Liouville Property of Harmonic Functions of Finite Energy for Dirichlet Forms

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Abstract A quasi-regular Dirichlet form is said to have a Liouville property if any associated harmonic function of finite energy is constant. We first examine this property for the energy form $\mathscr{E}^\rho$ on $\mathbb{R}^n$ generated by a positive function ρ. We next make a general consideration on a regular, strongly local and transient Dirichlet form $\mathscr{E}$ and an associated time changed symmetric diffusion process $\check{X}$ with finite lifetime. We show that $\check{X}$ always admits its one-point reflection $\check{X}^*$ at infinity by constructing the corresponding regular Dirichlet form. We then prove that, if $\mathscr{E}$ satisfies the Liouville property, a symmetric conservative diffusion extension Y of $\check{X}$ is unique up to a quasi-homeomorphism, and in fact, a quasi-homeomorphic image of Y equals the one-point reflection $\check{X}^*$ of $\check{X}$ at infinity.

Keywords Liouville property · Energy form · Strongly local transient Dirichlet form · One-point reflection at infinity · Symmetric extension

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1 Introduction

We consider a locally compact separable metric space E and a positive Radon measure m on E with full support. Given a regular Dirichlet form $(\mathscr{E}, \mathscr{F})$ on $L^2(E; m)$ with an associated Hunt process $X = (X_t, \zeta, \mathbf{P}_x)$ on E, let $\mathscr{F}_e$ and $\mathscr{F}^{\text{ref}}$ be its *extended Dirichlet space* and its *reflected Dirichlet space*, respectively. Then $\mathscr{F} \subset \mathscr{F}_e \subset \mathscr{F}^{\text{ref}}$ and the inner product $\mathscr{E}$ is extended from $\mathscr{F}$ to both spaces [6]. The notions of the extended and reflected Dirichlet spaces were introduced by Silverstein in [25, 26], respectively, in the same year 1974, but the latter notion was reformulated by Z.-Q. Chen [4] later in 1992 and further extended to a quasi-regular Dirichlet form in [6].

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Define the linear subspace $\mathscr{H}^*$ of $\mathscr{F}^{\mathrm{ref}}$ by

$$\mathscr{H}^* = \{u \in \mathscr{F}^{\mathrm{ref}} : \mathscr{E}(u, v) = 0 \quad \text{for any} \quad v \in \mathscr{F}_e\}.$$

$\mathscr{H}^*$ is the collection of X-harmonic functions u on E of finite energy $\mathscr{E}(u, u)$. We will be concerned with a specific *Liouville property*

$$\dim(\mathscr{H}^*) = 1 \tag{1}$$

of the form $\mathscr{E}$ and its probabilistic significance.

We first give two general remarks on the Liouville property (1). A Borel function h on E is said to be *X-harmonic* if it is specified and finite up to quasi equivalence and if for every relatively compact open subset $G \subset E$, $\mathbf{E}_x[|h(X_{\tau_G})|] < \infty$ and $h(x) = \mathbf{E}_x[h(X_{\tau_G}]$ for q.e. $x \in E$, where τ_G denotes the first exit time from G. By the next proposition, we only need to consider the transient form $\mathscr{E}$ to study the Liouville property (1).

Proposition 1.1 (i) *If $\mathscr{E}$ is irreducible and recurrent, then $\mathscr{E}$ enjoys the property* (1).
(ii) *If $\mathscr{E}$ is transient and if any bounded X-harmonic function on E is constant, then $\mathscr{E}$ enjoys the property* (1).

Proof (i) Suppose $(\mathscr{E}, \mathscr{F})$ is recurrent. Then $\mathscr{F}^{\mathrm{ref}} = \mathscr{F}_e$ by [6, Theorem 6.3.2]. Further $u \in \mathscr{F}_e$, $\mathscr{E}(u, u) = 0$ implies that the level set $\{x \in E : u(x) = c\}$ is invariant for each constant c by [6, Lemma 6.7.3]. Hence (1) follows from the irreducibility of $\mathscr{E}$.

The assertion (i) also follows from the identity $\mathscr{F}^{\mathrm{ref}} = \mathscr{F}_e$ and a Poincaré type inequality for $(\mathscr{F}_e, \mathscr{E})$ established in [17, Theorem 4.8.2] in the recurrent case, which requires an additional Sobolev type inequality holding for $(\mathscr{E}, \mathscr{F})$ however.
(ii) In view of [6, Remark 6.2.2], it holds under the transience of $\mathscr{E}$ that

$$\mathscr{H}^* = \{h = \mathbf{E}.[\varphi] : \varphi \in \mathbf{N}\},$$

for the space $\mathbf{N}$ of terminal random variables φ specified by [6, (6.2.1)]. For $\varphi \in \mathbf{N}$, let $\varphi_n = ((-n) \vee \varphi \wedge n$. Then $h_n(x) = \mathbf{E}_x[\varphi_n]$ is a bounded X-harmonic function and converges as $n \to \infty$ to h q.e. on E, yielding the assertion (ii). □

For an Euclidean domain $D \subset \mathbb{R}^n$, the *Beppo Levi space* and the *Sobolev space of order* (1, 2) are defined, respectively, by

$$\mathrm{BL}(D) = \{u \in L^2_{\mathrm{loc}}(D) : |\nabla u| \in L^2(D)\}, \quad H^1(D) = \mathrm{BL}(D) \cap L^2(D). \tag{2}$$

$\mathbf{D}(u, v)$ will denote the Dirichlet integral $\int_D \nabla u(x) \cdot \nabla v(x) dx$ of $u, v \in \mathrm{BL}(D)$. The space $\mathrm{BL}(D)$ is just the space of Schwartz distributions whose first order derivatives are in $L^2(D)$. It was introduced and profoundly studied by Deny-Lions [12] following the preceding works by Beppo Levi [21], Nikodym [24] and Deny [10]. This space was one of the original sources of the notion of the *Dirichlet space* introduced by

Beurling-Deny [2] in 1959 which was basically free from the choice of the underlying symmetrizing measure. Later on, the space $\mathrm{BL}(D)$ was designated as $L_2^1(D)$ by Maz'ja [23] and studied in a more general context of the spaces $L_p^\ell(D)$ for $p > 0$ and integers ℓ. However the space $\mathrm{BL}(D)$ bears its own independent potential theoretic and probabilistic significances from the beginning. See [3, 10, 11, 13, 15] in this connection.

Now suppose a domain $D \subset \mathbb{R}^n$ is either of continuous boundary or an extendable domain relative to $H^1(D)$. The symmetric form $\mathscr{E}$ with $\mathscr{D}(\mathscr{E}) = \mathscr{F}$ defined by

$$\mathscr{E} = \frac{1}{2}\mathbf{D}, \qquad \mathscr{F} = H^1(D), \tag{3}$$

is then a regular strongly local irreducible Dirichlet form on $L^2(\overline{D})$ and the associated diffusion X on $\overline{D}$ is by definition the *reflecting Brownian motion* (*RBM* in abbreviation). The extended Dirichlet space of $\mathscr{E}$ is denoted by $H_e^1(D)$ and called the *extended Sobolev space of order* 1. $\mathrm{BL}(D)$ is nothing but the reflected Dirichlet space of this form $\mathscr{E}$ [6, p. 273]. The space $\mathscr{H}^* = \mathrm{BL}(D) \ominus H_e^1(D)$ consists of those functions on D with finite Dirichlet integral such that they are not only harmonic on D in the ordinary sense but also their quasi continuous versions are harmonic with respect to the RBM Z on $\overline{D}$.

It was shown in [5, Theorem 3.5] that $\mathscr{E}$ fulfills the Liouville property (1) when $D \subset \mathbb{R}^n$ is a uniform domain in the sense of [27]. On the other hand, it is demonstrated in [8] that $\dim(\mathscr{H}^*) = N$ when $n \geq 3$ and D is a Lipschitz domain with N number of *Liouville branches* in the specific sense formulated there.

In the simplest case that $D = \mathbb{R}^n$ the whole space, $\mathscr{H}^*$ is just the space of harmonic functions on $\mathbb{R}^n$ with finite Dirichlet integrals. Brelot [3] first observed that the property (1) is valid, namely, any harmonic function on $\mathbb{R}^n$ with finite Dirichlet integral is constant. See [17, Example 1.5.3] in this connection. A simple question arises:

(Q) Is the property (1) still valid for the whole space $\mathbb{R}^n$ and for more general Dirichlet forms than $\frac{1}{2}\mathbf{D}$?

In Sects. 2 and 3, we shall consider a measurable function $\rho(x)$ on $\mathbb{R}^n$ such that

$$0 < \lambda_\ell \leq \rho(x) \leq \Lambda_\ell < \infty, \quad \text{for every } x \in B_\ell := \{|x| < \ell\}, \quad \ell > 0. \tag{4}$$

for constants λ_ℓ, Λ_ℓ depending on $\ell > 0$, and the associated spaces $\mathscr{F}^\rho$, $\mathscr{G}^\rho$ and form $\mathbf{D}^\rho$ defined respectively by

$$\mathscr{F}^\rho = \{u \in L^2(\mathbb{R}^n; \rho dx) \ : \ |\nabla u| \in L^2(\mathbb{R}^n; \rho dx)\}, \tag{5}$$

$$\mathscr{G}^\rho = \{u \in L^2_{\mathrm{loc}}(\mathbb{R}^n) \ : \ |\nabla u| \in L^2(\mathbb{R}^n; \rho dx)\}, \tag{6}$$

$$\mathbf{D}^\rho(u, v) = \int_{\mathbb{R}^n} \nabla u(x) \cdot \nabla v(x)\, \rho(x)\, dx. \tag{7}$$

In the next section, we show that the *energy form* $\mathcal{E}^\rho = (\mathbf{D}^\rho, \mathcal{F}^\rho)$ is a regular strongly local irreducible Dirichlet form on $L^2(\mathbb{R}^n; \rho dx)$ and the *weighted Beppo Levi space* $(\mathcal{G}^\rho, \mathbf{D}^\rho)$ is the reflected Dirichlet space of the energy form $\mathcal{E}^\rho$. If the constants λ_ℓ, Λ_ℓ are independent of $\ell > 0$, then the energy form $\mathcal{E}^\rho$ admits $H_e^1(\mathbb{R}^n)$ and $\mathrm{BL}(\mathbb{R}^n)$ as its extended Dirichlet space and reflected Dirichlet space, respectively, so that the answer to the question **(Q)** is affirmative.

In Sect. 3, we give also an affirmative answer to **(Q)** for the energy form $\mathcal{E}^\rho$ when $n \geq 2$ and $\rho(x)$ is any positive C^∞-function depending only on the radial part r of the variable $x \in \mathbb{R}^n$. Presently we have no example of the energy form $\mathcal{E}^\rho$ on $\mathbb{R}^n$ for $n \geq 2$ violating the Liouville property (1).

In Sects. 4, 5 and 6, we shall make a general consideration on a regular, strongly local and transient Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(E; m)$ and an associated diffusion process $X = (X_t, \zeta, \mathbf{P}_x)$ on E. X_t approaches to the point ∂ at infinity of E as $t \uparrow \zeta$ [6, Sect. 3.5]. But the lifetime ζ of X could be infinite and so, in place of X, we consider its time-changed process $\check{X} = (\check{X}_t, \check{\zeta}, \mathbf{P}_x)$ by means of its positive continuous additive functional whose Revuz measure ν is a finite measure on E charging no $\mathcal{E}$-polar set with full quasi-support. $\check{X}$ is ν-symmetric. As we see in Sect. 4, the lifetime $\check{\zeta}$ of $\check{X}$ is finite $\mathbf{P}_x$-a.s. for q.e. $x \in E$ and $\check{X}_t$ approaches to ∂ as $t \uparrow \check{\zeta}$.

Therefore the boundary problem of $\check{X}$ at ∂ looking for all possible Markovian extensions of $\check{X}$ beyond $\check{\zeta}$ makes perfect sense. A strong Markov process Y on a Lusin space $\widehat{E}$ is said to be an *extension* of $\check{X}$ if E is homeomorphically embedded into $\widehat{E}$ as an open subset, the part process of Y on E being killed upon leaving E is identical in law with $\check{X}$, and Y has no sojourn on $\widehat{E} \setminus E$, that is, Y spends zero Lebesgue amount of time on $\widehat{E} \setminus E$.

In Sect. 5, we show that the *time changed diffusion process* $\check{X}$ admits a ν-symmetric conservative diffusion extension $\check{X}^*$ from E to its one-point compactification $E^* = E \cup \{\partial\}$ by constructing a regular, strongly local, recurrent and irreducible Dirichlet form on $L^2(E^*; \nu)$, ν being extended to E^* by setting $\nu(\partial) = 0$. In accordance with [7], $\check{X}^*$ is called the *one point reflection* of $\check{X}$ at ∂.

Theorem 6.1 in Sect. 6 will state that, if $\mathcal{E}$ enjoys the Liouville property (1), then a ν-symmetric conservative diffusion extension Y of $\check{X}$ is unique and coincides with the one-point reflection $\check{X}^*$ of $\check{X}$ at ∂ up to a quasi-homeomorphism, namely, a quasi homeomorphic image of Y is identical with $\check{X}^*$, and furthermore the extended Dirichlet space of Y equals the reflected Dirichlet space $(\mathcal{F}^{\mathrm{ref}}, \mathcal{E}^{\mathrm{ref}})$ of $\mathcal{E}$ independently of the smooth measure ν employed in the time change.

The proof of Theorem 6.1 will make use of the following general observation. Owing to the works of S. Albeverio, Z.-M. Ma and M. Röckner [1, 22] and P.J. Fitzsimmons [14], the *quasi-regularity* of a Dirichlet form has been known to be not only a sufficient condition but also a necessary one for the existence of a properly associated right process. It is further shown by Z.-Q. Chen, Z.-M. Ma and M. Röckner [9] that a Dirichlet form is quasi-regular if and only if it is *quasi-homeomorphic* to a regular Dirichlet form on a locally compact separable metric space. On the other hand, it was known that, if two regular Dirichlet spaces are *equivalent* in the sense of [16] and [17, Appendix], then the equivalence can be induced by a certain quasi-homeomorphism of the underlying spaces. Hence the equivalence of two quasi-regular Dirichlet spaces can be induced by such a map as will be formulated in Theorem 6.2. In the specific setting of Theorem 6.1, the Dirichlet spaces of Y and $\check{X}^*$ are both quasi-regular and they can be shown to be equivalent if the Liouville property (1) is valid. Thus Theorem 6.1 follows from Theorem 6.2.

2 Energy Form $\mathscr{E}^\rho$ and Weighted Beppo Levi Space $\mathscr{G}^\rho$

For a fixed Borel function ρ on $\mathbb{R}^n$ satisfying (4), define $\mathscr{F}^\rho, \mathscr{G}^\rho, \mathbf{D}^\rho$ by (5), (6), (7), respectively. We put $\mathbf{D}_1^\rho(u,v) = \mathbf{D}^\rho(u,v) + \int_{\mathbb{R}^n} uv\rho dx,\ u, v \in \mathscr{F}^\rho$.

Proposition 2.1 *The energy form $\mathscr{E}^\rho = (\mathbf{D}^\rho, \mathscr{F}^\rho)$ is a regular strongly local and irreducible Dirichlet form on $L^2(\mathbb{R}^n; \rho dx)$.*

Proof Completeness: Suppose $\{u_k\} \subset \mathscr{F}^\rho$ is $\mathbf{D}_1^\rho$-Cauchy. There exists then $u \in L^2(\mathbb{R}^n; \rho dx)$ and u_k converges to u in $L^2(\mathbb{R}^n; \rho dx)$. For each $r > 0$, $\{u_k|_{B_r}\}$ is $\mathbf{D}$-Cauchy on B_r by (4) and so, $\partial_i u_k \to \partial_i u$ in $L^2(B_r)$, $1 \le i \le n$. One can find a subsequence $\{k_\ell\}$ such that

$$u_{k_\ell} \to u, \quad \partial_i u_{k_\ell} \to \partial_i u, \ 1 \le i \le n, \quad \text{a.e. on } \mathbb{R}^n, \quad \text{as } \ell \to \infty.$$

By Fatou's lemma

$$\mathbf{D}_1^\rho(u - u_m, u - u_m) \le \liminf_{\ell \to \infty} \mathbf{D}_1^\rho(u_{k_\ell} - u_m, u_{k_\ell} - u_m) \to 0, \ m \to \infty.$$

Regularity: Take any bounded $u \in \mathscr{F}^\rho$. For any $\varepsilon > 0$, we find $r > 0$ with

$$\int_{\mathbb{R}^n \setminus B_r} |\nabla u|^2 \rho dx < \varepsilon, \quad \int_{\mathbb{R}^n \setminus B_r} u^2 \rho dx < \varepsilon,$$

Choose $\varphi \in C_c^\infty(\mathbb{R}^n)$ with $\varphi = 1$ on B_r, $\varphi = 0$ on $\mathbb{R}^n \setminus B_{r+2}$, and $0 \le \varphi \le 1$, $0 \le |\nabla \varphi| \le 1$ on $\mathbb{R}^n$. Then

$$\mathbf{D}_1^\rho(u - u\varphi, u - u\varphi) \le 2 \int_{\mathbb{R}^n \setminus B_r} (1 - \varphi(x))^2 |\nabla u(x)|^2 \rho(x) dx$$
$$+ 2 \int_{\mathbb{R}^n \setminus B_r} u(x)^2 |\nabla \varphi(x)|^2 \rho(x) dx + \int_{\mathbb{R}^n \setminus B_r} u^2 \rho dx \le 5\varepsilon.$$

Since $u\varphi \in H_0^1(B_{r+2})$, there exists $f \in C_c^\infty(B_{r+2})$ with

$$\int_{B_{r+2}} \left[|\nabla(u\varphi - f)|^2 + (u\varphi - f)^2 \right] dx < \varepsilon / \Lambda_{r+2}.$$

Hence, by taking (4) into account,

$$\mathbf{D}_1^\rho(u-f,u-f) \le 2\mathbf{D}_1^\rho(u-u\varphi,u-u\varphi)+2\mathbf{D}_1^\rho(u\varphi-f,u\varphi-f) < 12\varepsilon,$$

Markov property, strong locality and irreducibility follow from Theorem 3.1.1, Excercise 3.1.1 and Corollary 4.6.4 of [17], respectively. □

We consider the quotient space $\dot{\mathscr{G}}^\rho = \mathscr{G}^\rho/\mathscr{N}$ of $\mathscr{G}^\rho$ relative to the space $\mathscr{N}$ of constant functions on $\mathbb{R}^n$.

Lemma 2.2 *$\dot{\mathscr{G}}^\rho$ is a Hilbert space with inner product $\mathbf{D}^\rho$.*
If $u_k \in \dot{\mathscr{G}}^\rho$ is $\mathbf{D}^\rho$-convergent to $u \in \dot{\mathscr{G}}^\rho$ as $k \to \infty$, then there are constants c_k such that $u_k - c_k$ converges to u in $L^2_{\rm loc}(\mathbb{R}^n)$ as $k \to \infty$.

Proof We use the Poincaré inequality holding for each ball B_r, $r > 1$:

$$\int_{B_r}(u(x)-\langle u\rangle_1)^2dx \le C_r\int_{B_r}|\nabla u(x)|^2dx, \quad u \in H^1(B_r), \tag{8}$$

where $\langle u\rangle_1 = |B_1|^{-1}\int_{B_1}u(x)dx$ and C_r is some positive constant (cf. [19, (7.45)]).

Let $\{u_k\} \subset \dot{\mathscr{G}}^\rho$ be $\mathbf{D}^\rho$-Cauchy. There exist then $f_i \in L^2(\mathbb{R}^n;\rho dx)$ such that $\partial_i u_k \to f_i$ in $L^2(\mathbb{R}^n;\rho dx)$ and hence in $L^2(B_r)$ as $k \to \infty$, $1 \le i \le n$, $r > 1$. Set $c_k = \langle u_k\rangle_1$. By (8), $\{u_k - c_k\}$ is $L^2(B_r)$-Cauchy for each $r > 1$. Let $u \in L^2_{\rm loc}(\mathbb{R}^n)$ be the limit function. Then, for any $\varphi \in C_c^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} f_i\varphi dx = \lim_{k\to\infty}\int_{\mathbb{R}^n}\partial_i(u_k-c_k)\cdot\varphi dx = -\int_{\mathbb{R}^n}(u_k-c_k)\partial_i\varphi dx = -\int_{\mathbb{R}^n}u\partial_i\varphi dx,$$

so that $f_i = \partial_i u$, $1 \le i \le n$. □

Let $(\mathscr{F}_e^\rho, \mathscr{E}^\rho)$ be the extended Dirichlet space of the energy form $\mathscr{E}^\rho$.

Corollary 2.3 *$\mathscr{F}_e^\rho \subset \mathscr{G}^\rho$ and $\mathscr{E}^\rho(u,u) = \mathbf{D}^\rho(u,u)$, $u \in \mathscr{F}_e^\rho$.*

Proof For $u \in \mathscr{F}_e^\rho$, there exist $u_k \in \mathscr{F}^\rho$, $k \ge 1$, which is $\mathbf{D}^\rho$-Cauchy and converge to u a.e. as $k \to \infty$ according to the definition [17, Sect 1.5]. By Lemma 2.2, $\{u_k\}$ is $\mathbf{D}^\rho$-convergent to some $v \in \mathscr{G}^\rho$ and, for some constants c_k and a subsequence $\{k_\ell\}$, $u_{k_\ell} + c_{k_\ell} \to v$ a.e. as $\ell \to \infty$. Hence $c = \lim_{\ell\to\infty}c_{k_\ell}$ exists and $u = v - c$ so that $u \in \mathscr{G}^\rho$ and $\{u_k\}$ is $\mathbf{D}^\rho$-convergent to u. □

For a general regular strongly local Dirichlet form $\mathscr{E} = (\mathscr{E}, \mathscr{F})$ on $L^2(E;m)$, the energy measure $\mu_{\langle u\rangle}$ is well defined for $u \in \mathscr{F}_{\rm loc}$ and, according to [6, Theorem 6.2.13], the reflected Dirichlet space of $\mathscr{E}$ can be defined by

$$\begin{cases}\mathscr{F}^{\rm ref} = \{u:\ \text{finite } m\text{-a.e.on } E,\ \tau_k u \in \mathscr{F}_{\rm loc},\ \sup_k \mu_{\langle\tau_k u\rangle}(E) < \infty\} \\ \mathscr{E}^{\rm ref}(u,u) = \lim_{k\to\infty}\mu_{\langle\tau_k u\rangle}(E), \quad u \in \mathscr{F}^{\rm ref},\end{cases} \tag{9}$$

where $\tau_k u(x) = (-k)\vee u(x)\wedge k$.

Let $(\mathscr{F}^{\rho,\mathrm{ref}}, \mathscr{E}^{\rho,\mathrm{ref}})$ be the reflected Dirichlet space of the energy form $\mathscr{E}^{\rho}$. By virtue of [6, Theorem 4.3.11], the energy measure of a bounded $u \in \mathscr{F}^{\rho}$ is given by $\mu_{\langle u\rangle}(dx) = |\nabla u|^2 \rho dx$. Thus we have from (9)

$$\begin{cases} \mathscr{F}^{\rho,\mathrm{ref}} = \{u : \text{ finite a.e.on } \mathbb{R}^n,\ \tau_k u \in H^1_{\mathrm{loc}}(\mathbb{R}^n),\ \sup_k \int_{\mathbb{R}^n} |\nabla \tau_k u|^2 \rho dx < \infty\} \\ \mathscr{E}^{\rho,\mathrm{ref}}(u,u) = \lim_{k\to\infty} \int_{\mathbb{R}^n} |\nabla \tau_k u|^2 \rho dx, \quad u \in \mathscr{F}^{\rho,\mathrm{ref}}. \end{cases} \tag{10}$$

Proposition 2.4 *It holds that*

$$\mathscr{F}^{\rho,\mathrm{ref}} = \mathscr{G}^{\rho}, \qquad \mathscr{E}^{\rho,\mathrm{ref}} = \mathbf{D}^{\rho}.$$

Proof From (10), we obviously have $\mathscr{G}^{\rho} \subset \mathscr{F}^{\rho,\mathrm{ref}}$ and $\mathscr{E}^{\rho,\mathrm{ref}}(u,u) = \mathbf{D}^{\rho}(u,u)$ for any $u \in \mathscr{G}^{\rho}$.

It remains to show that $\mathscr{F}^{\rho,\mathrm{ref}} \subset \mathscr{G}^{\rho}$, For any $u \in \mathscr{F}^{\rho,\mathrm{ref}}$, we see by Banach-Saks theorem (cf. [6, Theorem A.4.1]) that the Césaro mean sequence $\{f_\ell, \ell \geq 1\}$ of a suitable subsequence of $\{\tau_k u, k \geq 1\} \subset \mathscr{G}^{\rho}$ is $\mathbf{D}^{\rho}$-Cauchy and converges pointwise to u. By Lemma 2.2, there exist constants c_ℓ and $w \in \mathscr{G}^{\rho}$ such that $f_\ell - c_\ell \to w$ in $L^2(B_r)$ as $\ell \to \infty$ for each $r > 1$. Choose a subsequence $\{\ell_p\}$ such that $f_{\ell_p} - c_{\ell_p} \to w$ a.e. on $\mathbb{R}^n$ as $p \to \infty$. Then $c = \lim_{p\to\infty} c_{\ell_p}$ exists and $u = w + c \in \mathscr{G}^{\rho}$. □

3 Liouville Property of Rotation Invariant Energy Forms on $\mathbb{R}^n$

Theorem 3.1 *Let $\rho(x) = \eta(|x|)$, $x \in \mathbb{R}^n$, for a positive C^∞-function η on $[0,\infty)$ such that η is constant on $[0,\varepsilon)$ for some $\varepsilon > 0$.*
Then the energy form $\mathscr{E}^{\rho}$ satisfies the Liouville property (1) *when $n \geq 2$.*
When $n = 1$, $\dim(\mathscr{H}^*) = 2$ in transient case.

Proof In view of Propositions 1.1 and 2.1, it suffices to consider only the transient case in order to verify the Liouville property (1). According to Theorem 1.6.7 in the first edition of [17], $\mathscr{E}^{\rho}$ is transient if and only if

$$\int_1^\infty \frac{1}{\eta(r) r^{n-1}} dr < \infty. \tag{11}$$

In what follows, we assume that η satisfies condition (11).

It then follows from $1/r = (r^{n-3}\eta(r))^{1/2}(\eta(r)r^{n-1})^{-1/2}$ and the Schwarz inequality that

$$\int_1^\infty r^{n-3}\eta(r) dr = \infty. \tag{12}$$

We use the polar coordinate

$$x_1 = r\cos\theta_1,\ x_2 = r\sin\theta_1\cos\theta_2,\ x_3 = r\sin\theta_1\sin\theta_2\cos\theta_3,\ \dots,$$

$$x_{n-1} = r\sin\theta_1\sin\theta_2\dots\sin\theta_{n-2}\cos\theta_{n-1},\ x_n = r\sin\theta_1\sin\theta_2\dots\sin\theta_{n-2}\sin\theta_{n-1}.$$

Then, for $u, v \in C_c^1(\mathbb{R}^n)$,

$$\begin{aligned}\mathbf{D}^\rho(u,v) \qquad\qquad\qquad\qquad (13)\\ = \int_{(0,\infty)\times(0,\pi]^{n-2}\times[0,2\pi]} \left[u_r v_r + \frac{u_{\theta_1}v_{\theta_1}}{r^2} + \frac{u_{\theta_2}v_{\theta_2}}{r^2\sin^2\theta_1} + \dots + \frac{u_{\theta_{n-1}}v_{\theta_{n-1}}}{r^2\sin^2\theta_1\dots\sin^2\theta_{n-2}}\right]\\ \times\ \eta(r)r^{n-1}\sin^{n-2}\theta_1\dots\sin\theta_{n-2}dr d\theta_1\dots d\theta_{n-1}.\end{aligned}$$

For a C^∞-function u on $\mathbb{R}^n$, we denote by $I_\eta(u,u)$ the value of the integral of the right hand side of (13) for $v = u$.

By Proposition 2.4, the reflected Dirichlet space of $\mathscr{E}^\rho$ is given by $(\mathscr{G}^\rho, \mathbf{D}^\rho)$ where

$$\mathscr{G}^\rho = \{u \in L^2_{\rm loc}(\mathbb{R}^n) : \int_{\mathbb{R}^n} |\nabla u(x)|^2\eta(|x|)dx < \infty\}.$$

Since the extended Dirichlet space $(\mathscr{F}_e^\rho, \mathbf{D}^\rho)$ of the transient energy form $\mathscr{E}^\rho$ is a real Hilbert space possessing $C_c^\infty(\mathbb{R}^n)$ as its core, we have

$$\mathscr{H}^* = \{u \in \mathscr{G}^\rho : \mathbf{D}^\rho(u,v) = 0 \quad \text{for every} \quad v \in C_c^\infty(\mathbb{R}^n)\}. \tag{14}$$

By noting that $\rho(x) = \eta(|x|)$ is a C^∞-function on $\mathbb{R}^n$, we let

$$Lu(x) = \Delta u(x) + \nabla\log\rho(x)\cdot\nabla u(x), \quad x \in \mathbb{R}^n. \tag{15}$$

We say that u is $\mathscr{E}^\rho$*-harmonic* if

$$u \in C^\infty(\mathbb{R}^n), \quad Lu(x) = 0,\ x \in \mathbb{R}^n.$$

Equation (14) then implies that $u \in \mathscr{H}^*$ if and only if

$$u \text{ is } \mathscr{E}^\rho\text{-harmonic and} \quad I_\eta(u,u) < \infty. \tag{16}$$

It also follows from (13) that u is $\mathscr{E}^\rho$-harmonic if and only if

$$u \in C^\infty(\mathbb{R}^n),\quad \mathscr{L}u(r,\theta_1,\dots,\theta_{n-1}) = 0,\ (r,\theta_1,\dots,\theta_{n-1}) \in (0,\infty)\times(0,\pi)^{n-2}\times[0,2\pi], \tag{17}$$

where

$$
\begin{aligned}
&\mathscr{L}u(r,\theta_1,\ldots,\theta_{n-1}) \qquad (18)\\
&= \frac{1}{r^{n-1}}(u_r\cdot\eta(r)r^{n-1})_r + \frac{\eta(r)}{r^2\sin^{n-2}\theta_1}(u_{\theta_1}\sin^{n-2}\theta_1)_{\theta_1} + \frac{\eta(r)}{r^2\sin^2\theta_1\sin^{n-3}\theta_2}(u_{\theta_2}\sin^{n-3}\theta_2)_{\theta_2}\\
&+\ldots+\frac{\eta(r)}{r^2\sin^2\theta_1\ldots\sin^2\theta_{n-3}\sin\theta_{n-2}}(u_{\theta_{n-2}}\sin\theta_{n-2})_{\theta_{n-2}} + \frac{\eta(r)}{r^2\sin^2\theta_1\ldots\sin^2\theta_{n-2}}(u_{\theta_{n-1}})_{\theta_{n-1}}
\end{aligned}
$$

Now take any function $u \in \mathscr{H}^*$. We first claim that

$$u_{\theta_{n-1}} = 0. \qquad (19)$$

Put $w = u_{\theta_{n-1}}$. Then $w \in C^\infty(\mathbb{R}^n)$ because $w(x) = -u_{x_{n-1}}(x)x_n + u_{x_n}(x)x_{n-1}$, $x \in \mathbb{R}^n$. Due to the expression (18) of $\mathscr{L}$, we also have $\mathscr{L}w = (\mathscr{L}u)_{\theta_{n-1}} = 0$ on $(0,\infty)\times(0,\pi)^{n-2}\times[0,2\pi]$. Thus w satisfies (17) so that w is $\mathscr{E}^\rho$-harmonic.

For $B_r = \{x \in \mathbb{R}^n;\ |x| < r\}$ and the uniform probability measure $\Pi(d\xi)$ on ∂B_1, w therefore admits the Poisson integral formula

$$w(x) = \int_{\partial B_1} K_r(x, r\xi)w(r\xi)\Pi(d\xi).\quad x \in B_r, \qquad (20)$$

where $K_r(x, r\xi)$ is the Poisson kernel for B_r with respect to L, which is known to be continuous in $(x,\xi) \in B_r \times \partial B_1$ (cf. [20]). We also note that $K_r(0, r\xi) = 1$ for any $\xi \in \partial B_1$ by the rotation invariance of L around the origin 0.

Fix $a > 0$. It then holds For any $r > a$ that

$$K_r(x, r\xi_2) = \int_{\partial B_a} K_a(x, a\xi_1)K_r(a\xi_1, r\xi_2)\Pi(d\xi_1),\quad x \in B_a,\ \xi_2 \in \partial B_1.$$

Hence, if we let $\sup_{x\in B_{a/2},\xi_1\in\partial B_1} K_a(x, a\xi_1) = C_a < \infty$, then, for $x \in B_{a/2}$, $\xi_2 \in \partial B_1$,

$$K_r(x.r\xi_2) \le C_a\int_{\partial B_1} K_r(a\xi_1, r\xi_2)\Pi(d\xi_1) = C_aK_r(0, r\xi_2) = C_a,$$

and it follows from (20) that

$$|w(x)| \le C_a\int_{\partial B_1}|w(r\xi)|\Pi(d\xi),\quad x \in B_{a/2},\quad r > a.$$

Recall that $w = u_{\theta_{n-1}}$. We denote by σ_n the area of ∂B_1. We multiply the both hand side of the above inequality by $r^{n-3}\eta(r)$, integrate in r from a to R, apply the Schwarz inequality and finally use the expression (13) to get

$$|u_{\theta_{n-1}}(x)| \le \frac{C_a}{\sqrt{\sigma_n}} \left[\int_a^R r^{n-3}\eta(r)dr \right]^{-1/2} \cdot \sqrt{I_\eta(u,u)}, \quad x \in B_{a/2},$$

which tends to 0 as $R \to \infty$ by (12) and (16). Since $a > 0$ is arbitrary, we arrive at (19). Here the finite energy property for $u \in \mathscr{G}^\rho$ is crucially utilized.

It also holds that

$$u_{\theta_k} = 0 \quad \text{for any} \quad 1 \le k \le n-2. \tag{21}$$

In fact, if we let $\xi_i = \frac{x_i}{r}$, $1 \le i \le n$, $\xi = (\xi_1, \dots, \xi_n) \in \partial B_1$, then θ_k, $1 \le k \le n-2$, is an angle of two n-vectors $\xi^{(k)} = (\underbrace{0, \dots 0}_{k-1}, \xi_k, \dots, \xi_n)$, $\mathbf{e}_k = (\underbrace{0, \dots 0}_{k-1}, 1, 0, \dots, 0)$. Let O be an orthogonal matrix whose $(n-1)$-th and n-th row vectors are $\mathbf{e}_k$ and $\widehat{\mathbf{e}} = |\xi^{(k+1)}|^{-1} \cdot \xi^{(k+1)}$, respectively. We make the orthogonal transformation $\mathbf{y} = O\mathbf{x}$. Then θ_k equals an angle of two vectors on the (y_{n-1}, y_n)-plane in the new coordinate system $\mathbf{y}$ and (21) follows from (19) as the expression (7) of $\mathbf{D}^\rho(u, v)$ with $\rho(x) = \eta(|x|)$ remains valid for $\mathbf{y}$ in place of $\mathbf{x}$.

Thus u depends only on r and, in terms of a scale function $ds(r) = \dfrac{dr}{\eta(r)r^{n-1}}$ on $(0, \infty)$, (13) and (18) are reduced, respectively, to

$$I_\eta(u,u) = \sigma_n \int_0^\infty \left(\frac{du(r)}{ds(r)} \right)^2 ds(r), \qquad \mathscr{L}u(r) = \frac{1}{r^{n-1}} \frac{d}{dr} \cdot \frac{du(r)}{ds(r)}.$$

By (17), $\mathscr{L}u = 0$ so that $\frac{du(r)}{ds(r)}$ equals a constant C and $I_\eta(u,u) = \sigma_n C^2 s(0, \infty)$.

When $n \ge 2$, $s(0, \infty) = \infty$ and we get $C = 0$ from (16), yielding that u is constant. When $n = 1$ and $\rho(x) = \eta(|x|)$, $x \in \mathbb{R}$, for a positive continuous function η on $[0, \infty)$ with $\int_1^\infty \eta(x)^{-1}dx < \infty$, $u = c_1 + c_2 s$, $c_1, c_2 \in \mathbb{R}$, for $s(x) = \int_0^x \eta(|\xi|)^{-1}d\xi$, $x \in \mathbb{R}$. □

The condition for η to be a positive constant near 0 is just for simplicity and it can be weakened appropriately.

4 Strongly Local Transient Dirichlet Form $\mathscr{E}$ and a Time Change $\check{X}$ of the Associated Diffusion

Let $(E, m, \mathscr{E}, \mathscr{F})$ and $(\mathscr{F}_e, \mathscr{F}^{\mathrm{ref}})$ be as is stated in the beginning of Introduction. Once for all, we assume that the Dirichlet form $(\mathscr{E}, \mathscr{F})$ on $L^2(E; m)$ is transient and strongly local. Let $X = (X_t, \zeta, \mathbf{P}_x)$ be the associated diffusion process on E.

The lifetime ζ of X can be finite or infinite. Since X admits no killing inside E, we get from [6, Theorem 3.5.2 and Corollary 3.5.3]

$$\mathbf{P}_x(\lim_{t \to \zeta} X_t = \partial) = 1 \quad \text{q.e.} \quad x \in E, \tag{22}$$

$$\mathbf{P}_x(\lim_{t\to\zeta} u(X_t) = 0) = 1 \quad \text{q.e. } x \in E, \tag{23}$$

where ∂ is the point at infinity of E and u is any quasi continuous function belonging to the extended Dirichlet space $\mathcal{F}_e$.

In the remainder of this paper, we fix an arbitrary positive finite measure ν on E charging no $\mathcal{E}$-polar set such the the quasi-support of ν equals E. Let A be the positive continuous additive functional of X with Revuz measure ν. A typical example of such a measure ν is $\nu(dx) = f(x)m(dx)$ for a strictly positive Borel function f on E with $\int_E f dm < \infty$ and, in this case, $A_t = \int_0^{t\wedge\zeta} f(X_s)ds,\ t \geq 0$.

Let $\check{X} = (\check{X}_t, \check{\zeta}, \mathbf{P}_x)$ be the time changed process of X by means of A:

$$\check{X}_t = X_{\tau_t}, \quad \tau_t = \inf\{s : A_s > t\}, \qquad \check{\zeta} = A_\zeta.$$

$\check{X}$ is a diffusion process on E symmetric with respect to the measure ν and the Dirichlet form $\check{\mathcal{E}} = (\check{\mathcal{E}}, \check{\mathcal{F}})$ of $\check{X}$ on $L^2(E;\nu)$ is given by

$$\check{\mathcal{E}} = \mathcal{E}, \qquad \check{\mathcal{F}} = \mathcal{F}_e \cap L^2(E;\nu), \tag{24}$$

which is strongly local and regular [6, p. 183].

Proposition 4.1 (i) *It holds that*

$$\mathbf{P}_x(\check{\zeta} < \infty,\ \lim_{t\uparrow\check{\zeta}} \check{X}_t = \partial) = \mathbf{P}_x(\check{\zeta} < \infty) = 1 \ \textit{for q.e. } x \in E. \tag{25}$$

(ii) *The extended and reflected Dirichlet spaces of* $(\check{\mathcal{E}}, \check{\mathcal{F}})$ *equal* $(\mathcal{F}_e, \mathcal{E})$ *and* $(\mathcal{F}^{\text{ref}}, \mathcal{E}^{\text{ref}})$, *respectively.*

Proof (i) It suffices to show that

$$\mathbf{P}_x(A_\infty < \infty) = 1 \qquad \text{for q.e. } x \in E. \tag{26}$$

Take a strictly positive Borel function h on E with $\int_E h(x)m(dx)dx < \infty$.. By the transience of X and [6, Proposition 2.1.3], $Rh(x) < \infty$ for m-a.e. $x \in E$ where R is the 0-order resolvent of X. For integer $\ell \geq 1$, let $A_\ell = \{x \in E : Rh(x) \leq \ell\}$. Then $R((I_{A_\ell}h)(x) \leq \ell$ for any $x \in E$.

From [6, (4.1.3)], we have for each $\ell \geq 1$

$$\int_{A_\ell} \mathbf{E}_x[A_\infty]h(x)m(dx) = \langle R(I_{A_\ell}h), \nu\rangle \leq \ell\nu(E) < \infty.$$

As $m(E \setminus (\cup_{\ell=1}^\infty A_\ell)) = 0$, it follows that $\mathbf{E}_x[A_\infty] < \infty$ m-a.e. $x \in E$ and hence q.e. $x \in E$ by [6, Theorem A.2.13 (v)], yielding (26).

(ii) is a consequence of the invariance of the extended and reflected Dirichlet spaces under a time change by means of a fully supported positive Radon measure charging no $\mathcal{E}$-polar set [6, Corollary 5.2.12 and Proposition 6.4.6]. □

Since the lifetime $\check{\zeta}$ of the time changed diffusion $\check{X}$ is finite $\mathbf{P}_x$-a.s. for q.e. $x \in E$ by the above proposition, the boundary problem concerning possible Markovian extensions of $\check{X}$ beyond its lifetime $\check{\zeta}$ makes a perfect sense. For different choices of ν, the diffusions $\check{X}$ share a common geometric structure related each other only by time changes. So the study of the boundary problem for $\check{X}$ as we shall engage in the next two sections is a good way to make a closer look at the behavior of the diffusion process X around ∂.

5 One-Point Reflection of $\check{X}$ at ∂

Denote by E^* the one-point compactification $E \cup \{\partial\}$ of E. The measure ν is extended from E to E^* by setting $\nu(\{\partial\}) = 0$. In this section, we construct a ν-symmetric conservative diffusion extension of $\check{X}$ from E to E^* by constructing a regular strongly local Dirichlet form on $L^2(E^*; \nu)$. Note that $L^2(E^*; \nu)$ can be identified with $L^2(E; \nu)$.

Recall that the reflected Dirichlet space $(\mathcal{F}^{\mathrm{ref}}, \mathcal{E}^{\mathrm{ref}})$ of the regular strongly local Dirichlet form $\mathcal{E} = (\mathcal{E}, \mathcal{F})$ on $L^2(E; m)$ is given by (9). On account of the property [6, Theorem 4.3.10] of the energy measure $\mu_{\langle u \rangle}$ for $u \in \mathcal{F}_{\mathrm{loc}}$, this means that

$$1 \in \mathcal{F}^{\mathrm{ref}}, \qquad \mathcal{E}^{\mathrm{ref}}(1, 1) = 0. \tag{27}$$

Furthermore $\mathcal{F}_e$ does not contain a non-zero constant function because of the transience of $\mathcal{E}$. In what follows, every function in the space $\mathcal{F}_e$ is taken to be $\mathcal{E}$-quasi-continuous.

Let us define

$$\begin{cases} \mathcal{F}_e^* = \{u + c : u \in \mathcal{F}_e,\ c \in \mathbb{R}\}, \\ \mathcal{E}^*(u_1 + c_1, u_2 + c_2) = \mathcal{E}(u_1, u_2),\ u_i \in \mathcal{F}_e,\ c_i \in \mathbb{R},\ i = 1, 2. \end{cases} \tag{28}$$

$(\mathcal{F}_e^*, \mathcal{E}^*)$ is a subspace of $(\mathcal{F}^{\mathrm{ref}}, \mathcal{E}^{\mathrm{ref}})$.

Theorem 5.1 (i) *Define $\check{\mathcal{F}}^* = \mathcal{F}_e^* \cap L^2(E; \nu)$. The form $\check{\mathcal{E}}^* = (\mathcal{E}^*, \check{\mathcal{F}}^*)$ is then a regular strongly local Dirichlet form on $L^2(E^*; \nu)$.*
(ii) *The extended Dirichlet space of $\check{\mathcal{E}}^*$ equals $(\mathcal{F}_e^*, \mathcal{E}^*)$. $\check{\mathcal{E}}^*$ is recurrent.*
(iii) *Let $\check{X}^* = (\check{X}_t^*, \mathbf{P}_x^*)$ be the diffusion process on E^* associated with $\check{\mathcal{E}}^*$. The part of $\check{X}^*$ on E being killed upon hitting ∂ is then identical in law with the time changed diffusion $\check{X}$. $\check{X}^*$ is conservative and irreducible.*

Proof (i) *Completeness.* Suppose $w_n = u_n + c_n \in \check{\mathcal{F}}^*$, $n \geq 1$, are $\mathcal{E}_1^*$-Cauchy. Then $\{u_n\} \subset \mathcal{F}_e$ is an $\mathcal{E}$-Cauchy sequence. Due to the transience, it is $\mathcal{E}$-convergent to some $u \in \mathcal{F}_e$ and some subsequence $\{u_{n_k}\}$ of it converges to u q.e. on E in view of [6, Theorem 2.3.2]. $\{w_n\}$ is $L^2(E; \nu)$-convergent to some $w \in L^2(E : \nu)$ and a subsequence $\{w_{n'_k}\}$ of $\{w_{n_k}\}$ converges to w ν-a.e. on E. Hence $\lim_{k\to\infty} c_{n'_k} = c$ exists and $w = u + c$. Hence $\{w_n\}$ is $\mathcal{E}_1^*$-convergent to $w \in \check{\mathcal{F}}^*$.

Markov property. This follows from $(0 \vee w) \wedge 1 = [(-c) \vee u] \wedge (1-c) + c$ for $w = u + c \in \check{\mathscr{F}}_e^*$.
Regularity. For any bounded $w = u + c \in \check{\mathscr{F}}_e^*$, choose $u_k \in \mathscr{F}_e \cap C_c(E)$ $\mathscr{E}$-converging to u. We may assume that $\{u_k\}$ are uniformly bounded by [17, Theorem 1.4.2]. By [6, Theorem 2.3.2], a subsequence $\{\widetilde{u}_k\}$ of $\{u_k\}$ converges q.e. to u. Since $\int_E u_k^2 d\nu$ is uniformly bounded, the Césaro mean $\{f_k\}$ of a suitable subsequence of $\{\widetilde{u}_k\}$ converges to $f \in L^2(E;\nu)$. Since f_k converges to u ν-a.e., $f = u$. Then $f_k + c \in \check{\mathscr{F}}_e^* \cap C(E^*)$ is $\check{\mathscr{E}}_1^*$-convergent to $u + c = w$.
Strong locality. Suppose, for $w_i = u_i + c_i$, $u_i \in \mathscr{F}_e \cap C_c(E)$, $i = 1, 2$, that w_1 is constant in a neighborhood of $\mathrm{Supp}(w_2)$. When $c_2 = 0$, $\mathscr{E}^*(w_1, w_2) = 0$ by the strong locality of $\mathscr{E}$. When $c_2 \neq 0$, the set $U = E^* \setminus \mathrm{Supp}(w_2)$ is either empty or non-empty relatively compact open subset of E. In the former case, $\mathscr{E}^*(w_1, w_2) = 0$. In the latter case, $u_2 = -c_2$ on U, while $\mathrm{Supp}(u_1) \subset U$ and $\mathscr{E}^*(w_1, w_2) = \mathscr{E}(u_1, u_2) = 0$ again. Therefore $\mathscr{E}^*$ is strongly local on account of [17, Theorem 3.1.2].
(ii) The inclusion $\subset$ can be shown by using [6, Theorem 3.2.3]. Conversely, for any $u \in \mathscr{F}_e^*$, its truncations are in $\check{\mathscr{F}}^*$ and convergent to u pointwise and in $\mathscr{E}^*$. Hence u is in the extended Dirichlet space of $\check{\mathscr{E}}^*$. Since $1 \in \mathscr{F}_e^*$ and $\mathscr{E}^*(1, 1) = 0$, $\check{\mathscr{E}}^*$ is recurrent.
(iii) By virtue of [17, Theorem 4.4.3], the part of $\check{\mathscr{E}}^*$ on E admits $\mathscr{F}_e \cap C_c(E)$ as its core. So it coincides with the Dirichlet form (24) on $L^2(E;\nu)$ associated with $\check{X}$. Hence $\check{X}^*$ is an extension of $\check{X}$.

Since $\check{\mathscr{E}}^*$ is recurrent, $\check{X}^*$ is recurrent and in particular conservative.

In order to verify the irreducibility of $\check{X}^*$, the resolvents of $\check{X}^*$, $\check{X}$ are denoted by R_α^*, R_α, respectively, and we let $u_\alpha(x) = \mathbf{E}_x[e^{-\zeta}]$, $x \in E$. $(f.g)$ will stand for the integral $\int_E fgd\nu$. By the strong Markov property of $\check{X}^*$ at the hitting time of ∂, we have for any bounded Borel function f on E,

$$R_\alpha^* f(x) = R_\alpha f(x) + u_\alpha(x)\, R_\alpha^* f(\partial), \quad x \in E. \tag{29}$$

By (25), $u_\alpha(x) > 0$ for q.e. $x \in E$ and $1 - u_\alpha(x) = \alpha R_\alpha 1(x)$, $x \in E$. By integrating the both hand sides of (29) by ν, we thus get $R_\alpha^* f(\partial) = \dfrac{(u_\alpha, f)}{\alpha(u_\alpha, 1)}$. Hence it follows from (29) that

$$(I_A, R_\alpha^* I_B) \geq \frac{(u_\alpha, I_A)(u_\alpha, I_B)}{\alpha(u_\alpha, 1)} > 0,$$

for any Borel sets $A, B \subset E$ with positive ν-measure, yielding the irreducibility of $\check{X}^*$. □

In accordance with [7], we call $\check{X}^*$ the *one-point reflection of* $\check{X}$ *at* ∂.

This theorem is a generalization of Theorem 3.2 in [18] where a stronger assumption of a Poincaré inequality for $\mathscr{E}$ was made. The first construction of such a one-point reflection at ∂ goes back to [15, Sect. 8] where $\check{X} = X$ and X was the absorbing Brownian motion on an arbitrary bounded domain of $\mathbb{R}^n$.

As has been presented in [18] and [6, Sects. 7.5 and 7.6], there is an alternative quite different way to construct a one-point reflection $\check{X}^*$ of $\check{X}$ at ∂ by using a Poisson point process of excursions of $\check{X}$ around ∂, which makes the structure of the constructed process $\check{X}^*$ more transparent but requires a certain regularity condition on the resolvent of $\check{X}$ in the construction. Notice that $\check{X}^*$ becomes irreducible, while $\check{X}$ may not be. See Example 6.2 in [18] in this connection.

In the next section, we shall show that, if $\mathscr{E}$ satisfies the Liouville property (1), then any symmetric conservative diffusion extension of $\check{X}$ must equal $\check{X}^*$ up to a quasi-homeomorphism.

6 Liouville Property of $\mathscr{E}$ and Uniqueness of a Symmetric Conservative Diffusion Extension of $\check{X}$

Let $\widehat{E}$ be a Lusin space into which E is homeomorpically embedded as an open subset. The measure ν on E is extended to $\widehat{E}$ by setting $\nu(\widehat{E} \setminus E) = 0$. Let $Y = (Y_t, \zeta^Y, \mathbf{P}_x^Y)$ be any ν-symmetric conservative diffusion process on $\widehat{E}$ whose part process on E being killed upon leaving E is identical in law with $\check{X}$. We denote by $(\mathscr{E}^Y, \mathscr{F}^Y)$ the Dirichlet form of Y on $L^2(\widehat{E}; \nu)$. We call Y a *ν-symmetric conservative diffusion extension* of $\check{X}$.

Theorem 6.1 *Suppose $\mathscr{E}$ satisfies the Liouville property* (1). *Then we have the following:*
(i) *As Dirichlet forms on $L^2(E, \nu)$,*

$$(\mathscr{E}^Y, \mathscr{F}^Y) = (\mathscr{E}^*, \check{\mathscr{F}}^*). \tag{30}$$

(ii) *The extended Dirichlet space of $(\mathscr{E}^Y, \mathscr{F}^Y)$ equals $(\mathscr{F}_e^*, \mathscr{E}^*) = (\mathscr{F}^{\mathrm{ref}}, \mathscr{E}^{\mathrm{ref}})$.*
(iii) *Y under $\mathbf{P}_{g\cdot\nu}^Y$ and $\check{X}^*$ under $\mathbf{P}_{g\cdot\nu}^*$ share the same finite dimensional distributions for any non-negative $g \in L^2(E; \nu)$.*
(iv) *A quasi-homeomorphic image of Y in the specific sense described in Theorem 6.2 below is identical with $\check{X}^*$.*

Proof of Theorem 6.1 **(i), (ii), (iii)** We use basic results due to Albeverio-Ma-Röckner [1, 22], Fitzsimmons [14] and Chen-Ma-Röckner [9] being formulated in [6]: $\mathscr{E}^Y$ is a quasi-regular Dirichlet form on $L^2(\widehat{E}; \nu)$ and Y is properly associated with it [6, Theorem 1.5.3]. By considering the image by the quasi-homeomorphsim j in [6, Theorem 3.1.13], we can therefore assume that $\widehat{E}$ is a locally compact separable metric space, ν is a fully supported positive Radon measure on $\widehat{E}$, $(\mathscr{E}^Y, \mathscr{F}^Y)$ is a regular Dirichlet form on $L^2(\widehat{E}; \nu)$ and Y is an associated diffusion Hunt process on $\widehat{E}$. E is now quasi-open and hence q.e. finely open in $\widehat{E}$.

Since $\check{X}$ is the part on E of Y, we can use [6, Theorem 3.3.8] to characterize its Dirichlet form (24) as

$$\check{\mathscr{F}} = \{u \in \mathscr{F}^Y : u = 0 \quad \text{q.e. on} \quad \widehat{E} \setminus E\}, \quad \check{\mathscr{E}} = \mathscr{E}^Y \text{ on } \check{\mathscr{F}} \times \check{\mathscr{F}},$$

where every function in $\mathscr{F}^Y$ is taken to be quasi continuous. This means that $\check{\mathscr{F}}$ is an ideal of $\mathscr{F}^Y$; if $u \in \check{\mathscr{F}}_b$, $v \in \mathscr{F}^Y_b$, then $uv \in \check{\mathscr{F}}_b$, in other words, $\mathscr{F}^Y$ is a Silverstein extension of $\check{\mathscr{F}}$.

We can then invoke [6, Theorem 6.6.9] about the maximality of the reflected Dirichlet space of $\check{\mathscr{E}}$ which equals $(\mathscr{F}^{\text{ref}}, \mathscr{E}^{\text{ref}})$ by virtue of Proposition 4.1 (ii) Thus we have

$$\mathscr{F}^Y \subset \mathscr{F}_a^{\text{ref}} \; (= \mathscr{F}^{\text{ref}} \cap L^2(E; \nu)),$$

But under the present assumption (1).

$$\mathscr{F}^{\text{ref}} = \mathscr{F}^*_e, \quad \mathscr{E}^{\text{ref}} = \mathscr{E}^*, \quad \mathscr{F}_a^{\text{ref}} = \check{\mathscr{F}}^*, \tag{31}$$

so that $\mathscr{F}^Y \subset \check{\mathscr{F}}^*$. As Y is assumed to be conservative while $\check{X}$ has a finite lifetime in view of Proposition 4.1 (i), $\check{\mathscr{F}}$ is a proper subspace of $\mathscr{F}^Y$. Hence we must have the identity $\mathscr{F}^Y = \check{\mathscr{F}}^*$. Since Y is a diffusion with no killing inside $\widehat{E}$, the regular Dirichlet form $(\mathscr{E}^Y, \mathscr{F}^Y)$ is strongly local so that $\mathscr{E}^Y(1, 1) = 0$. Consequently, for $w = u + c,\ u \in \check{\mathscr{F}},\ \mathscr{E}^Y(w, w) = \mathscr{E}^Y(u, u) = \check{\mathscr{E}}(u, u) = \mathscr{E}^*(w, w)$, yielding (30).

The assertion (ii) follows from (30), (31) and Theorem 5.1 (ii).

By (30), Y and $\check{X}$ generate the same strongly continuous Markovian semigroup on $L^2(E; \nu)$ yielding the assertion (iii). □

Here we give one remark on the above proof. Let $(\mathscr{E}, \mathscr{F})$ be a quasi-regular Dirichlet form and κ be the killing measure in the Beurling-Deny representation of $\mathscr{E}$. Theorem 6.6.9 in the book [6] by Z.-Q.Chen and the present author states that, among all of Silverstein extensions of $(\mathscr{E}, \mathscr{F})$, its reflected Dirichlet space is the maximal one. Actually this statement holds true under the condition that

$$\kappa = 0. \tag{32}$$

But the condition (32) is missing in that statement of [6]. We would like to take this opportunity to correct it by requiring the condition (32). Of course, (32) is fulfilled by the present strongly local Dirichlet form $(\check{\mathscr{E}}, \check{\mathscr{F}})$. In this connection, see also the proof of Theorem 7.1.6 in [6].

In accordance with [17, A.4], we say that a quadruplet $(E, m, \mathscr{E}, \mathscr{F})$ is a *Dirichlet space* if E is a Hausdorff topological space with a countable base, m is a σ-finite positive Borel measure on E and $\mathscr{E}$ with domain $\mathscr{F}$ is a Dirichlet form on $L^2(E; m)$. The inner product in $L^2(E; m)$ is denoted by $(\cdot, \cdot)_E$. For a given Dirichlet space $(E, m, \mathscr{E}, \mathscr{F})$, the notions of an *$\mathscr{E}$-nest*, an *$\mathscr{E}$-polar set*, an *$\mathscr{E}$-quasi-continuous numercal function* and '*$\mathscr{E}$-quasi-everywhere*' ('$\mathscr{E}$-q.e.' in abbreviation) are defined as in [6, Definition 1.2.12]. The *quasi-regularity* of the Dirichlet space is defined just as in [6, Definition 1.3.8]. We note that the space $\mathscr{F}_b = \mathscr{F} \cap L^\infty(E; m)$ is an algebra.

Given two Dirichlet spaces

$$(E, m, \mathcal{E}, \mathcal{F}), \qquad (\tilde{E}, \tilde{m}, \tilde{\mathcal{E}}, \tilde{\mathcal{F}}), \tag{33}$$

we call them *equivalent* if there is an algebraic isomorphism Φ from $\mathcal{F}_b$ onto $\tilde{\mathcal{F}}_b$ preserving three kinds of metrics: for $u \in \mathcal{F}_b$

$$\|u\|_\infty = \|\Phi u\|_\infty, \ (u, u)_E = (\Phi u, \Phi u)_{\tilde{E}}, \ \mathcal{E}(u, u) = \tilde{\mathcal{E}}(\Phi u, \Phi u).$$

One of the two equivalent Dirichlet spaces is called a *representation* of the other.

The underlying spaces E, $\tilde{E}$ of two Dirichlet spaces (33) are said to be *quasi-homeomorphic* if there exist $\mathcal{E}$-nest $\{F_n\}$, $\tilde{\mathcal{E}}$-nest $\{\tilde{F}_n\}$ and a one to one mapping q from $E_0 = \cup_{n=1}^\infty F_n$ onto $\tilde{E}_0 = \cup_{n=1}^\infty \tilde{F}_n$ such that the restriction of q to each F_n is a homeomorphism onto $\tilde{F}_n$. $\{F_n\}$, $\{\tilde{F}_n\}$ are called the *nests attached to the quasi-homemorphism* q. Any quasi-homeomorphism is quasi-notion-preserving.

We say that the equivalence Φ of two Dirichlet spaces (33) *is induced by a quasi-homeomorphism* q of the underlying spaces if

$$\Phi u(\tilde{x}) = u(q^{-1}(\tilde{x})), \quad u \in \mathcal{F}_b, \quad \tilde{m}\text{−a.e. } \tilde{x}.$$

Then $\tilde{m}$ is the image measure of m by q and $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ is the image Dirichlet form of $(\mathcal{E}, \mathcal{F})$ by q.

Theorem 6.2 *Assume that two Dirichlet spaces* (33) *are quasi-regular and that they are equivalent. Let* $X = (X_t, \mathbb{P}_x)$ *(resp.* $\tilde{X} = (\tilde{X}_t, \tilde{\mathbb{P}}_x)$*) be an* m*-symmetric right process on* E *(resp. an* $\tilde{m}$*-symmetric right process on* $\tilde{E}$*) properly associated with* $(\mathcal{E}, \mathcal{F})$ *on* $L^2(E; m)$ *(resp.* $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ *on* $L^2(\tilde{E}; \tilde{m})$*). Then the equivalence is induced by a quasi-homeomorphism* q *with attached nests* $\{F_n\}$, $\{\tilde{F}_n\}$ *such that* $\tilde{X}$ *is the image of* X *by* q *in the following sense: there exist an* m*-inessential Borel subset* N *of* E *containing* $\cap_{n=1}^\infty F_n^c$ *and an* $\tilde{m}$*-inessential Borel subset* $\tilde{N}$ *of* $\tilde{E}$ *containing* $\cap_{n=1}^\infty \tilde{F}_n^c$ *so that* q *is one to one from* $E \backslash N$ *onto* $\tilde{E} \backslash \tilde{N}$ *and*

$$\tilde{X}_t = q(X_t), \qquad \tilde{\mathbb{P}}_{\tilde{x}} = \mathbb{P}_{q^{-1}\tilde{x}}, \qquad \tilde{x} \in \tilde{E} \backslash \tilde{N}. \tag{34}$$

A proof of this theorem can be carried out as is explained at the end of Introduction. See Appendix of [8] for the details.

Proof of Theorem 6.1 **(iv)** By Theorem 6.1 (i), the two Dirichlet spaces

$$(\widehat{E}, \nu, \mathcal{E}^Y, \mathcal{F}^Y), \qquad (E^*, \nu, \mathcal{E}^*, \check{\mathcal{F}}^*)$$

are equivalent in the above sense by the identity map Φ from $\mathcal{F}_b^Y$ onto $\check{\mathcal{F}}_b^*$. Hence Theorem 6.1 (iv) follows from Theorem 6.2.

To be more precise, there exist a quasi-homeomorphism q with attached $\mathcal{E}^Y$-nest $\{F_n\}$ on $\widehat{E}$ and $\mathcal{E}^*$-nest $\{\widetilde{F}_n\}$ on E^*, a ν-inessential Borel set N with $\cap_{n=1}^\infty F_n^c \subset N \subset \widehat{E}$

for Y and a ν-inessential Borel set $\widetilde{N}$ with $\cap_{n=1}^{\infty}\widetilde{F}_n^c \subset \widetilde{N} \subset E^*$ for $\check{X}^*$ such that q is one to one from $\widehat{E} \setminus N$ onto $E^* \setminus \widetilde{N}$ and

$$\check{X}_t^* = q(Y_t), \qquad \tilde{\mathbf{P}}_{\tilde{x}}^* = \mathbf{P}_{q^{-1}\tilde{x}}^Y, \qquad \tilde{x} \in E^* \setminus \tilde{N}. \qquad \square$$

We note that the third assertion (iii) of Theorem 6.1 follows from the fourth one (iv) because the above map q preserves the ν-measure.

An analogous theorem to Theorem 6.1 has appeared in [5, Theorem 3.4] for the reflecting X on the closure of an Euclidean domain.

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