{i\theta})\big|_{r=1},$$

where G^0 is the Green function (the 0-order resolvent density) of the ABM on $\mathbb{A}_q$, φ is the hitting probability of $\mathbb{D}_q$ for the ABM on $\mathbb{D}$, and p is the period of φ around $\mathbb{D}_q$. Due to the rotational symmetry, the second term of the right hand side is independent of θ, and $K^*(z, \zeta)$ is a harmonic function in $z \in \mathbb{A}_q$ taking a constant $1/(2\pi)$ on $\partial\mathbb{D}_q$ for each $\theta \in [0, 2\pi)$.

Consider any non-negative continuous function ϕ on $[0, 2\pi)$ with $\int_0^{2\pi} \phi(\theta)d\theta = 1$ and let $u(z) = \int_0^{2\pi} K^*(z, e^{i\theta})\phi(\theta)d\theta,\ z \in \mathbb{A}_q$. Then u is harmonic on $\mathbb{A}_q$, taking a constant $1/(2\pi)$ on $\partial\mathbb{D}_q$ and taking the value $\phi(\theta)$ at each $e^{i\theta} \in \partial\mathbb{D}$. By virtue of Proposition 2.2 (ii), $f(z) = \frac{1}{2\pi}\int_0^{2\pi} \phi(\theta)\mathcal{K}_q(z, e^{i\theta})d\theta$ is an analytic function on $\mathbb{A}_q$ whose real part $\Re f(z)$ possesses the same boundary value on $\partial\mathbb{A}_q$ as u. Therefore $\Re f(z) = u(z),\ z \in \mathbb{A}_q$. By making $\phi \to \delta_{\theta_0}$ for a fixed $\theta_0 \in [0, 2\pi)$, we conclude that $\frac{1}{2\pi}\Re\mathcal{K}_q(z, e^{i\theta_0}) = K^*(z, e^{i\theta_0})$, that is to say, $\frac{1}{2\pi}\mathcal{K}_q(z, e^{i\theta_0}),\ 0 \le \theta_0 < 2\pi$, is nothing but a BMD-Schwarz kernel for the annulus $\mathbb{A}_q$.

6 K-L equation on circularly slit annulus in terms of BMD Schwarz kernel

A domain D of the form $D = \mathbb{A}_q \setminus \bigcup_{j=1}^{N-1} C_j$ is called a *circularly slit annulus* if $\mathbb{A}_q = \{z \in \mathbb{C} : q \le |z| < 1\}$ is an annulus for some $q \in (0, 1)$ and C_j are mutually disjoint concentric circular slits contained in $\mathbb{A}_q$. We denote by $\mathcal{D}$ the collection of all circularly slit annuli. The Komatu-Loewner equation for $\mathcal{D}$ has been formulated by Komatu [K3] and Bauer-Friedrich [BF2]. In this section, we make their descriptions of the equation more precise in terms of a normalized BMD Schwarz kernel introduced below.

We fix $D = \mathbb{A}_Q \setminus \bigcup_{j=1}^{N-1} C_j \in \mathcal{D}$ and consider a Jordan arc $\gamma : [0, t_\gamma] \mapsto \overline{D}$ with $\gamma(0) = \partial\mathbb{D},\ \gamma(0, t_\gamma] \subset D$. According to [K3], we can then find a strictly increasing function $\alpha :\ [0, t_\gamma] \mapsto [Q, Q_\gamma],\ (\alpha(t_\gamma) = Q_\gamma)$ such that, for $q = \alpha(t)$, there exists a unique conformal map

$$g_q :\ D \setminus \gamma[0, t] \ \mapsto\ D_q = \mathbb{A}_q \setminus \bigcup_{j=1}^{N-1} C_j(q) \in \mathcal{D}, \quad \text{with} \quad g_q(Q) = q.$$

α may not be continuous as in the annulus case of §3. Nevertheless we can reparametrize the curve γ as $\{\widetilde{\gamma}(q) : q \in \mathrm{dom}(\widetilde{\gamma})\}$ by setting $\widetilde{\gamma}(q) = \gamma(\alpha^{-1}(q))$, where $\mathrm{dom}(\widetilde{\gamma}) = \alpha[0, t_\gamma] \subset [Q, Q_\gamma]$.

For $D = \mathbb{A}_q \setminus \bigcup_{j=1}^{N-1} C_j \in \mathcal{D}$, let $K^*_D(z, \zeta),\ z \in D,\ \zeta \in \partial\mathbb{D}$, be the BMD Poisson kernel for D. A *BMD Schwarz kernel* $\mathcal{S}_D(z, \zeta)$ for D is by definition a function analytic in $z \in D$ satisfying $\Re\mathcal{S}_D(z, \zeta) = K^*_D(z, \zeta)$. For each $\zeta \in \partial\mathbb{D}$, $\mathcal{S}_D(\cdot, \zeta)$ exists uniquely up to an imaginary additive constant owing to the zero period property of a BMD harmonic function (cf. [CFR]). Let us denote by $\widehat{\mathcal{S}}_D(z, \zeta)$ the BMD Schwarz kernel subjected to a normalization

$$\Im\widehat{\mathcal{S}}_D(q, \zeta) = 0, \quad \text{for any} \quad \zeta \in \partial\mathbb{D}. \tag{6.1}$$

Any BMD Schwarz kernel $\mathcal{S}_D(z, \zeta)$ gives rise to a normalized one by

$$\widehat{\mathcal{S}}_D(z, \zeta) = \mathcal{S}_D(z, \zeta) - i\Im\mathcal{S}_D(q, \zeta),\ z \in D,\ \zeta \in \partial\mathbb{D}. \tag{6.2}$$

If D is just an annulus $\mathbb{A}_q$ with no circular slit, then we see by virtue of the result in the preceding section that its normalized BMD Schwarz kernel equals the *normalized Villat's kernel* multiplied by $\frac{1}{2\pi}$:

$$\widehat{\mathcal{S}}_{\mathbb{D}}(z,\zeta) = \frac{1}{2\pi}[\mathcal{K}_q(z,\zeta) - i\Im\mathcal{K}_q(q,\zeta)]. \tag{6.3}$$

Take $0 \leq t^* < t \leq t_\gamma$ and put $q = \alpha(q),\ q^* = \alpha(t^*)$. Then $Q \leq q^* < q \leq Q_\gamma$. Define $g_{q^*q} = g_{q^*} \circ g_q^{-1}$ which maps D_q conformally onto $D_{q^*} \setminus S_{q^*q}$ and satisfies

$$g_{q^*q}(q) = q^*, \tag{6.4}$$

where $S_{q^*q} = g_{q^*}\gamma[t^*,t]$. Let

$$\lambda(q) = g_q(\widetilde{\gamma}(q)) \tag{6.5}$$

that is located on an outer circle of D_q. The pre-image $g^{-1}_{q^*q}(S_{q^*q})$ of S_{q^*q} is a subarc $\{e^{i\theta} :\ \beta_1(t^*,t) < \theta < \beta_2(t^*,t)\}$ of the outer circle of D_q containing the point $\lambda(q)$.

Now $\log\left|\dfrac{g_{q^*q}(z)}{z}\right|,\ z \in D_q$, is harmonic on D_q as the imaginary part of the well defined analytic function $\log\dfrac{g_{q^*q}(z)}{z}$ on D_q and takes a constant value on each circular slit $C_j(q)$. Therefore we can verify just as in [CFR, §6.3] that

$$\log\left|\frac{g_{q^*q}(z)}{z}\right| = \int_{\partial\mathbb{D}}\left|\log\frac{g_{q^*q}(\zeta)}{\zeta}\right| K_q^*(z,\zeta)s(d\zeta),\ z \in D_q, \tag{6.6}$$

where $K_q^*(z,\zeta)$ is the BMD Poisson kernel for the circularly slit annulus D_q. Hence we get

$$\log\frac{g_{q^*q}(z)}{z} = \int_{\beta_0(t^*,t)}^{\beta_1(t^*,t)} \log|g_{q^*q}(e^{i\varphi})|\widehat{\mathcal{S}}_q(z,e^{i\varphi})d\varphi + ic, \tag{6.7}$$

for the normalized BMD Schwarz kernel $\widehat{\mathcal{S}}_q$ and for some real constant c.

By substituting $z = q$ in (6.7), we obtain from (6.4)

$$\log\frac{q^*}{q} = \int_{\beta_0(t^*,t)}^{\beta_1(t^*,t)} \log|g_{q^*q}(e^{i\varphi})|\widehat{\mathcal{S}}_q(q,e^{i\varphi})d\varphi + ic,$$

which implies that $c = 0$ on account of (6.1).

On the other hand, the Cauchy integral theorem applied to the analytic function $\log\dfrac{g_{q^*q}(z)}{z}$ on the circularly slit annulus D_q yields just as in [BF2, §3.2]

$$\log\frac{q^*}{q} = \int_{\beta_0(t^*,t)}^{\beta_1(t^*,t)} \log|g_{q^*q}(e^{i\varphi})|d\varphi. \tag{6.8}$$

The integrand on the right hand side of (6.8) being uniformly bounded, we get the left continuity of $q = \alpha(t)$ by letting $t^* \uparrow t$ in (6.8).

We next substitute $z = g_q(w)$ into the identity (6.7) with $c = 0$. We then divide the resulting the both hand sides of the resulting identity by those of (6.8) and let $t^* \uparrow t$ in getting the following theorem.

Theorem 6.1 *$q = \alpha(t)$ is left continuous in $t \in (0, t_\gamma]$.*
$g_q(z)$ is left-differentiable in q and it holds for $z \in D \setminus \gamma[0,t]$ that

$$\frac{\partial^- \log g_q(z)}{\partial \log q} = 2\pi \widehat{\mathcal{S}}_q(g_q(z), \lambda(q)), \;\; q \in \alpha(0, t_\gamma] \subset (Q, Q_\gamma], \;\; g_Q(z) = z, \quad (6.9)$$

where the left hand side denotes the left derivative.

Remark 6.2 In the special case that $N = 1$, D is just an annulus $\mathbb{A}_Q$ and the equation (6.9) is reduced to

$$\frac{\partial^- \log g_q(z)}{\partial \log q} = \mathcal{K}_q(g_q(z), \lambda(q)) - i\Im\mathcal{K}_q(q, \lambda(q))]. \; q \in \alpha(0, t_\gamma], \;\; g_Q(z) = z, \quad (6.10)$$

by virtue of (6.3), which actually holds in the true derivative sense as has been proved in Theorem 3.1 by making use of the kernel convergence theorem for annuli formulated in Appendix.

In the case where $N > 1$ so that the degree of the multiplicity of the circularly slit annulus D is equal or greater than 3, the problem of proving the equation (6.9) to be a genuine ODE remains open, although Komatu [K3] tried to do so by an induction in $N \geq 1$ not quite successfully.

7 Appendix: Carathéodory-Komatu convergence theorem for annuli

As in §3, we use the notations $\mathbb{D}(z, r) = \{w \in \mathbb{C} : |w - z| < r\}$, $\mathbb{A}(s, t) = \{w \in \mathbb{C} : s < |w| < t\}$ for $r > 0$, $0 < s < t$.

Consider the following two conditions on a doubly connected domain D in $\mathbb{C}$:
(i) $D \subset \mathbb{A}(1, a)$ for some $a > 1$,
(ii) D admits $\partial\mathbb{D}(0, 1)$ as one of the boundary components of D.

We let $\mathcal{D} = \{D : D$ is a doubly connected domain satisfying (i) and (ii)$\}$.

For a sequence $\{D_n\}$ in $\mathcal{D}$, we define its kernel as follows. Suppose that $D_0 \subset \cap_{n=1}^\infty D_n$ for some $D_0 \in \mathcal{D}$. Then the *kernel* of $\{D_n\}$ is defined as the maximal doubly connected domain D in $\mathbb{D}(0,1)^c$ such that D satisfies (ii) and any compact subset of D is contained in D_n for sufficiently large n. Otherwise, the kernel is defined to be $\partial\mathbb{D}(0,1)$. A sequence $\{D_n\}$ in $\mathcal{D}$ is said to be *convergent* to D in the sense of *kernel convergence*, if D is the kernel of $\{D_n\}$ and the kernel of any subsequence of $\{D_n\}$ coincides with D. A sequence $\{D_n\}$ in $\mathcal{D}$ is said to be *uniformly bounded* if $D_n \subset \mathbb{A}(1,a),\ n \geq 1$, for some $a > 1$.

It is known that if there exists a conformal map from D onto D' with $D, D' \in \mathcal{D}$, then D and D' admit an identical modulus and the map extends homeomorphically from $\partial\mathbb{D}(\mathbf{0},1) \cup D$ onto $\partial\mathbb{D}(\mathbf{0},1) \cup D'$.

A version of the following theorem was presented in [K1], [K2] without proof only by mentioning its similarity to a proof of Carathéodory's kernel convergence theorem for a disk. But we give a proof for completeness.

Theorem 7.1 (*Carathéodory-Komatu Convergence Theorem*) *Let $\{D_n\}$ be a uniformly bounded sequence of doubly connected domains in $\mathcal{D}$ and let $\{R_n\}$ be a sequence with $R_n > 1,\ n \geq 1$, such that there is a conformal map F_n from $\mathbb{A}(1,R_n)$ onto D_n satisfying $F_n(1) = 1$ for every n. Then the kernel convergence of $\{D_n\}$ to a doubly connected domain D in $\mathcal{D}$ implies that the sequence $\{R_n\}$ converges to R yielding the modulus of D to be $\log R$ and that the sequence $\{F_n\}$ converges locally uniformly to a conformal map F from $\mathbb{A}(1,R)$ onto D.*

Proof. The assumption of the uniform boundedness of $\{D_n\}$ and the kernel convergence of $\{D_n\}$ to $D \in \mathcal{D}$ imply that $\partial D_n \subset \overline{\mathbb{A}(1,a)} \setminus \mathbb{A}(1,b),\ n \geq 1$, for some a, b with $1 < b \leq a$. Due to the monotonicity of the moduli (cf. [G2, [V,1,Theorem 3]), we then have $b \leq R_n \leq a$.

As $\{F_n\}$ is a normal family, there exist a positive number $R' > 1$ and a subsequence $\{n_k\}$ of $\{n\}$ such that $\lim_{k\to\infty} R_{n_k} = R'$ and $\{F_{n_k}\}$ converges locally uniformly to some analytic function F on $\mathbb{A}(1,R')$, which is non-constant because $\dfrac{1}{2\pi i}\int_{|z|=(R'+1)/2} d\log F(z)dz = \lim_{k\to\infty} \dfrac{1}{2\pi i}\int_{|z|=(R'+1)/2} d\log F_{n_k}(z)dz = \lim_{k\to\infty} \dfrac{1}{2\pi}\int_{|z|=(R'+1)/2} d\arg F_{n_k}(z) = 1$. By virtue of Hurwitz's theorem, we can deduce from the univalence of $\{F_{n_k}\}$ that F is an injective map from $\mathbb{A}(1,R')$ to its image $F(\mathbb{A}(1,R'))$.

It holds that $F(\mathbb{A}(1,R')) \subset D$. In fact, for any $\zeta \in \mathbb{A}(1,R')$, there exists $\delta > 0$ with $\mathbb{D}(\zeta,\delta) \subset \mathbb{A}(1,R')$. Then $\mathbb{D}(\zeta,\delta) \subset \mathbb{A}(1,R_n)$ from some n on. Since the univalence of the function F implies that the coefficient c_1 in the Taylor expansion of $F(z) - F(\zeta) = c_1(z-\zeta) + \cdots$ around ζ does not vanish, we can deduce $|F'_{n_k}(\zeta)| \geq c$ holding for some $c > 0$ and for sufficiently large k from the local uniform convergence of $\{F_{n_k}\}$ to F combined with Cauchy's integral expressions of $F'(\zeta)$ and $F'_{n_k}(\zeta)$. Hence there exists $\rho > 0$ such that $\mathbb{D}(F_{n_k}(\zeta), \rho) \subset F_{n_k}(\mathbb{D}(\zeta, \delta)) \subset D_{n_k}$ from some k on by Koebe 1/4 theorem. Since $\lim_{n_k \to \infty} F_{n_k}(\zeta) = F(\zeta)$, we have $\mathbb{D}(F(\zeta), \rho/2) \subset D_{n_k}$ from some k on, and consequently $F(A(1, R')) \subset D$ because D is also the kernel of D_{n_k}.

Denote by H_n the inverse of F_n. Since the family $\{H_n\}$ is uniformly bounded, we may assume that $\{H_{n_k}\}$ is a locally uniformly convergent sequence by taking a suitable subsequence of $\{n_k\}$ if necessary. Since D is also the kernel of $\{D_{n_k}\}$, we can see that, for any $w \in D$, $w \in D_{n_k}$ for sufficiently large k and $H(w) = \lim_{k\to\infty} H_{n_k}(w)$ is well defined with $1 \leq |H(w)| \leq R'$. Further H is non-constant because of $\frac{1}{2\pi i}\int_{|w|=r} d\log H(w) = \lim_{k\to\infty} \frac{1}{2\pi i}\int_{|w|=r} d\log H_{n_k}(w) = \lim_{k\to\infty} \frac{1}{2\pi}\int_{|w|=r} d\arg H_{n_k}(w) = 1$ for some $r > 1$ satisfying $\partial\mathbb{D}(0, r) \subset \cap_{n=1}^{\infty} D_n$.

Therefore, by applying the open mapping theorem to the analytic function H together with the pointwise convergence of $\{H_{n_k}\}$ to H as $k \to \infty$, we see that, for any fixed $w \in D$, there exists a positive number δ such that $H_{n_k}(w) \in \mathbb{D}(H(w), \delta) \subset \mathbb{A}(1, R')$ for sufficiently large k.

If F omits the value w, we have the following contradiction:

$$0 = \frac{1}{2\pi i}\int_{C_{H(w),\delta}} \frac{F'(z)}{F(z) - w} dz = \lim_{k\to\infty} \frac{1}{2\pi i}\int_{C_{H(w),\delta}} \frac{F'_{n_k}(z)}{F_{n_k}(z) - w} dz = 1,$$

where $C_{H(w),\delta} = \partial\mathbb{D}(H(w), \delta)$ with counterclockwise orientation. Accordingly, F takes the value w at some point in $\mathbb{D}(H(w), \delta)$. By combining this with $F(A(1, R')) \subset D$ and the univalence of F, we conclude that F is a conformal map from $\mathbb{A}(1, R')$ onto D.

Owing to the uniqueness of the modulus of the domain D, we have $R = \lim_{k\to\infty} R_{n_k}$ independently of the choice of $\{n_k\}$. Further, $F = \lim_{k\to\infty} F_{n_k}$ gives a conformal map from $\mathbb{A}(1, R)$ onto D. As $F(1) = 1$, F is uniquely determined independently of the choice of $\{n_k\}$ (cf. [G2, V,1, Theorem 2]), yielding the desired conclusion. □

Consider q^* with $0 < q^* < 1$ and a sequence $\{q_n\}$ satisfying $q^* < q_n < 1$ for each n.

Corollary 7.2 *Let $\{h_n\}$ be a sequence of univalent functions satisfying the following conditions :*

(i) *Each* h_n *is a surjective map from a domain* E_n *to* $\mathbb{A}(q_n, 1)$ *with* $E_n \subset \mathbb{A}(q^*, 1)$.

(ii) $E_n \subset E_{n+1}$ *for every* n *and* $\cup_{n=1}^{\infty} E_n = \mathbb{A}(q^*, 1)$.

(iii) *Each* E_n *has* $\partial\mathbb{D}(\mathbf{0}, q^*)$ *as one of its boundary components.*

(iv) $h_n(q^*) = q_n$ *for every* n.

Then $\lim_{n\to\infty} q_n = q^*$ *and* $\{h_n\}$ *converge locally uniformly to the identity map on* $\mathbb{A}(q^*, 1)$.

Proof. We denote the inverse function of h_n by g_n and define a conformal map F_n from $\mathbb{A}(1, \dfrac{1}{q_n})$ onto D_n satisfying $F_n(1) = 1$ by $F_n(z) = \dfrac{1}{q^*} g_n(q_n z)$ for each n, where $D_n = \{\dfrac{z}{q^*} \in \mathbb{C} : z \in E_n\} \cap \mathbb{A}(1, \dfrac{1}{q^*})$. Then the kernel convergence of the sequence $\{D_n\}$ in $\mathcal{D}$ to $\mathbb{A}(1, \dfrac{1}{q^*}) \in \mathcal{D}$ follows from (ii). Since the modulus of $\mathbb{A}(1, \dfrac{1}{q^*})$ equals q^*, we can apply Theorem 7.1 to deduce that $\lim_{n\to\infty} q_n = q^*$ and that $\{F_n\}$ converges to a conformal map F from $\mathbb{A}(1, \frac{1}{q^*})$ onto itself locally uniformly on $\mathbb{A}(1, \frac{1}{q^*})$. Since $F(1) = 1$, we get $F(z) = z,\ z \in \mathbb{A}(1, \frac{1}{q^*})$, that yields the desired conclusion. □

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