



One-point reflection

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Abstract

We examine symmetric extensions of symmetric Markov processes with one boundary point. Relationship among various normalizations of local times, entrance laws and excursion laws is studied. Dirichlet form characterization of elastic one-point reflection of symmetric Markov processes is derived. We give a direct construction of Walsh's Brownian motion as a one-point reflection together with its Dirichlet form characterization. This yields directly the analytic characterization of harmonic and subharmonic functions for Walsh's Brownian motion, recently obtained by Fitzsimmons and Kuter (2014) using a different method. We further study as a one-point reflection two-dimensional Brownian motion with darning (BMD).

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1. Introduction

Boundary theory for one-dimensional diffusions is now well understood thanks to the fundamental works of Feller, Itô and McKean. Much less is known for boundary theory of multi-dimensional diffusions and of Markov processes with discontinuous sample paths. Recently, satisfactory progress has been made for symmetric Markov processes, and more generally, for

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strong Markov processes having weak dual, with finitely many boundary points. See [25,13,14,9–11] and the references therein. In these works, Markovian extensions of the minimal processes are carried out through Poisson point processes of excursions via entrance laws and exit systems. They are further characterized in terms of resolvent representations, and in the symmetric case, in terms of Dirichlet forms.

In this paper, we examine symmetric extensions of symmetric Markov processes with one boundary point. To be more specific, let E be a locally compact separable metric space and m be a positive Radon measure on E of full support. We fix a non-isolated point $a \in E$ with $m\{a\} = 0$ and put $E_0 = E \setminus \{a\}$. Let $X^0 = (X_t^0, \zeta^0, \mathbf{P}_x^0)$ be an m symmetric Borel standard process on E_0 admitting no killing inside E_0 such that

$$(A.1) \quad \varphi(x) = \mathbf{P}_x^0(\zeta^0 < \infty, X_{\zeta^0-}^0 = a) > 0 \text{ for any } x \in E_0.$$

We call a Borel standard process X on E a *one-point extension* of X^0 to E or a *one-point reflection* of X^0 at a if X is m -symmetric and of no killings on $\{a\}$, and its subprocess obtained by killing upon hitting $\{a\}$ (called the *part* of X on E_0) is identical with X^0 in law. It is proved in [11, Theorem 7.5.4] that a one-point reflection X of X^0 at a is unique in law and the point a is regular for itself with respect to X in this case. The existence of the one-point reflection X of X^0 at a is also shown in [11, Theorem 7.5.6] for a Hunt process X^0 satisfying additional conditions (A.2), (A.3) as well as (A.4) in non-diffusion case that are stated in Section 3 of the present paper. Furthermore X becomes automatically a diffusion on E if so is X^0 on E_0 .

The purpose of this paper is four-fold. First, we work with a general Borel right process X on E and a point $a \in E$ which is X -regular for itself. We clarify the scaling relation between local time at a and its corresponding entrance law (Theorem 2.1). We then give a characterization of a local time being normalized in terms of its associated excursion law in the context of one-dimensional reflecting diffusion on an interval $[0, r_0)$ (Theorem 3.1).

Second, we deal with in Section 4 an *elastic one-point reflection* X , namely, a one-point extension of a symmetric Hunt process X^0 on E_0 to E that allows killings at a . Such an extension has been constructed in [13] in a more general setting by modifying the excursion law in a way to allow a jump to the cemetery. In particular, we derive in Section 4 a Dirichlet form characterization (Theorem 4.2) for the elastic one-point reflection, generalizing the corresponding one for the one-point reflection first obtained in [10]. We also clarify in this section that the law of the one-point reflection is not affected if the canonical entrance law in the construction is replaced with its constant multiple.

The third goal is to identify in Section 5 Walsh's Brownian motion X with the one-point reflection of a certain symmetric diffusion X^0 and determine its Dirichlet form. Walsh's Brownian motion, introduced in a descriptive way in [36], is a continuous Markov process on the plane \mathbb{R}^2 that lives on rays $\{R_\theta, 0 \leq \theta < 2\pi\}$ emanating from the origin $\mathbf{0}$. It behaves like one-dimensional Brownian motion along the ray away from the origin, and when it hits the origin $\mathbf{0}$, it “goes” in a random direction θ according to a probability measure $\eta(d\theta)$ on $[0, 2\pi)$. There have been several characterizations or constructions of it afterwards [34,2,35], and in particular Barlow–Pitman–Yor [1] gave a rigorous construction of its semigroup.

Let X^0 be the Brownian motion on each ray R_θ being killed upon hitting $\mathbf{0}$. X^0 is a diffusion on $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ that can be easily verified to be symmetric with respect to the product measure $m = \lambda \times \eta$ on $(0, \infty) \times [0, 2\pi)$ where λ is the Lebesgue measure on $(0, \infty)$. Clearly X^0 satisfies conditions (A.1), (A.2), (A.3). So there exists a unique one-point reflection X of X^0 at $\mathbf{0}$. We call X *Walsh's Brownian motion*. The semigroup $\{T_t, t > 0\}$ constructed in [1] is a Feller semigroup on $C_\infty(\mathbb{R}^2)$

whose associated Hunt process Y on \mathbb{R}^2 has X^0 as its part on $\mathbb{R}^2 \setminus \{\mathbf{0}\}$. The m -symmetry of T_t can be also readily seen so that Y becomes a diffusion automatically and coincides with X in law.

Such a straightforward definition of Walsh's Brownian motion along with its Dirichlet form characterization given in Theorem 5.2 of this paper are not only more direct but also capture the essence of its intuitive picture. The Dirichlet form characterization together with results from [7,15] quickly gives the analytic characterization of harmonicity and subharmonicity for Walsh's Brownian motion, which recovers the main result of a recent paper [19] by Fitzsimmons and Kuter.

Instead of taking a point a of E and putting $E_0 = E \setminus \{a\}$, we may take any compact subset K of E , put $E_0 = E \setminus K$, and consider the topological space $E_0^* = E_0 \cup \{a^*\}$ obtained from E by rendering the set K as a single point a^* . The restriction of m to E_0 is denoted by m_0 which is extended to E_0^* by setting $m_0(\{a^*\}) = 0$. For a given m -symmetric diffusion X on E , its part X^0 on E_0 is known to be m_0 -symmetric. As will be formulated in Section 6 under certain conditions on X and X^0 , there exists a unique one-point reflection X^* of X^0 at a^* . This procedure of getting X^* from X (or from X^0) is called *darning a hole* K .

When E is a domain of the complex plane \mathbb{C} and X is the absorbing Brownian motion on E , the diffusion X^* on $(E \setminus K) \cup \{a^*\}$ so obtained is called a *Brownian motion with darning* (BMD). In Section 6, we consider a special case where K is a connected set containing at least two points so that $\mathbb{C} \setminus K$ connected. The fourth goal of this paper is to show the conformal invariance of BMD and to identify, in this special case, BMD with the *excursion reflected Brownian motion* (ERBM) introduced by Lawler [30] and Drenning [16]. This is done by combining a quite analogous consideration to Section 5 with the conformal invariance of BMD. BMD has been formulated and characterized in a more general setting allowing many holes in E rather than a single hole K together with its Dirichlet form characterization [8,12]. It has played important roles in the study of Komatu–Loewner equations for multiply connected planar domains [12,23]. In Section 6, we also consider a *Walsh's Brownian motion with darning* (WBMD) obtained from Walsh's Brownian motion X on \mathbb{R}^2 by rendering a star-like compact set $K = \{(r, \theta) : r \leq \psi(\theta), \theta \in [0, 2\pi)\}$ into a single point. In contrast with a BMD, a WBMD is highly reducible until approaching to the hole K .

We end this introduction by pointing out that we can consider other types of Markov processes in the plane or higher dimensional spaces by adopting censored stable processes [5], for instance, in place of the absorbing Brownian motions in Sections 5 and 6.

2. Normalized local time and canonical entrance law

Let $X = (X_t, \zeta, \mathbf{P}_x)$ be a Borel right process on a Lusin space E with lifetime ζ and with cadlag paths up to ζ . We fix a point $a \in E$ which is X -regular for itself, that is,

$$\mathbf{P}_a(\sigma_a = 0) = 1. \quad (2.1)$$

Here $\sigma_a = \inf\{s > 0 : X_s = a\}$. To avoid triviality, we assume that $\mathbf{P}_x(\sigma_a < \infty)$ is not identically zero on $E_0 := E \setminus \{a\}$. Denote by $X^0 = (X_t^0, \zeta^0, \mathbf{P}_x^0)$ the part process of X killed upon leaving E_0 . The transition semigroup of X (resp. X^0) is denoted by $\{P_t\}$ (resp. $\{P_t^0\}$).

Since a is a regular point, there exists a positive continuous additive functional (PCAF) ℓ_t of X supported by $\{a\}$ in the sense that $\int_0^t I_{\{a\}}(X_s) d\ell_s = \ell_t$ (cf. [4, Theorem V.3.13]). A PCAF supported by $\{a\}$ is called a *local time* of X at the point a . Clearly, if ℓ is a local time at a , then so is $C\ell_t$ for any positive constant C .

Given a local time ℓ_t at a , its inverse $\tau_t = \inf\{s \geq 0 : \ell_s > t\}$ gives rise to an *excursion point process* $(\mathbf{p}, \mathbf{P}_a)$ in the following manner [28,14]. Let Ω be the space of cadlag paths from $[0, \infty)$ to $E \cup \Delta$, where Δ is an isolated point added to E serving as a cemetery point. Define the operators θ_t, k_t and i_t on Ω by

$$(\theta_t \omega)(s) = \omega(t + s), \quad (k_t \omega)(s) = \begin{cases} \omega(s) & s < t \\ \Delta & s \geq t \end{cases}, \quad i_t = k_{\sigma_a} \circ \theta_t.$$

We introduce spaces of excursions around $\{a\}$ by

$$\begin{cases} W = \{k_{\sigma_a}(\omega) : \omega \in \Omega, \sigma_a > 0\}, \\ W^+ = \{\omega \in W : \sigma_a(\omega) < \infty\}, \quad W^- = \{\omega \in W : \sigma_a(\omega) = \infty, \zeta > 0\}. \end{cases} \quad (2.2)$$

Then $W = W^+ \cup W^- \cup \{\partial\}$, where ∂ denotes the path identically taking value Δ . Define

$$\mathcal{D}_{\mathbf{p}(\omega)} = \{s \in (0, \infty) : \tau_{s-}(\omega) < \tau_s(\omega)\}, \quad \mathbf{p}_s(\omega) = i_{\tau_{s-}} \quad \text{for } s \in \mathcal{D}_{\mathbf{p}(\omega)}.$$

Note that $\{\mathbf{p}_s(\omega) : s \in \mathcal{D}_{\mathbf{p}(\omega)}\} \subset W$ and $\{\mathbf{p}_s(\omega) : s \in \mathcal{D}_{\mathbf{p}(\omega)}, s < \ell_\infty\} \subset W^+$. When $T = \ell_\infty < \infty$, then $\tau_{T-} < \infty$ and $\tau_T = \infty$. In this case, $T \in \mathcal{D}_{\mathbf{p}}$ and $\mathbf{p}_T(\omega) = \theta_{\tau_{T-}}(\omega) \in W^- \cup \{\partial\}$. Thus T is the only time of the occurrence of a non-returning excursion. By [18, Proposition 2.10], T is exponentially distributed with some parameter $\delta \in [0, \infty)$ under \mathbf{P}_a .

It follows from [32, Sections 1–2] (see also the proof of [14, Lemma 3.1]) that, under probability measure \mathbf{P}_a , \mathbf{p} is an absorbed Poisson point process with absorbing time T in the sense of Meyer [32].

Let \mathbf{n} be the characteristic measure of $(\mathbf{p}, \mathbf{P}_a)$, which is a σ -finite measure on the excursion space W . Define

$$\nu_t(B) = \mathbf{n}(w \in W : w(t) \in B, t < \sigma_a), \quad B \in \mathcal{B}(E_0).$$

Then $\{\nu_t\}$ is then an X^0 -entrance law in the sense that

$$\nu_s P_t^0 = \nu_{s+t} \quad \text{for } s, t > 0. \quad (2.3)$$

The family of measures $\{\nu_t\}$ is called the *entrance law associated with the local time ℓ_t* . Note that, if $\{\nu_t\}$ is an X^0 -entrance law, then so is $\{C\nu_t\}$ for any constant $C > 0$.

By the X^0 -entrance law $\{\nu_t\}$ associated with ℓ , the above characteristic measure \mathbf{n} is uniquely determined by

$$\begin{aligned} & \int_W f_1(w(t_1)) f_2(w(t_2)) \cdots f_n(w(t_n)) \mathbf{n}(dw) \\ &= \nu_{t_1} f_1 P_{t_2-t_1}^0 f_2 \cdots P_{t_{n-1}-t_{n-2}}^0 f_{n-1} P_{t_n-t_{n-1}}^0 f_n. \end{aligned} \quad (2.4)$$

The σ -finite measure \mathbf{n} is sometimes called the *excursion law associated with ℓ* .

Let m be an X -excessive measure, that is, m is σ -finite and $mP_t \leq m$ for every $t > 0$. The existence of m is a mild assumption; see [18, Theorem 4.5]. The *Revuz measure* ν_A of a PCAF A_t of X relative to an X -excessive measure m is defined by

$$\lim_{t \downarrow 0} \frac{1}{t} \mathbf{E}_m \left[\int_0^t f(X_s) dA_s \right] = \nu_A(f), \quad f \in \mathcal{B}_+(E).$$

By assumption (2.1), the set $\{a\}$ is not semipolar so that the Dirac measure δ_a concentrated on $\{a\}$ is smooth and admits a unique PCAF ℓ_t of X whose Revuz measure relative to an excessive

measure m is δ_a [17]:

$$\lim_{t \downarrow 0} \frac{1}{t} \mathbf{E}_m \left[\int_0^t f(X_s) d\dot{\ell}_s \right] = f(a), \quad f \in \mathcal{B}_+(E). \quad (2.5)$$

The PCAF $\dot{\ell}_t$ is called the *normalized local time of X at a relative to m* .

From now on, we assume that there exist a σ -finite measure m on E and a Borel right process $\widehat{X} = (\widehat{X}_t, \widehat{\zeta}, \widehat{\mathbf{P}}_x)$ on E such that X and \widehat{X} are in *weak duality relative to m* : for the transition function $\{\widehat{P}_t\}$ of \widehat{X} ,

$$\int_E \widehat{P}_t f(x) g(x) m(dx) = \int_E f(x) P_t g(x) m(dx) \quad \text{for every } t > 0 \text{ and } f, g \in \mathcal{B}^+(E). \quad (2.6)$$

Clearly, m has to be X -excessive. Consider the hitting probability of a for \widehat{X} :

$$\widehat{\varphi}(x) =: \widehat{\mathbf{P}}_x(\sigma_a < \infty), \quad x \in E.$$

Let m_0 be the restriction of m to $E_0 = E \setminus \{a\}$. Then $\widehat{\varphi} \cdot m_0$ is *purely X^0 -excessive* in the sense that $\lim_{t \uparrow \infty} (\widehat{\varphi} \cdot m_0) \cdot P_t^0 = 0$. By a theorem of P. Fitzsimmons [26], there exists a unique X^0 -entrance law $\{\mu_t\}$ such that

$$\widehat{\varphi} \cdot m_0 = \int_0^\infty \mu_t dt; \quad (2.7)$$

in other words,

$$\mu_t(dx) dt = \widehat{\mathbf{P}}_x(\sigma_a \in dt) m_0(dx). \quad (2.8)$$

We call $\{\mu_t\}$ the *canonical entrance law of X^0 at the point a relative to m_0* .

If

$$\widehat{\mathbf{P}}_x(\sigma_a < \infty \text{ and } \widehat{X}_{\sigma_a-} = a) = \widehat{\mathbf{P}}_x(\sigma_a < \infty) \quad \text{for } x \in E_0, \quad (2.9)$$

then $\widehat{\varphi}(x) = \widehat{\mathbf{P}}_x^0(\widehat{X}_{\widehat{\zeta}^0-}^0 = a, \widehat{\zeta}^0 < \infty)$ and consequently

$$\mu_t(dx) dt = \widehat{\mathbf{P}}_x^0(\widehat{\zeta}^0 \in dt, \widehat{X}_{\widehat{\zeta}^0-}^0 = a) m_0(dx). \quad (2.10)$$

Thus under assumption (2.9), the canonical entrance law $\{\mu_t\}$ of X^0 at a is determined by the dual minimal process \widehat{X}^0 .

Theorem 2.1. *Let $\dot{\ell}$ be the normalized local time of X at a relative to m and $\{\mu_t\}$ be the canonical entrance law of X^0 at a relative to m_0 . Then*

- (i) $\{\mu_t\}$ is the entrance law associated with $\dot{\ell}$.
- (ii) For any constant $C > 0$, $\{C^{-1}\mu_t\}$ is the entrance law associated with $C\dot{\ell}$.

Proof. When X is an m -symmetric diffusion, (i) of Theorem 2.1 was first proved in [25, Theorem 3.1, Remark 4.2]. In [25], the local time was always assumed to be a normalized one but a similar proof works to obtain also (ii) (by taking $CL(m_0, \psi)^{-1}$ in place of $L(m_0, \psi)^{-1}$ on the third line of page 452 of [25]).

In the present generality, (i) of Theorem 2.1 was first established in [14, Theorem 3.2] using the notion of an exit system (Q^*, ℓ) due to Maisonneuve [31] defined as follows. Fix any local

time ℓ of X at a . Let $M(\omega)$ be the closure of the visiting time set $\{t \in [0, \infty) : X_t(\omega) = a\}$. The complement of $M(\omega)$ in $[0, \infty)$ is a disjoint union of open intervals called *excursion intervals*. Denote by $G(\omega)$ the collection of left end points of those intervals. Then, by [31], there exists a σ -finite measure Q^* on W characterized by

$$\mathbf{E}_x \left[\sum_{s \in G} Z_s \cdot \Gamma \circ i_s \right] = Q^*(\Gamma) \cdot \mathbf{E}_x \left[\int_0^\infty Z_s d\ell_s \right], \quad x \in E, \quad (2.11)$$

for every non-negative predictable process Z_s and every non-negative random variable Γ on W . Define

$$\nu_t(B) = Q^*(w \in W : w(t) \in B, t < \sigma_a), \quad B \in \mathcal{B}(E_0).$$

$\{\nu_t, t > 0\}$ is then an X^0 -entrance law, which we call the *entrance law induced by the exit system* (Q^*, ℓ) . In [14, Proposition 2.1], the entrance law induced by the exit system $(Q^*, \dot{\ell})$ was identified with the canonical entrance law $\{\mu_t, t > 0\}$ of X^0 at a relative to m_0 . As a consequence, the entrance law induced by the exit system $(\frac{1}{C}Q^*, C\dot{\ell})$ can be identified with $\{\frac{1}{C}\mu_t, t > 0\}$ for any positive constant C . On the other hand, the excursion law \mathbf{n} associated with the local time ℓ can be identified with Q^* characterized by (2.11):

$$Q^* = \mathbf{n}. \quad (2.12)$$

Identity (2.12) was established in [14, Theorem 3.2] in the case that $\ell = \dot{\ell}$ but exactly the same proof works for any local time ℓ . In particular, the entrance law associated with a local time ℓ coincides with the entrance law induced by the exit system (Q^*, ℓ) and so we get the desired conclusion (i) and (ii). \square

As a matter of fact, Theorem 2.1 holds for any Borel right process X on E and for any point $a \in E$ that is X -regular for itself. Indeed, for each choice of an X -excessive measure m , it is known that m automatically admits an m -dual moderate Markov process as a substitute of \hat{X} in the above [17,18] so that the canonical entrance law $\{\mu_t\}$ of X^0 at a relative to m_0 still makes sense by the characterization (2.7). Fitzsimmons and Gettoor have identified in [18, (3.23)] the canonical entrance law with the entrance law induced by the exit system $(Q^*, \dot{\ell})$, and so (i) and (ii) of Theorem 2.1 follow from the identity (2.12).

From (2.11) and (2.12), we have the following characterization of the excursion law \mathbf{n} associated with a local time ℓ of X at a :

$$\mathbf{E}_x \left[\sum_{s \in G} Z_s \cdot \Gamma \circ i_s \right] = \mathbf{n}(\Gamma) \cdot \mathbf{E}_x \left[\int_0^\infty Z_s d\ell_s \right], \quad x \in E, \quad (2.13)$$

for every non-negative predictable process Z_s and every non-negative random variable Γ on W . This will be utilized in Section 3.

Example 2.2. Let $X = (X_t, \mathbf{P}_x)$ be one-dimensional reflecting Brownian motion on $E = [0, \infty)$ and $X^0 = (X_t^0, \zeta^0, \mathbf{P}_x^0)$ be the absorbing Brownian motion on $E_0 = (0, \infty)$. It is known that $a = 0$ is X -regular (i.e. $\mathbf{P}_0(\sigma_0 = 0) = 1$) and the Lebesgue measure $m = dx$ (resp. $m_0 = dx$) on $[0, \infty)$ (resp. $(0, \infty)$) is X (resp. X^0)-excessive.

Let $\dot{\ell}_t$ be the normalized local time of X at 0 relative to m and $\{\mu_t\}$ be the canonical entrance law of X^0 at 0 relative to m_0 .

Since $\mathbf{P}_x(\sigma_0 \in dt) = \mathbf{P}_0^{\text{BM}}(\sigma_x \in dt) = \frac{1}{(2\pi t^3)^{1/2}} x e^{-x^2/(2t)} dt$, we have from (2.8) or (2.10),

$$\mu_t(dx) = \frac{1}{(2\pi t^3)^{1/2}} x e^{-x^2/(2t)} m_0(dx). \quad (2.14)$$

By virtue of Theorem 2.1, $\{\mu_t\}$ is the entrance law of X at 0 associated with $\dot{\ell}_t$. Consequently, by Theorem 2.1(ii), $\{\tilde{\mu}_t := 2\mu_t\}$ is the entrance law of X at 0 associated with $\dot{\ell}_t := \frac{1}{2}\dot{\ell}_t$. Note that $\tilde{\ell}_t$ has been a *standard choice* of the local time of X at 0 by the following reason. The Dirichlet form of X on $L^2([0, \infty); dx)$ is $(\frac{1}{2}\mathbf{D}, H^1(0, \infty))$. For $u(x) = x$, which is in $H_{\text{b,loc}}^1(0, \infty)$, and for $v \in C_c^1([0, \infty))$, we have $v_{(u)}(dx) = dx$ and

$$\frac{1}{2} \int_0^\infty u'v' dx = \frac{1}{2} \int_0^\infty v' dx = -\frac{1}{2}v(0) = -\left\langle \frac{1}{2}\delta_0, v \right\rangle.$$

Thus Fukushima's decomposition [24, Theorem 5.5.5] yields the well-known *Lévy–Skorohod decomposition* of the RBM X [29, Section 4]:

$$X_t - X_0 = B_t + \tilde{\ell}_t, \quad (2.15)$$

where B_t is Brownian motion on \mathbb{R} with $B_0 = 0$. The measures $\tilde{\mu}_t = 2\mu_t$ has been adopted as a *standard choice* of the entrance law of the RBM on $[0, \infty)$ in [20,27,33,37]. The standard ones $\tilde{\ell}_t$, $\tilde{\mu}_t$ can be also considered as normalized and canonical ones relative to $2m$, respectively.

[28,3] adopted a different normalization of the local time: the *Blumenthal–Gettoor local time* $\widehat{\ell}_t$ at a characterized as in [4, pp. 217] by

$$\mathbf{E}_x \left[\int_0^\infty e^{-t} d\widehat{\ell}_t \right] = \mathbf{E}_x [e^{-\sigma_a}].$$

In the above example, let $C\delta_0$, $C > 0$, be the Revuz measure of $\widehat{\ell}_t$ relative to m . Denote by $R_\alpha(x, y)$ the resolvent density of the RBM X on $[0, \infty)$. By [24, (5.1.14)], we have for $u_1(x) = \mathbf{E}_x[e^{-\sigma_0}]$

$$u_1(y) = CR_1(0, y) = CR_1(y, 0) = Cu_1(y)R_1(0, 0) = \sqrt{2}Cu_1(y), \quad y > 0.$$

Hence $C = \frac{1}{\sqrt{2}}$ and $\widehat{\ell}_t = \frac{1}{\sqrt{2}}\dot{\ell}_t$. The entrance law associated with $\widehat{\ell}$ is therefore $\{\sqrt{2}\mu_t\}$.

3. Normalized excursion laws for reflecting diffusions

Let s and m be a canonical scale and a canonical measure on $(0, r_0)$, where $r_0 \in (0, \infty]$; s is a strictly increasing continuous function with $s(0+) = 0$ and m is a positive Radon measure of full support. We assume that the left boundary 0 is regular, while the right boundary r_0 is non-regular with respect to (s, m) . We extend the measure m to $[0, r_0)$ by setting $m(\{0\}) = 0$.

Define

$$\begin{cases} \mathcal{E}^s(u, v) = \int_0^{r_0} \frac{du}{ds} \frac{dv}{ds} ds \\ \mathcal{F}^s = \{u : \text{absolutely continuous in } ds \text{ on } (0, r_0) \text{ with } \mathcal{E}^s(u, u) < \infty\}. \end{cases}$$

$(\mathcal{E}^s, \mathcal{F}^s \cap L^2([0, r_0); m))$ is a regular symmetric strongly local irreducible Dirichlet form on $L^2([0, r_0), m)$. Since r_0 is assumed to be non-regular, its extended Dirichlet space is given by

$$\{u \in \mathcal{F}^s : u(r_0-) = 0 \text{ if } s(r_0) < \infty\}, \quad (3.1)$$

in view of [11, Theorem 2.2.11]. Let $X = (X_t, \zeta, \mathbf{P}_x)$ be the associated m -symmetric diffusion process on $[0, r_0)$ and $X^0 = (X_t^0, \zeta^0, \mathbf{P}_x^0)$ be the subprocess of X killed upon hitting the point 0. X^0 is then the minimal diffusion for (s, m) and X is the reflecting extension of X^0 at 0 (see [11, Example 3.5.7, Section 7.3 (3)]). The point 0 is X -regular for itself.

Take any local time ℓ of X at 0 and let \mathbf{n} be the excursion law associated with ℓ . By the strong Markov property of \mathbf{n} , we have for $0 < x < y < r_0$,

$$\begin{aligned} \mathbf{n}(\sigma_y < \infty) &= \mathbf{n}(\sigma_x < \infty, \sigma_y \circ \theta_{\sigma_x} < \infty) = \mathbf{n}\left(\mathbf{P}_{X_{\sigma_x}^0}^0(\sigma_y < \infty); \sigma_x < \infty\right) \\ &= \mathbf{P}_x^0(\sigma_y < \infty)\mathbf{n}(\sigma_x < \infty) = \frac{s(x)}{s(y)} \mathbf{n}(\sigma_x < \infty). \end{aligned}$$

So $c = s(x)\mathbf{n}(\sigma_x < \infty)$ does not depend on $x \in (0, r_0)$ and $\mathbf{n}(\sigma_x < \infty) = \frac{c}{s(x)}$, $x \in (0, r_0)$.

In his study of general boundary conditions for linear diffusions using excursion theory, K. Yano [38, (2.34)] adopts a special normalization $\mathbf{n}(\sigma_x < \infty) = \frac{1}{s(x)}$, $0 < x < r_0$, for the excursion law \mathbf{n} of X , which will now be shown to characterize the normalization of the local time of X at 0 given in the preceding section.

Theorem 3.1. *The local time ℓ at 0 of X is the normalized one relative to m if and only if the excursion law \mathbf{n} associated with ℓ satisfies*

$$\mathbf{n}(\sigma_x < \infty) = \frac{1}{s(x)} \quad \text{for any } x \in (0, r_0). \quad (3.2)$$

Proof. We first note that

$$\mathbf{P}_0(\sigma_x < \infty) = 1 \quad \text{for any } x \in (0, r_0). \quad (3.3)$$

This can be seen by looking at the transformed process $Y = s(X)$ of X by the scale function s . Let $I = [0, b)$ with $b = s(r_0)$ and

$$\mathbf{D}(u, v) = \int_I u'v' dx, \quad \text{BL}(I) = \{u : \text{absolutely continuous on } I, \mathbf{D}(u, u) < \infty\}.$$

Define $Y = (Y_t, \zeta, \{\mathbf{P}_x^Y\}_{x \in I})$ by $Y_t = s(X_t)$, $\mathbf{P}_x^Y = \mathbf{P}_{s^{-1}(x)}$. By virtue of [21, Lemma 4.3] and by taking (3.1) into account, one concludes that Y is symmetric with respect to the measure $\tilde{m} = 2m \circ s^{-1}$ on I , and its Dirichlet form $(\mathcal{E}^Y, \mathcal{F}^Y)$ on $L^2(I; \tilde{m})$ together with its extended Dirichlet space \mathcal{F}_e^Y admits the expression

$$\begin{aligned} \mathcal{E}^Y &= \frac{1}{2} \mathbf{D}, \quad \mathcal{F}^Y = \mathcal{F}_e^Y \cap L^2(I; \tilde{m}), \\ \mathcal{F}_e^Y &= \{u \in \text{BL}(I) : u(b-) = 0 \text{ whenever } b < \infty\}. \end{aligned}$$

This implies that Y can be obtained from the reflecting Brownian motion Z on $[0, \infty)$ by killing Z upon hitting the point b whenever $b < \infty$ followed by a time change using a strictly increasing PCAF in the strict sense with Revuz measure \tilde{m} . Therefore we have for $\sigma_x^Y = \inf\{t > 0 : Y_t = x\}$

$$\mathbf{P}_0(\sigma_x^Y < \infty) = \mathbf{P}_0^Y(\sigma_x^Y < \infty) = 1 \quad \text{for any } x \in (0, b),$$

which yields (3.3).

In view of [22, Lemma 4.1], for every $\alpha > 0$, the resolvent kernel R_α of X admits the reproducing kernel $g_\alpha(x, y)$ of $(\mathcal{F}^s \cap L^2([0, r_0); m), \mathcal{E}_\alpha^s)$ as its density relative to m that satisfies

$\sup\{g_\alpha(x, y) : x \in J, y \in [0, r_0)\} < \infty$ for any compact interval $J \subset [0, r_0)$. Hence we can apply the strict decomposition theorem [24, Theorem 5.5.5] to the function $s \in \mathcal{F}_{b,loc}^s$. The energy measure $d\mu_{(s)}$ equals $2ds$ on $[0, r_0)$. For $v \in C_c^1(I)$, $v \circ s \in \mathcal{F}^s$ and $\mathcal{E}^s(s, v \circ s) = \mathbf{D}(x, v) = -v(0) = -v(s(0))$. Since such function $v \circ s$ is \mathcal{E}_1^s -dense in \mathcal{F}^s , we have the decomposition

$$s(X_t) - s(X_0) = M_t + \dot{\ell}_t, \quad t \in [0, \zeta), \quad \mathbf{P}_x\text{-a.e. for any } x \in [0, r_0), \quad (3.4)$$

where M is a local MAF of X and $\dot{\ell}$ is a normalized local time of X at 0 relative to m . Let $\{T_n; n \geq 1\}$ be a sequence of stopping times so that $T_n \rightarrow \infty$ and $\{M_{t \wedge T_n}; t \geq 0\}$ is a martingale under \mathbf{P}_0 for every $n \geq 1$. Then we have by (3.4), $\mathbf{E}_0[s(X_{\sigma_x \wedge t \wedge T_n})] = \mathbf{E}_0[\dot{\ell}_{\sigma_x \wedge t \wedge T_n}]$. Because of (3.3), it follows from the bounded convergence theorem and monotone convergence theorem that

$$s(x) = \mathbf{E}_0[s(X_{\sigma_x})] = \lim_{n \rightarrow \infty} \lim_{t \rightarrow \infty} \mathbf{E}_0[s(X_{\sigma_x \wedge t \wedge T_n})] = \lim_{n \rightarrow \infty} \lim_{t \rightarrow \infty} \mathbf{E}_0[\dot{\ell}_{\sigma_x \wedge t \wedge T_n}] = \mathbf{E}_0[\dot{\ell}_{\sigma_x}].$$

On the other hand, by (2.13) and (3.3), we have for the excursion law \mathbf{n} associated with $\dot{\ell}$,

$$1 = \mathbf{P}_0(\sigma_x < \infty) = \mathbf{E}_0 \left[\sum_{s \in G} \mathbf{1}_{\{s < \sigma_x\}} \mathbf{1}_{\{\sigma_x \circ \theta_s < \sigma_0 \circ \theta_s\}} \right] = \mathbf{E}_0[\dot{\ell}_{\sigma_x}] \cdot \mathbf{n}(\sigma_x < \infty). \quad (3.5)$$

We thus have (3.2). Conversely, if ℓ is any local time at 0 of X with its associated excursion law \mathbf{n} , then the same calculation as that in (3.5) yields $\mathbf{E}_0[\ell_{\sigma_x}] \cdot \mathbf{n}(\sigma_x < \infty) = 1$. Since both ℓ and $\dot{\ell}$ are local times of X at 0, $\ell = c\dot{\ell}$ for some constant $c > 0$. This implies that

$$1 = c \mathbf{E}_0[\dot{\ell}_{\sigma_x}] \cdot \mathbf{n}(\sigma_x < \infty) = cs(x) \cdot \mathbf{n}(\sigma_x < \infty).$$

Hence (3.2) holds if and only if $c = 1$, namely $\ell = \dot{\ell}$. \square

4. Elastic one-point reflection

In this section, we first review a construction in [14] of a one-point extension X possibly with a killing at the boundary point a of a general symmetric Markov process X^0 using the canonical entrance law of X^0 at a . The process X can also be called an *elastic one-point reflection* of X^0 at a . We then study basic properties of the constructed X .

Let E be a locally compact separable metric space and m a positive Radon measure with full support on E . Let $E_\Delta = E \cup \Delta$ be the one point compactification of E . When E is compact, Δ is added as an isolated point. We fix a non-isolated point $a \in E$ with $m(\{a\}) = 0$. We consider an m -symmetric Hunt process $X^0 = (X_t^0, \zeta^0, \mathbf{P}_x^0)$ on $E_0 = E \setminus \{a\}$ satisfying the following three conditions. The quantities related to X^0 are designated by having the superscript 0. The resolvent of X^0 is denoted by G_α^0 .

(A.1) $\varphi(x) =: \mathbf{P}_x^0(\zeta^0 < \infty, X_{\zeta^0-}^0 = a) > 0$ for any $x \in E_0$.

(A.2) $\mathbf{P}_x^0(X_{\zeta^0-}^0 \in \{a, \Delta\}) = 1$ for any $x \in E$, regardless the length of ζ^0 .

(A.3) There exists a neighborhood of a such that $\inf_{x \in V} G_1^0 \varphi(x) > 0$ for any compact set $V \subset U \setminus \{a\}$.

When X^0 is not a diffusion, we assume an additional condition (A.4) on the jumping measure J_0 of X^0 that is given by $J_0(dx, dy) = N(x, dy)\mu_H(dx)$ in terms of the Lévy system (N, H) of X^0 (cf. [11]):

(A.4) Either $E \setminus U$ is compact for any neighborhood U of a in E , or for any neighborhood U_1 of a in E , there exists an open neighborhood U_2 of a in E with $\overline{U_2} \subset U_1$ such that

$$J_0(U_2 \setminus \{a\}, E_0 \setminus U_1) < \infty.$$

Let $\{\mu_t\}$ be the canonical entrance law of X^0 at a relative to m :

$$\int_0^\infty \mu_t dt = \varphi \cdot m. \quad (4.1)$$

Introduce the excursion space by

$$\begin{aligned} W &= \{w : \text{cadlag function from } (0, \zeta(w)) \text{ to } E_0 \\ &\quad \text{for some } \zeta(w) \in (0, \infty], w(0+) = a, w(\zeta(w)-) \in \{a, \Delta\}\} \\ W^+ &= \{w \in W : \zeta(w) < \infty, w(\zeta(w)-) = a\}, \quad W^- = W \setminus W^+, \end{aligned}$$

and define a measure \mathbf{n} on W by

$$\begin{aligned} &\int_W f_1(w(t_1)) f_2(w(t_2)) \cdots f_n(w(t_n)) \mathbf{n}(dw) \\ &= \mu_{t_1} f_1 P_{t_2-t_1}^0 f_2 \cdots P_{t_{n-1}-t_{n-2}}^0 f_{n-1} P_{t_n-t_{n-1}}^0 f_n. \end{aligned} \quad (4.2)$$

For functions u, v on E_0 , denote the integral $\int_{E_0} u(x)v(x)m(dx)$ by (u, v) . The *energy functional* $L^0(\varphi, v)$ of the X^0 -excessive measure $\varphi \cdot m$ and an X^0 -excessive function v is defined by

$$L^0(\varphi, v) = \lim_{t \downarrow 0} \frac{1}{t} (\varphi - P_t^0 \varphi, v).$$

By [14, Lemma 5.5], we then have

$$L^0(\varphi, 1 - \varphi) = \mathbf{n}(W^-) < \infty. \quad (4.3)$$

The equality in the above follows from (4.1) and $\mathbf{n}(\zeta > t; W^-) = \langle \mu_t, 1 - \varphi \rangle$, while the finiteness is due to our assumptions (A.3), (A.4) on X^0 .

We add to W a path ∂ identically equal to Δ and extend \mathbf{n} to $W \cup \{\partial\}$ by setting $\mathbf{n}(\{\partial\}) = \kappa$ for a fixed value $\kappa \geq 0$ indicating a killing rate at a . We then consider a Poisson point process \mathbf{p} defined on a certain probability space (Ω, \mathbf{P}) taking values in $W \cup \{\partial\}$ with the characteristic measure \mathbf{n} . Let \mathbf{p}^+ and \mathbf{p}^- be the range restrictions of \mathbf{p} to W^+ and $W^- \cup \{\partial\}$, respectively.

We define a subordinator J by

$$J(s) = \sum_{r \leq s} \zeta(\mathbf{p}_r^+) \quad \text{for } s > 0, \text{ and } J(0) = 0.$$

Let T be the first time of occurrence of the point process $\{\mathbf{p}_s^-, s > 0\}$. Then J and T are independent and T is exponentially distributed with rate $\mathbf{n}(W^-) + \kappa$.

By piecing together the returning excursions \mathbf{p}_s^+ at each jumping time of J until the time T and then joining the path \mathbf{p}_T^- , we can obtain an E -valued continuous process $(X_t^a, \zeta_\omega, \mathbf{P})$ with $X_0^a = a$ and $X_{\zeta_\omega-} \in \{a, \Delta\}$. Define

$$G_\alpha f(a) = \mathbf{E} \left[\int_0^{\zeta_\omega} e^{-\alpha t} f(X_t^a) dt \right], \quad \alpha > 0, f \in \mathcal{B}(E).$$

According to this way of constructing X_t^a , we have

$$G_\alpha f(a) = \frac{\mathbf{n}(f_\alpha)}{\alpha \int_0^\infty e^{-\alpha t} \mathbf{n}(t < \zeta < \infty, w(\zeta-) = a) dt + \mathbf{n}(W^-) + \kappa}, \quad (4.4)$$

where $f_\alpha = \int_0^{\zeta(w)} e^{-\alpha t} f(w(t)) dt$, $w \in W$. See Sections 5.1 and 5.2 of [14] for more details.

Theorem 4.1. (i) By patching X^0 and (X_t^a, \mathbf{P}) together, we have an m -symmetric right process $X = (X_t, \zeta, \mathbf{P}_x)$ on E possessing the resolvent

$$\begin{cases} G_\alpha f(a) = \frac{(u_\alpha, f)}{\alpha(u_\alpha, \varphi) + L^0(\varphi, 1 - \varphi) + \kappa}, \\ G_\alpha f(x) = G_\alpha^0 f(x) + u_\alpha(x) G_\alpha f(a), \quad x \in E_0. \end{cases} \quad (4.5)$$

(ii) The part process of X on E^0 equals X^0 . The singleton $\{a\}$ is regular for itself relative to X . The process X admits no sojourn at a nor jump from E_0 to a .

(iii) The sample path $\{X_t, 0 \leq t < \zeta\}$ is cadlag on $[0, \zeta)$, continuous when $X_t = a$ and satisfies

$$X_{\zeta-} \in \{a, \Delta\} \quad \text{when } \zeta < \infty.$$

(iv) If X^0 is a diffusion on E_0 , then X is a diffusion on E .

(v) For constants $C > 0$ and $\tilde{\kappa} \geq 0$, let $X^{C, \tilde{\kappa}}$ be the symmetric right process on E obtained as above like X but with the canonical entrance law $\{\mu_t\}$ and κ being replaced by $\{C^{-1}\mu_t\}$ and $\tilde{\kappa}$, respectively. Then, the statements (i)–(iv) hold for $X^{C, \tilde{\kappa}}$ but with κ in (4.5) being replaced by $C\tilde{\kappa}$. In particular, $X^{C, \tilde{\kappa}}$ has the same distribution as X if and only if $\tilde{\kappa} = C^{-1}\kappa$.

Proof. (i)–(iv) are taken from Theorem 5.15 in [14, Section 5], which was formulated under a more general setting that X^0 is a Borel standard process on E_0 possessing an m -weakly dual Borel standard process \hat{X}^0 on E_0 .

(v) If the canonical entrance law $\{\mu_t\}$ is replaced by $\{C^{-1}\mu_t\}$, then the characteristic measure \mathbf{n} specified by (4.2) is changed into $C^{-1}\mathbf{n}$ and so (v) follows from (4.4). \square

A right process $X = (X_t, \zeta, \mathbf{P}_x)$ on E is called a *one-point extension* or *one-point reflection* of X^0 if its part on E_0 equals X^0 , X is m -symmetric and X admits no killing at $\{a\}$. By virtue of [11, Theorem 7.5.4], a one-point reflection X of X^0 is unique in law. Theorem 4.1 with $\kappa = 0$ provides us with a construction of a one-point reflection X of X^0 . Theorem 4.1(iv) in particular implies that the one-point reflection of X^0 becomes automatically a diffusion process on E .

We next study the Dirichlet form and the L^2 -infinitesimal generator of the symmetric right process X on E constructed in Theorem 4.1(i) from X^0 . We recall some concepts introduced in [11] in relation to the process X^0 .

Let $(\mathcal{E}^0, \mathcal{F}^0)$ be the Dirichlet form of X^0 on $L^2(E_0, m)$ and $((\mathcal{F}^0)^{\text{ref}}, \mathcal{E}^{0, \text{ref}})$ be its reflected Dirichlet space; see [11] for related definitions. Define the active reflected Dirichlet space $(\mathcal{F}^0)_a^{\text{ref}}$ to be $(\mathcal{F}^0)^{\text{ref}} \cap L^2(E_0; m)$.

We then consider an operator \mathcal{L} on $L^2(E_0, m)$ specified by

$$f \in \mathcal{D}(\mathcal{L}) \quad \text{with } \mathcal{L}f = g \in L^2(E_0, m)$$

if and only if

$$f \in (\mathcal{F}^0)_a^{\text{ref}} \quad \text{with } \mathcal{E}^{0, \text{ref}}(f, v) = -(g, v), \quad v \in \mathcal{F}^0.$$

Denote by u_α the α -order approach probability of X^0 to a :

$$u_\alpha(x) = \mathbf{E}_x^0 \left[e^{-\alpha \zeta^0}; X_{\zeta^0-}^0 = a \right].$$

The flux $\mathcal{N}(f)(a)$ of $f \in \mathcal{D}(\mathcal{L})$ at a is then defined by

$$\mathcal{N}(f)(a) = \mathcal{E}^{0,\text{ref}}(f, u_\alpha) + (\mathcal{L}f, u_\alpha),$$

which is independent of $\alpha > 0$.

We say that an X^0 -q.e. finely continuous function f on E_0 has an X^0 -fine limit value $c \in \mathbb{R}$ at a if

$$\mathbf{P}_x^0 \left(\lim_{t \uparrow \zeta^0} f(X_t^0) = c \mid \zeta^0 < \infty, X_{\zeta^0-}^0 = a \right) = 1 \quad \text{for q.e. } x \in E_0. \quad (4.6)$$

Theorem 4.2. Let X be the m -symmetric right process on E constructed in Theorem 4.1 from X^0 .

- (i) Let $(\mathcal{E}, \mathcal{F})$ be the Dirichlet form of X on $L^2(E; m)$ and $(\mathcal{F}_e, \mathcal{E})$ be its extended Dirichlet space. Then \mathcal{F} is the linear subspace of $(\mathcal{F}^0)_a^{\text{ref}}$ spanned by \mathcal{F}^0 and u_α , and \mathcal{F}_e is the linear subspace of $(\mathcal{F}^0)^{\text{ref}}$ spanned by \mathcal{F}_e^0 and φ . Moreover, for $f = f_0 + c\varphi$, $f_0 \in \mathcal{F}_e^0$, $c \in \mathbb{R}$,

$$\mathcal{E}(f, f) = \mathcal{E}^{0,\text{ref}}(f, f) + c^2\kappa. \quad (4.7)$$

Denote by k the killing measure on E in the Beurling–Deny decomposition of \mathcal{E} . Then

$$k(\{a\}) = \kappa. \quad (4.8)$$

- (ii) Suppose the active reflected Dirichlet space $(\mathcal{F}^0)_a^{\text{ref}}$ of X^0 satisfies the following condition:

$$\text{If } f \in (\mathcal{F}^0)_a^{\text{ref}} \text{ admits an } X^0\text{-fine limit value } 0 \text{ at } a, \text{ then } f \in \mathcal{F}^0. \quad (4.9)$$

Let \mathcal{A} be the L^2 -infinitesimal generator of X . Then $f \in \mathcal{D}(\mathcal{A})$ if and only if $f \in \mathcal{D}(\mathcal{L})$, f admits an X^0 -fine limit value $f(a)$ at a and satisfies the boundary condition

$$\mathcal{N}(f)(a) + \kappa f(a) = 0. \quad (4.10)$$

In this case, $\mathcal{A}f = \mathcal{L}f$.

Proof. (i) Since X is an m -symmetric Borel right process, $(\mathcal{E}, \mathcal{F})$ is quasi-regular Dirichlet form on $L^2(E; m)$ and the transfer method as in the proof of [11, Theorem 7.5.4] applies. As the single point set $\{a\}$ is not m -polar, we have

$$\mathcal{F} = \{f_0 + cu_\alpha : f_0 \in \mathcal{F}^0, c \in \mathbb{R}\}, \quad \mathcal{F}_e = \{f_0 + c\varphi : f_0 \in \mathcal{F}_e^0, c \in \mathbb{R}\}.$$

On the other hand, we can verify as in [25, (2.19)] that

$$G_\alpha f(a) = \frac{(u_\alpha, f)}{\alpha(u_\alpha, \varphi) + \mathcal{E}(\varphi, \varphi)} \quad \text{for } f \in C_c(E). \quad (4.11)$$

In fact, we have for $w = G_\alpha f$, $g \in \mathcal{F}$ the equation $\mathcal{E}(w, g) + \alpha(w, g) = (f, g)$.

By Lemma 2.1.15 of [11], we can find a uniformly bounded sequence $g_n \in \mathcal{F}$ such that

$$\lim_{n \rightarrow \infty} g_n(x) = \varphi(x) \quad m\text{-a.e.}, \quad \lim_{n \rightarrow \infty} \mathcal{E}(g_n - \varphi, g_n - \varphi) = 0.$$

Substituting g_n in the above equation and letting $n \rightarrow \infty$, we get

$$\mathcal{E}(w, \varphi) + \alpha(w, \varphi) = (f, \varphi). \quad (4.12)$$

As $w = G_\alpha f \in \mathcal{F} \subset \mathcal{F}_e$, there is some $u_0 \in \mathcal{F}_e^0$ so that $w = u_0 + c\varphi$ with $c = w(a) = G_\alpha f(a)$. By the strong Markov property, we also have $w = G_\alpha^0 f + cu_\alpha$. Putting the above expressions for w in the first and second terms on the left hand side of (4.12), we get

$$c\mathcal{E}(\varphi, \varphi) + c\alpha(u_\alpha, \varphi) = (f, \varphi) - \alpha(G_\alpha^0 f, \varphi).$$

As the right hand side equals (f, u_α) , we have (4.11).

A comparison of (4.11) with (4.5) yields

$$\mathcal{E}(\varphi, \varphi) = L^0(\varphi, 1 - \varphi) + \kappa. \quad (4.13)$$

In order to deduce (4.7) and (4.8) from (4.13), we use again the transfer method: without loss of generality, we can assume that $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(E; m)$ associated with a Hunt process X on E .

We first note that the local part $\mu_{\langle \varphi \rangle}^c$ of the associated energy measure of $\varphi \in \mathcal{F}_e$ does not charge on the one-point set $\{a\}$:

$$\mu_{\langle \varphi \rangle}^c(\{a\}) = 0. \quad (4.14)$$

To see this, take a function $g \in \mathcal{F} \cap C_c(E)$ with $g = 1$ in a neighborhood of a and put $\tilde{\varphi}(x) = \varphi(x)g(x)$, $x \in E$. Then $\tilde{\varphi} \in \mathcal{F}_b$ and $\mu_{\langle \tilde{\varphi} \rangle}^c(\{a\}) = \mu_{\langle \tilde{\varphi} \rangle}^c(\{a\}) = 0$ in view of [11, Theorem 4.3.10]. Owing to the energy image density property formulated in [11, Theorem 4.3.8], $\mu_{\langle \tilde{\varphi} \rangle}^c(\tilde{\varphi}^{-1}(1)) = 0$, which implies (4.14) because $\{a\} \subset \tilde{\varphi}^{-1}(1)$.

In view of Theorem 4.1 (ii), X admits no jump from E_0 to a so that Theorem 7.1.6 of [11] applies. By combining this theorem with (4.14), we can conclude that $\mathcal{F}_e \subset (\mathcal{F}^0)^{\text{ref}}$ and, for $f \in \mathcal{F}_e$, $f = f_0 + c\varphi$, $f_0 \in \mathcal{F}_e^0$, $c \in \mathbb{R}$,

$$\mathcal{E}(f, f) = \mathcal{E}^{0, \text{ref}}(f, f) + c^2 k(\{a\}), \quad \mathcal{E}^{0, \text{ref}}(f, f) = \mathcal{E}^0(f_0, f_0) + c^2 V(\{a\}), \quad (4.15)$$

where $V(\{a\})$ is the supplementary Feller measure at the point a as the boundary of E_0 . According to the definition [11, (5.5.7)] of V , we have $V(\{a\}) = L^0(\varphi, 1 - \varphi)$. Therefore we are led to (4.7) and (4.8) from (4.13) and (4.15).

(ii) This follows from [11, Theorem 7.3.5] on account of (i) and (4.14). \square

When $\kappa > 0$, the process X constructed in Theorem 4.1 can be obtained from the one-point reflection of X^0 by killing it with rate κ at a by virtue of (4.8) and [11, Theorem 5.1.2]. Thus when $\kappa > 0$, we can call X the *elastic reflection of X^0 at a* .

When X^0 is the absorbing Brownian motion on $(0, \infty)$, then $\mathcal{N}(f)(0) = -\frac{1}{2}f'(0)$. If we employ the standard choice $\tilde{\mu}_t = 2\mu_t$ of the entrance law in the construction of X , then κ in the expression (4.5) of the resolvent of X is changed into $\frac{1}{2}\kappa$ by Theorem 4.1(v) so that its boundary condition (4.10) is reduced to $f'(0) = \kappa f(0)$, the *elastic boundary condition* with killing rate κ . For $\kappa = 0$, the constructed process X^r is the RBM on $[0, \infty)$.

Let X^r be the RBM on $[0, \infty)$, $\tilde{\ell}_t = \frac{1}{2}\dot{\ell}_t$ be its standard local time at 0 and T be an independent exponential holding time with rate κ . In [29] the diffusion X on $[0, \infty)$ with the stated elastic boundary condition was constructed by killing X^r upon the first time that $\tilde{\ell}_t$ exceeds T , in other words, at time $J(T-)$ for the inverse J of $\tilde{\ell}$. Since the entrance law of the characteristic measure \mathbf{n} for J equals $\{\tilde{\mu}_t\}$, this procedure matches what we have done above in constructing X from X^0 by starting with $\{\tilde{\mu}_t\}$ instead of $\tilde{\ell}$.

5. Walsh's Brownian motion as a one-point reflection

Consider the case where $E = \mathbb{R}^2$, $a = \mathbf{0}$ the origin and $E_0 = \mathbb{R}^2 \setminus \{\mathbf{0}\}$. The punctured plane E_0 can be represented in terms of the polar coordinate: $E_0 = \{x = (r, \theta) : r \in (0, \infty), \theta \in [0, 2\pi)\}$. Let $\lambda(dr) = dr$ be the Lebesgue measure on $(0, \infty)$ and η be an arbitrary probability measure on $[0, 2\pi)$. We equip $E_0 = (0, \infty) \times [0, 2\pi)$ with the product measure $m = \lambda \times \eta$. For simplicity, we assume that $S = \text{Supp}[\eta] = [0, 2\pi)$. (Otherwise it suffices to set $E_0 = (0, \infty) \times S$.)

Let $X^0 = (X_t^0, \zeta^0, \mathbf{P}_x^0)$ be the diffusion process on E_0 that behaves, when started from $x = (r, \theta) \in E_0$, as the one-dimensional absorbing Brownian motion on the ray R_θ connecting (r, θ) with the origin $\mathbf{0}$. Denote by $\{p_t^0\}$ the transition semigroup of the absorbing Brownian motion on $(0, \infty)$ that is symmetric with respect to $\lambda(dr) = dr$. For a function f on E_0 , write $f_\theta(r) = f(r, \theta)$, $r > 0$, $\theta \in [0, 2\pi)$. Then the transition semigroup $\{P_t^0\}$ of X^0 satisfies $(P_t^0 f)(r, \theta) = (p_t^0 f_\theta)(r)$ and we have by Fubini's theorem that for every $f \in \mathcal{B}_+(E_0)$

$$\begin{aligned} \int_{E_0} f(x) P_t^0 g(x) m(dx) &= \int_0^{2\pi} \eta(d\theta) \int_0^\infty f_\theta(r) (p_t^0 g_\theta)(r) dr \\ &= \int_0^{2\pi} \eta(d\theta) \int_0^\infty (p_t^0 f_\theta)(r) g_\theta(r) dr = \int_{E_0} P_t^0 f(x) g(x) m(dx). \end{aligned}$$

Thus X^0 is m -symmetric. Clearly X^0 satisfies conditions (A.1)–(A.3). We extend m on $E_0 = \mathbb{R}^2 \setminus \{\mathbf{0}\}$ to $E = \mathbb{R}^2$ by setting $m(\mathbf{0}) = 0$. Therefore there exists a unique one-point reflection $X = (X_t, \zeta, \mathbf{P}_x)$ of X^0 at $\mathbf{0}$. On account of (2.10) and (2.14), the X^0 -canonical entrance law $\{\mu_t\}$ to produce X from X^0 is given by

$$\mu_t(dx) = \frac{1}{(2\pi t^3)^{1/2}} r e^{-r^2/(2t)} dr \cdot \eta(d\theta). \quad (5.1)$$

By virtue of Theorem 4.1(v) with $\kappa = \tilde{\kappa} = 0$, we can take any positive constant multiple of μ_t as an entrance law.

We call X *Walsh's Brownian motion* (WBM in abbreviation). The entrance law (5.1) and the construction of X from X^0 given in Section 4 (with $\kappa = 0$) fit well the intuitive description of the motion that Walsh gave in [36, Epilogue], which says it moves like a one-dimensional Brownian motion along the ray but, upon hitting the origin, it reflects in a random direction θ with a given distribution η on $[0, 2\pi)$. Describing the motion in terms of Itô's excursion theory, he argued, would give the reader a feeling of watching a butterfly being assassinated with an elephant gun. Now the notion of the one-point reflection provides us with a neat apparatus to capture the butterfly alive.

Walsh's Brownian motion has previously been rigorously constructed or characterized by several authors using different approaches [34, 2, 35]. In particular Barlow–Pitman–Yor [1] gave a rigorous construction of its semigroup. Let $\{p_t^r\}$ be the semigroup of the reflecting Brownian motion on $[0, \infty)$ that is symmetric with respect to $\lambda(dr) = dr$. Define for $f \in C_\infty(\mathbb{R}^2)$

$$\begin{aligned} P_t f(\mathbf{0}) &= p_t^r \bar{f}(0), & P_t f(r, \theta) &= p_t^r \bar{f}(r) + p_t^0 (f_\theta - \bar{f})(r), \\ \bar{f}(r) &= \frac{1}{2\pi} \int_0^{2\pi} f(r, \theta) d\theta. \end{aligned} \quad (5.2)$$

It was then shown in [1] that $\{P_t\}$ is a Feller semigroup on $C_\infty(\mathbb{R}^2)$ and that the associated Hunt process Y on \mathbb{R}^2 admits X^0 as its part on E_0 , namely, X^0 is obtained from Y by killing upon hitting $\mathbf{0}$.

In the same way as we have verified for P_t^0 in the above, we can use Fubini's theorem again to check that the semigroup P_t defined by (5.2) is symmetric with respect to $m = \lambda \times \eta$. Therefore Y is a one-point reflection of X^0 at $\mathbf{0}$ so that it automatically becomes a diffusion on \mathbb{R}^2 and coincides in law with our X .

Let $(\mathcal{E}^0, \mathcal{F}^0)$ be the Dirichlet form of X^0 on $L^2(\mathbb{R}^2 \setminus \{\mathbf{0}\}, m)$. Using $\{P_t^0\}$, we immediately obtain

$$C_c^\infty(\mathbb{R}^2 \setminus \{\mathbf{0}\}) \subset \mathcal{F}^0, \quad \mathcal{E}^0(f, g) = \frac{1}{2} \int_0^{2\pi} \mathbf{D}(f_\theta, g_\theta) \eta(d\theta), \quad (5.3)$$

where $\mathbf{D}(f, g) = \int_0^\infty f'g' dx$.

We introduce the following spaces of functions on the interval $(0, \infty)$:

$$\begin{cases} \text{BL}((0, \infty)) = \{f : \text{absolutely continuous on } (0, \infty), \mathbf{D}(f, f) < \infty\}, \\ H_{0e}^1((0, \infty)) = \{f \in \text{BL}((0, \infty)) : f(0+) = 0\}, \\ H^1((0, \infty)) = \text{BL}((0, \infty)) \cap L^2((0, \infty)), \\ H_{0e}^1((0, \infty)) = H_{0e}^1((0, \infty)) \cap L^2((0, \infty)). \end{cases}$$

Notice that, in view of [11, p 352], the Dirichlet space (resp. the extended Dirichlet space) of $X^0|_{R_\theta}$ for each $\theta \in [0, 2\pi)$ is given by $H_{0e}^1((0, \infty))$ (resp. $H_{0e}^1((0, \infty))$) which is a Hilbert space with inner product $\frac{1}{2}\mathbf{D}_2$ (resp. $\frac{1}{2}\mathbf{D}$). Here $\mathbf{D}_\alpha(f, g) = \mathbf{D}(f, g) + \alpha \int_0^\infty fg dx$, $\alpha > 0$.

The Dirichlet form $(\mathcal{E}^0, \mathcal{F}^0)$ is transient as it is associated with the transient semigroup $\{P_t^0\}$ of X^0 . Denote by $(\mathcal{F}_e^0, \mathcal{E}^0)$ its extended Dirichlet space. Define the space (\mathcal{G}, a) by

$$\mathcal{G} = \left\{ f : f_\theta \in H_{0e}^1((0, \infty)), \text{ for } \eta\text{-a.e. } \theta, \int_0^{2\pi} \mathbf{D}(f_\theta, f_\theta) \eta(d\theta) < \infty \right\}. \quad (5.4)$$

$$a(f, g) = \frac{1}{2} \int_0^{2\pi} \mathbf{D}(f_\theta, g_\theta) \eta(d\theta), \quad f, g \in \mathcal{G}. \quad (5.5)$$

We denote by G^0 the 0-order resolvent of X^0 : $G^0 f(x) = \int_0^\infty P_t^0 f(x) dt$ for any bounded Borel function f on $\mathbb{R}^2 \setminus \{\mathbf{0}\}$.

The next theorem gives explicit description the Dirichlet space \mathcal{F}^0 , extended Dirichlet space \mathcal{F}_e^0 and reflected Dirichlet space $\mathcal{F}^{0, \text{ref}}$ of X^0 .

Theorem 5.1. (i) *It holds that*

$$(\mathcal{F}_e^0, \mathcal{E}^0) = (\mathcal{G}, a). \quad (5.6)$$

Moreover, $C_c^\infty(\mathbb{R}^2 \setminus \{\mathbf{0}\})$ is \mathcal{E}^0 -dense in \mathcal{F}_e^0 .

(ii) *It holds that*

$$\mathcal{F}^0 = \left\{ f : f_\theta \in H_{0e}^1((0, \infty)) \text{ for } \eta\text{-a.e. } \theta, \int_0^{2\pi} \mathbf{D}_1(f_\theta, f_\theta) \eta(d\theta) < \infty \right\}. \quad (5.7)$$

$$\mathcal{E}_\alpha^0(f, g) = \frac{1}{2} \int_0^{2\pi} \mathbf{D}_{2\alpha}(f_\theta, g_\theta) \eta(d\theta) \quad \text{for } f, g \in \mathcal{F}^0, \alpha > 0. \quad (5.8)$$

Moreover, $C_c^\infty(\mathbb{R}^2 \setminus \{\mathbf{0}\})$ is \mathcal{E}_1^0 -dense in \mathcal{F}^0 .

(iii) Let $((\mathcal{F}^0)^{\text{ref}}, \mathcal{E}^{0,\text{ref}})$ be the reflected Dirichlet space of $(\mathcal{E}^0, \mathcal{F}^0)$ and $(\mathcal{F}^0)_a^{\text{ref}} = (\mathcal{F}^0)^{\text{ref}} \cap L^2(\mathbb{R}^2 \setminus \{\mathbf{0}\})$ be its active reflected Dirichlet space. Then

$$(\mathcal{F}^0)^{\text{ref}} = \left\{ f : f_\theta \in \text{BL}((0, \infty)) \text{ for } \eta\text{-a.e. } \theta, \int_0^{2\pi} \mathbf{D}(f_\theta, f_\theta) \eta(d\theta) < \infty \right\}. \quad (5.9)$$

$$(\mathcal{F}^0)_a^{\text{ref}} = \left\{ f : f_\theta \in H^1((0, \infty)) \text{ for } \eta\text{-a.e. } \theta, \int_0^{2\pi} \mathbf{D}_1(f_\theta, f_\theta) \eta(d\theta) < \infty \right\}. \quad (5.10)$$

$$\mathcal{E}^{0,\text{ref}}(f, g) = \frac{1}{2} \int_0^{2\pi} \mathbf{D}(f_\theta, g_\theta) \eta(d\theta) \quad \text{for } f, g \in (\mathcal{F}^0)^{\text{ref}}. \quad (5.11)$$

Proof. When the support of the measure η is a finite set, this theorem has been proved in [10, Section 7.6(3)]. But here η is a general probability measure on $[0, 2\pi)$.

(i) We first establish (5.6). By virtue of [11, Theorem 2.1.14], it suffices to prove that

(a) \mathcal{G} is a real Hilbert space with inner product a .

(b) For any $f \in C_c(\mathbb{R}^2 \setminus \{\mathbf{0}\})$,

$$G^0 f \in \mathcal{G} \quad \text{and} \quad a(G^0 f, v) = \int_{(\mathbb{R}^2 \setminus \{\mathbf{0}\})} f v dm \quad \text{for any } v \in \mathcal{G}. \quad (5.12)$$

To prove (a), suppose $\{u_n\}$, $n \geq 1$, is an a -Cauchy sequence in \mathcal{G} . Taking a subsequence if needed, we may and do assume that

$$a(u_{n+1} - u_n, u_{n+1} - u_n) \leq 2^{-n}.$$

We then have

$$\int_0^{2\pi} \sum_{n=1}^{\infty} \left(\int_0^{\infty} \left(\frac{\partial}{\partial r} (u_{n+1} - u_n)(r, \theta) \right)^2 dr \right)^{1/2} \eta(d\theta) < \infty.$$

This in particular implies that there is a set $A \subset [0, 2\pi)$ with $\eta(A) = 0$ so that, for every $\theta \in [0, 2\pi) \setminus A$, $u_n(\cdot, \theta) \in H_{0e}^1((0, \infty))$ for any $n \geq 1$ and

$$\sum_{n=1}^{\infty} \left(\int_0^{\infty} \left(\frac{\partial}{\partial r} (u_{n+1} - u_n)(r, \theta) \right)^2 dr \right)^{1/2} < \infty.$$

It follows that for each $\theta \in [0, 2\pi) \setminus A$, $u_n(r, \theta)$ converges locally uniformly (this follows from Cauchy–Schwarz inequality) on $[0, \infty)$ and in \mathbf{D} -norm to some function $v_\theta \in H_{0e}^1((0, \infty))$. Moreover, by Fatou's Lemma, we have

$$\int_0^{2\pi} \mathbf{D}(v_\theta, v_\theta) \eta(d\theta) < \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} \int_0^{2\pi} \mathbf{D}(u_{n,\theta} - v_\theta, u_{n,\theta} - v_\theta) \eta(d\theta) = 0,$$

in other words, $v(r, \theta) = v_\theta(r) \in \mathcal{G}$ and u_n is a -convergent to v .

To show (b), we consider the reproducing kernel $\{g^0(r, s), 0 < r, s < \infty\}$ of the space $(H_{0e}^1((0, \infty)), \frac{1}{2}\mathbf{D})$ (cf. [22]):

$$g^0(\cdot, s) \in H_{0e}^1((0, \infty)) \quad \text{and} \quad \frac{1}{2} \mathbf{D}(g^0(\cdot, s), v) = v(s) \\ \text{for } s \in (0, \infty), v \in H_{0e}^1((0, \infty)).$$

$g^0(r, s)$ is positive and continuous on $(0, \infty) \times (0, \infty)$. For any $f \in C_c(\mathbb{R}^2 \setminus \{\mathbf{0}\})$, $(G^0 f)_\theta(r) = \int_0^\infty g^0(r, s) f_\theta(s) ds$ so that $(G^0 f)_\theta(0+) = 0$ and

$$\frac{1}{2} \mathbf{D}((G^0 f)_\theta, (G^0 f)_\theta) = \int_0^\infty \int_0^\infty g^0(s, s') f_\theta(s) f_\theta(s') ds ds'.$$

Hence $G^0 f \in \mathcal{G}$ and $a(G^0 f, G^0 f) = \int_{\mathbb{R}^2 \setminus \{\mathbf{0}\}} f(x)(G^0 f)(x) m(dx)$. The equation in (5.12) can be obtained in the same way.

Next suppose $u \in \mathcal{G}$ satisfies $a(u, v) = 0$ for any $v \in C_c^\infty(\mathbb{R}^2 \setminus \{\mathbf{0}\})$. Taking $v(r, \theta) = \phi(r)\psi(\theta)$ with $\phi \in C_c^\infty(0, \infty)$ and $\psi \in C^\infty(\mathbb{S})$, where \mathbb{S} is the unit circle, we have

$$\int_0^{2\pi} \mathbf{D}(u_\theta, \phi) \psi(\theta) \eta(d\theta) = 0.$$

Since this holds for any $\psi \in C^\infty(\mathbb{S})$, there exists $A \subset \mathbb{S}$ with $\eta(A) = 0$ such that, for every $\theta \in \mathbb{S} \setminus A$, $u_\theta \in H_{0e}^0((0, \infty))$ and $\mathbf{D}(u_\theta, \phi) = 0$ for a countable family of $\phi \in C_c^\infty(0, \infty)$ uniformly dense in $C_c^\infty(0, \infty)$. It follows that $u_\theta = 0$, $\theta \in \mathbb{S} \setminus A$ and hence $u = 0$ m -a.e. This proves that $C_c^\infty(\mathbb{R}^2 \setminus \{\mathbf{0}\})$ is a -dense in \mathcal{G} .

(ii) (5.7) and (5.8) follow from (i) as $\mathcal{F}^0 = \mathcal{F}_e^0 \cap L^2(\mathbb{R}^2 \setminus \{\mathbf{0}\}, m)$ and $\mathcal{E}_\alpha^0(u, v) = \mathcal{E}^0(u, v) + \alpha \int_{\mathbb{R}^2 \setminus \{\mathbf{0}\}} uv dm$. The last statement can be shown as the proof of the corresponding one in (i) by using (5.8) in place of (5.5).

(iii) Follows immediately from (ii), [11, Theorem 6.2.5], and the fact that $(\mathcal{E}^0, \mathcal{F}^0)$ is a local Dirichlet form. \square

Define the approaching probability to $\mathbf{0}$ by X^0 and its α -order version by

$$\begin{cases} \varphi(x) = \mathbf{P}_x^0(\zeta^0 < \infty, X_{\zeta^0-}^0 = \mathbf{0}) \\ u_\alpha(x) = \mathbf{E}_x^0[e^{-\alpha\zeta^0}; X_{\zeta^0-}^0 = \mathbf{0}], \end{cases} \quad x \in \mathbb{R}^2 \setminus \{\mathbf{0}\}, \alpha > 0.$$

Let $(\mathcal{E}, \mathcal{F})$ be the Dirichlet form of the Walsh's Brownian motion X on $L^2(\mathbb{R}^2; m)$ and $(\mathcal{F}_e, \mathcal{E})$ be its extended Dirichlet space. By virtue of Theorem 4.2 with $\kappa = 0$, \mathcal{F}_e (resp. \mathcal{F}) is the linear subspace of $\mathcal{F}^{0, \text{ref}}$ (resp. $\mathcal{F}_a^{0, \text{ref}}$) spanned by \mathcal{F}_e^0 and φ (resp. \mathcal{F}^0 and u_α). Since φ equals identically 1 on $\mathbb{R}^2 \setminus \{\mathbf{0}\}$, we can deduce from Theorem 5.1 the following theorem.

Theorem 5.2. *It holds that*

$$\mathcal{F}_e = \left\{ f : \text{for } \eta\text{-a.e. } \theta, f_\theta \in \text{BL}((0, \infty)) \text{ and } f_\theta(0+) = c \right. \\ \left. \text{for some } c \text{ independent of } \theta, \int_0^\infty \mathbf{D}(f_\theta, f_\theta) \eta(d\theta) < \infty \right\}, \quad (5.13)$$

$$\mathcal{E}(f, g) = \frac{1}{2} \int_0^{2\pi} \mathbf{D}(f_\theta, g_\theta) \eta(d\theta), \quad f, g \in \mathcal{F}_e. \quad (5.14)$$

$$\mathcal{F} = \left\{ f : \text{for } \eta\text{-a.e. } \theta, f_\theta \in H^1((0, \infty)) \text{ and } f_\theta(0+) = c \right. \\ \left. \text{for some } c \text{ independent of } \theta, \int_0^\infty \mathbf{D}_1(f_\theta, f_\theta) \eta(d\theta) < \infty \right\}, \quad (5.15)$$

$$\mathcal{E}_\alpha(f, g) = \frac{1}{2} \int_0^{2\pi} \mathbf{D}_{2\alpha}(f_\theta, g_\theta) \eta(d\theta), \quad f, g \in \mathcal{F}. \quad (5.16)$$

Moreover

$$\mathcal{F} = \{f = f_0 + cu_\alpha : f_0 \in \mathcal{F}^0, c \in \mathbb{R}\}, \quad (5.17)$$

and, $(\mathcal{E}, \mathcal{F})$ is a strongly local, irreducible recurrent and regular Dirichlet form on $L^2(\mathbb{R}^2; m)$.

Property (5.15) is a consequence of (5.13) and $\mathcal{F} = \mathcal{F}_e \cap L^2(\mathbb{R}^2; m)$. The function $u_\alpha(r, \theta) = u_\alpha(r)$ is independent of θ and $u_\alpha(r)$, $r \in (0, \infty)$, is a decreasing function belonging to the space $H^1((0, \infty))$ so that $u_\alpha \in C_\infty(\mathbb{R}^2)$. Therefore the regularity of the Dirichlet form $(\mathcal{E}, \mathcal{F})$ follows from (5.17) and Theorem 5.1(ii).

Remark 5.3. Let A be a non-empty Borel subset of $[0, 2\pi)$. Then $\Gamma = (0, \infty) \times A \subset E_0$ is X^0 -finely open, and $m(\Gamma) = 0$ if and only if $\eta(A) = 0$. Hence Γ is m -polar with respect to X^0 if and only if $\eta(A) = 0$ in view of [11, Theorem A.2.13]. Moreover, it is easy to see from the description of X , a set $\Gamma \subset E_0$ is m -polar if and only if there is a Borel set $A \subset [0, 2\pi)$ with $\eta(A) = 0$ so that $\Gamma \subset (0, \infty) \times A$. Take any function f in the space $(\mathcal{F}^0)_a^{\text{ref}}$ of (5.10). We can then see from the above observation that f has an X^0 -fine limit value $c \in \mathbb{R}$ at $\mathbf{0}$ in the sense of (4.6) if and only if $f_\theta(0+) = c$ for η -a.e. θ . In particular, the space $(\mathcal{F}^0)_a^{\text{ref}}$ satisfies the condition (4.9) imposed in Theorem 4.2(ii) on account of the description (5.7) of the space \mathcal{F}^0 . \square

Let \mathcal{A} be the L^2 -infinitesimal generator of X . Theorem 4.2(ii) therefore applies and we can deduce from Theorem 5.1 the following characterization of \mathcal{A} :

Theorem 5.4. $f = f(r, \theta) \in \mathcal{D}(\mathcal{A})$ if and only if the following three conditions are satisfied:

- (i) f belongs to the space \mathcal{F} specified by (5.15).
- (ii) f'_θ is absolutely continuous for η -a.e. θ and $f''_\theta \in L^2(\mathbb{R}^2; m)$,
- (iii) $\lim_{\epsilon \downarrow 0} \int_0^{2\pi} f'_\theta(\epsilon) \eta(d\theta) = 0$.

In this case, $\mathcal{A}f = \frac{1}{2} f''_\theta$.

When the support of the measure η is a finite set, this theorem has been proved in [11, Section 7.6 (3)]. A similar method of the proof works in interpreting the zero flux condition (4.10) with $\kappa = 0$ as the one stated in Theorem 5.4(iii).

With the Dirichlet form characterization, Theorem 5.2, of Walsh's Brownian motion X in hands, we can apply results from [7, 15] to give an analytic characterization of harmonic and subharmonic functions of X . The following definition is taken from [15, Definition 2.2], tailored to Walsh's Brownian motion.

Definition 5.5 (Sub/Super-Harmonicity). Let D be an open set in E . We say that a nearly Borel function u defined on E is *subharmonic* (resp. *superharmonic*) in D if u is q.e. finely continuous in D , and for any relatively compact open subset U with $\bar{U} \subset D$ and for q.e. $x \in U$, $\mathbf{E}_x[|u|(X_{\tau_U})] < \infty$ and $u(x) \leq \mathbf{E}_x[u(X_{\tau_U})]$ (resp. $u(x) \geq \mathbf{E}_x[u(X_{\tau_U})]$). A nearly Borel measurable function u on E is said to be *harmonic in D in the weak sense* if u is both superharmonic and subharmonic in D .

Theorem 5.6. Let D be an open subset of E . Suppose that a nearly Borel $u \in L^\infty_{\text{loc}}(D; m)$. The following are equivalent.

- (i) u is subharmonic in D ;
- (ii) $u \in (\mathcal{F}_D)_{\text{loc}}$ and $\mathcal{E}(u, v) \leq 0$ for every non-negative $v \in \mathcal{F} \cap C_c(D)$;

(iii) For η -a.e. $\theta \in [0, 2\pi)$, $r \rightarrow u(r, \theta)$ is convex on each connected open interval of $\{r \in (0, \infty) : (r, \theta) \in D\}$, and, if $\mathbf{0} \in D$, $\frac{\partial u}{\partial r}(0+, \theta)$ exists, and

$$\int_0^{2\pi} \frac{\partial u}{\partial r}(0+, \theta) \eta(d\theta) \geq 0. \quad (5.18)$$

Proof. The equivalence of (i) and (ii) is a special case of a more general result established in [15, Theorem 2.7], which extends [7, Theorem 2.11] from harmonic functions to subharmonic functions. It remains to show that (ii) is equivalent to (iii).

Suppose (ii) holds. For every $(r_0, \theta_0) \in D \setminus \{\mathbf{0}\}$, take $v(r, \theta) = \phi(r)\psi(\theta)$ with non-negative $\phi \in C_c^\infty(0, \infty)$ and $\psi \in C^\infty(\mathcal{S})$ so that the support of v is contained in $D \setminus \{\mathbf{0}\}$. Note that the linear span of such functions is \mathcal{E}_1^0 -dense in \mathcal{F}_D^0 . Then

$$0 \geq \mathcal{E}(u, v) = \int_0^{2\pi} \mathbf{D}(u(\cdot, \theta), \phi) \psi(\theta) \eta(d\theta).$$

This implies that for η -a.e. $\theta \in [0, 2\pi)$, $\frac{\partial^2 u(r, \theta)}{\partial r^2} \geq 0$ in the distributional sense on $\{r \in (0, \infty) : (r, \theta) \in D\}$ and so $r \mapsto u(r, \theta)$ is convex there. Suppose now $\mathbf{0} \in D$. since for η -a.e. $\theta \in [0, 2\pi)$, $r \rightarrow u(r, \theta)$ is convex on $\{r \in (0, \infty) : (r, \theta) \in D\}$, $\frac{\partial u}{\partial r}(0+, \theta)$ as the decreasing limit of $\frac{\partial u}{\partial r}(\varepsilon, \theta)$ as $\varepsilon \rightarrow 0+$ exists. For every $\varepsilon > 0$, let $v_\varepsilon(r, \theta) = \phi_\varepsilon(r)$, where $\phi_\varepsilon \in C_c^\infty[0, \varepsilon]$ with $\phi_\varepsilon(0) = 1$, $\phi'_\varepsilon(r) \leq 1$ and $\int_0^\infty \phi'_\varepsilon(r) dr = -1$. Clearly $\text{supp}[v_\varepsilon] \subset D$ when ε is sufficiently small. So by the monotone convergence theorem, we have

$$\begin{aligned} \int_0^{2\pi} \frac{\partial u}{\partial r}(0+, \theta) \eta(d\theta) &= - \lim_{\varepsilon \rightarrow 0+} \int_0^{2\pi} \int_0^\infty \frac{\partial u}{\partial r}(r, \theta) \phi'_\varepsilon(r) dr \eta(d\theta) \\ &= - \lim_{\varepsilon \rightarrow 0+} \mathcal{E}(u, v_\varepsilon) \geq 0. \end{aligned}$$

This establishes (iii). By reversing the above argument, we conclude from Theorem 5.2 that (iii) implies (ii). \square

Taking $D = E$ in Theorem 5.6 recovers the main results (Theorems 3.1 and 4.1) of [19], which are proved by using a different method.

6. One-point reflections with darning

Let (E, m) be as in Section 3 and $X = (X_t, \zeta, \mathbf{P}_x)$ be an m -symmetric Hunt process on E with continuous sample paths whose Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(E; m)$ is regular and irreducible. We assume that X admits no killing inside E . We further assume the following condition on the transition function $\{P_t, t > 0\}$ of X :

$$P_t(x, \cdot) \text{ is absolutely continuous with respect to } m \text{ for every } t > 0 \text{ and } x \in E. \quad (6.1)$$

Consider a compact subset K of E and put $E_0 = E \setminus K$. We extend the topological space E_0 to $E_0^* = E_0 \cup \{a^*\}$ by adding an extra point a^* to E_0 whose topology is prescribed as follows: a subset U of E_0^* containing the point a^* is an open neighborhood of a^* if there is an open set $U_1 \subset E$ containing K such that $U_1 \cap E_0 = U \setminus \{a^*\}$. In other words, E_0^* is obtained from E by identifying the points of K into a single point a^* . The restriction of the measure m to E_0 will be denoted by m_0 , which is then extended to E_0^* by setting $m_0(\{a^*\}) = 0$.

Let $X^0 = (X_t^0, \zeta^0, \mathbf{P}_x^0)$ be the part process of X on E_0 , which is known to be an m_0 -symmetric diffusion process on E_0 . Denote by $\{G_\alpha^0, \alpha > 0\}$ the resolvent of X^0 . In parallel to (A.1), (A.3) in Section 3, we make the following two assumptions on X^0 :

(A^o.1) $\varphi^K(x) =: \mathbf{P}_x^0(\zeta^0 < \infty, X_{\zeta^0-}^0 \in K) > 0$ for any $x \in E_0$.

(A^o.3) There exists a neighborhood U of K such that $\inf_{x \in V} G_1^0 \varphi^K(x) > 0$ for any compact set $V \subset U \setminus K$.

Since X^0 admits no killing inside E_0 , it holds that

$$\varphi^K(x) = \mathbf{P}_x(\sigma_K < \infty) \quad x \in E_0. \quad (6.2)$$

Theorem 6.1. *Under the above assumptions on X and X^0 , there exists a unique one-point reflection $X^* = (X_t^*, \zeta^*, \mathbf{P}_x^*)$ on $E_0^* = E_0 \cup \{a^*\}$ of X^0 . Moreover, X^* is a diffusion process on E_0^* .*

Proof. As for the existence of X^* , it suffices to verify that X^0 satisfies conditions (A.1)–(A.3) in Section 3 but with a^* in place of a . Then X^* on E_0^* can be constructed just as in Section 3 for $\kappa = 0$ and it is a diffusion process on E_0^* . (A.1), (A.3) with a^* in place of a follow from the present assumptions (A^o.1), (A^o.3), respectively. Let us verify that X^0 also enjoys the property

(A^o.2) $\mathbf{P}_x^0(X_{\zeta^0-}^0 \in K \cup \{\Delta\}) = 1$ for every $x \in E_0$ regardless the length of ζ^0 .

This property implies (A.2) with a^* in place of a .

By the absolute continuity assumption (6.1), it is enough to show (A^o.2) holding for q.e. $x \in E_0$. By the irreducibility assumption, X is either recurrent or transient. In the recurrent case, $\mathbf{P}_x(\sigma_K < \infty) = 1$ for q.e. $x \in E$ by (A^o.1) and (6.2) and so $\mathbf{P}_x^0(\zeta^0 < \infty, X_{\zeta^0-}^0 \in K) = 1$ for q.e. $x \in E_0$. In the transient case, we have $\mathbf{P}_x(X_{\zeta-} = \Delta) = 1$ for q.e. $x \in E$ on account of [11, Theorem 3.5.2] and the assumption that X admits no killing inside E , and accordingly (A^o.2) holds for q.e. $x \in E_0$. \square

We call the procedure of obtaining X^* from X as in Theorem 6.1 *darning* or *collapsing* or *shorting* the hole K .

Remark 6.2. (i) As we saw in the above proof, without assuming the absolute continuity condition (6.1) on X , property (A^o.2) remains valid for q.e. $x \in E_0$ so that a one-point reflection X^* on E_0^* of X^0 can be constructed for q.e. starting points $x \in E_0$. Theorem 7.5.9 of [11] is formulated in this way.

(ii) As the function φ^K in (A^o.1) is X^0 -purely excessive, there exists a unique X^0 -entrance law $\{\mu_t^K, t > 0\}$ such that

$$\varphi^K \cdot m_0 = \int_0^\infty \mu_t^K dt. \quad (6.3)$$

We call $\{\mu_t^K\}$ the *canonical entrance law* of X^0 for the set K relative to m_0 . The construction of X^* in Theorem 6.1 is carried out as in Section 3 using an excursion law \mathbf{n} around a^* being defined by (4.2) with μ_t^K in place of μ_t .

(iii) The process X^* in Theorem 6.1 enjoys the properties in Theorem 4.2 for $\kappa = 0$ with a due replacement of involved functions. In particular, the Dirichlet form $(\mathcal{E}, \mathcal{F})$ of X^* on $L^2(E_0^*; m_0)$ can be described as follows. Let $(\mathcal{E}^0, \mathcal{F}^0)$ be the Dirichlet form of X^0 on

$L^2(E_0; m_0)$. Then \mathcal{F}_e is the linear subspace of $(\mathcal{F}^0)^{\text{ref}}$ spanned by \mathcal{F}_e^0 and φ^K , \mathcal{F} is the linear subspace of $(\mathcal{F}^0)^{\text{ref}}$ spanned by \mathcal{F}^0 and $u_\alpha^K(x) := \mathbf{E}_x^0[e^{-\alpha\zeta^0}; X_{\zeta^0-}^0 \in K]$, $x \in E_0$, and

$$\mathcal{E}(f, g) = \mathcal{E}^{0, \text{ref}}(f, g), \quad f, g \in \mathcal{F}_e. \quad (6.4)$$

Example 6.3 (Relation Between BMD and ERBM). We consider a domain E in the complex plane \mathbb{C} and the absorbing Brownian motion $Z = (Z_t, \zeta, \mathbf{P}_z)$ on E . Let K be a continuum (a compact set containing at least two points) contained in E . As is well-known, K is non-polar and each point of K is regular for K with respect to Z . Let $Z^0 = (Z_t^0, \zeta^0, \mathbf{P}_z^0)$ be the part process of Z on $E_0 = E \setminus K$. We then consider the topological space $E_0^* = E_0 \cup \{a^*\}$ obtained from E by identifying the points of K into a single point a^* . We denote by m_0 the Lebesgue measure on E_0 which is extended to E_0^* by setting $m_0(\{a^*\}) = 0$. Clearly Z and Z^0 satisfy all assumptions stated preceding to Theorem 6.1. Hence Theorem 6.1 applies in getting the one-point reflection $Z^* = (Z_t^*, \zeta^*, \mathbf{P}_z^*)$ of Z^0 at a^* .

We call Z^* a *Brownian motion with darning* (BMD). For BMD, $\mathbb{C} \setminus K$ can be disconnected. This is the case, for example, when K is the unit circle in \mathbb{C} . In fact, in [7,11], K itself can even be disconnected.

When $\mathbb{C} \setminus K$ is connected, a closely related diffusion on E_0^* called an *excursion reflected Brownian motion* (ERBM) has been introduced by G. Lawler in [30] in a descriptive way using excursions of the reflected Brownian motion outside a disk and subsequently constructed as a Feller process via its semigroup by S. Drenning [16]. We shall show that it is actually identical with our BMD.

First consider a special case that $E = \mathbb{C}$ and $K = \overline{\mathbb{D}}$ for a unit disk \mathbb{D} centered at $\mathbf{0}$. We put $D = \mathbb{C} \setminus \overline{\mathbb{D}}$. In this special case, the transition semigroup of the ERBM is specified by [16] in an analogous manner to Barlow–Pitman–Yor’s semigroup (5.2) of Walsh’s Brownian motion as follows. We write $z \in D$ as $z = (s, \theta) = se^{i\theta}$ and consider the Lebesgue measure $m_0(dz) = sdsd\theta$ on D . Let P_t^r (resp. P_t^r) be the transition semigroup of the absorbing (resp. reflecting) Brownian motion on D (resp. \overline{D}). Notice that the density function of P_t^r with respect to m_0 is symmetric and rotation invariant: $p_t^r(se^{i\theta}, s'e^{i\theta'}) = p_t^r(s, s'e^{i(\theta'-\theta)})$. The same is true for the density function of P_t^0 . Hence, if $f(s, \theta) = f(s)$ is independent of θ , then so is $(P_t^r f)(s, \theta)$, being expressed as $\int_1^\infty q_t^r(s, s')f(s')s'ds'$ by a symmetric kernel $q_t^r(s, s')$. The same is true for $(P_t^0 f)(s, \theta)$.

As in [16], define

$$(P_t f)(s, \theta) = P_t^r \bar{f}(s) + P_t^0(f - \bar{f})(s, \theta), \quad \bar{f}(s) = \frac{1}{2\pi} \int_0^{2\pi} f(s, \theta)d\theta. \quad (6.5)$$

Let $D^* = D \cup \{0^*\}$ be the space obtained from \mathbb{C} by rendering \mathbb{D} into a single point 0^* and

$$C_\infty(D^*) = \{f \in C_\infty(\overline{D}) : f(1+, \theta) \text{ is independent of } \theta\}.$$

For $f \in C_\infty(D^*)$, we let $f(0^*) = f(1+, \theta) (= \bar{f}(1+))$. By the above observation, it is easy to see that $\{P_t, t > 0\}$ defined by (6.5) is a Feller semigroup making the space $C_\infty(D^*)$ invariant. Let Z^* be the associated Hunt process on D^* . Then, just as in [1, Lemma 2.3], we can verify that the part of Z^* on E_0 equals Z^0 in law. Furthermore, the above observation combined with the m_0 -symmetry of P_t^0 and the Fubini theorem readily implies the symmetry of P_t :

$$\int_D f P_t g dm_0 = \int_D P_t f g dm_0.$$

Therefore Z^* is a one-point extension of Z^0 to D^* , and by the uniqueness, Z^* is automatically a diffusion process on D^* and it is nothing but the BMD.

When E is a domain of \mathbb{C} and $K(\subset E)$ is a compact continuum such that $\mathbb{C} \setminus K$ is connected, the ERBM on $E_0^* = (E \setminus K) \cup \{a^*\}$ is defined in [30,16] as the image process of the above Z^* by a conformal map from $\mathbb{C} \setminus \overline{\mathbb{D}}$ onto $\mathbb{C} \setminus K$ with a due time change followed by a killing upon leaving the set E . Hence the ERBM is identical with the BMD in this case too by virtue of the conformal invariance of the BMD that will be presented in Theorem 6.4 in detail.

As is stated in the Introduction, the BMD has been formulated by allowing many holes in a planar domain E rather than one hole K with a characterization based on a unique existence theorem in [11, Section 7.7] that extends Theorem 6.1. The notion of ERBM for a multiply connected planar domain has also been introduced by G. Lawler [30] and S. Drenning [16]. However, when the number of holes is equal to or greater than 2, no precise semigroup characterization like (6.5) of ERBM seems to be available. \square

We now give a precise statement on a conformal invariance of BMD. Let E (resp. \widehat{E}) be an unbounded domain of \mathbb{C} and K (resp. \widehat{K}) a non-polar compact subset of E (resp. \widehat{E}). Suppose ϕ is a conformal map from $E_0 = E \setminus K$ onto $\widehat{E}_0 = \widehat{E} \setminus \widehat{K}$ sending ∞ to ∞ . Denote by $E_0^* = E_0 \cup \{a^*\}$ (resp. $\widehat{E}_0^* = \widehat{E}_0 \cup \{\widehat{a}^*\}$) be the topological space obtained from E (resp. \widehat{E}) by rendering K (resp. \widehat{K}) into a single point a^* (resp. \widehat{a}^*). The conformal mapping ϕ is then extended to be a homeomorphic map from E_0^* onto \widehat{E}_0^* . Denote by λ_0 (resp. $\widehat{\lambda}_0$) the Lebesgue measure on E_0 (resp. \widehat{E}_0), which is extended to E_0^* (resp. \widehat{E}_0^*) by setting $\lambda_0(\{a^*\}) = 0$ (resp. $\widehat{\lambda}_0(\{\widehat{a}^*\}) = 0$).

Theorem 6.4 (Conformal Invariance of BMD). *Let $Z^* = (Z_t^*, \zeta^*, \mathbf{P}_z^*)$ and $\widehat{Z}^* = (\widehat{Z}_t^*, \widehat{\zeta}^*, \widehat{\mathbf{P}}_z^*)$ be BMD on E_0^* and \widehat{E}_0^* , respectively. Set*

$$m(dz) = |\phi'(z)|^2 \mathbf{1}_{E_0}(z) \lambda_0(dz) \quad \text{and} \\ A_t = \int_0^{t \wedge \zeta^*} |\phi'(Z_s^*)|^2 \mathbf{1}_{E_0}(Z_s^*) ds \quad \text{for } t \geq 0. \quad (6.6)$$

Then $\{A_t, t \geq 0\}$ is a PCAF of Z^ in the strict sense with full support E_0^* with m as its Revuz measure relative to λ_0 . Let $\tau_t = \inf\{s > 0 : A_s > t\}$ for $t \geq 0$. Then for every $w \in \widehat{E}_0^*$,*

$$\left(\left\{ \phi(Z_{\tau_t}^*), t \geq 0 \right\}, A_{\zeta^*}^*, \mathbf{P}_{\phi^{-1}(w)}^* \right) \\ \text{has the same distribution as } (\{\widehat{Z}_t^*, t \geq 0\}, \widehat{\zeta}^*, \widehat{\mathbf{P}}_w^*). \quad (6.7)$$

Proof. (i) Since $\phi'(z) \neq 0$ on E_0 , A defined by (6.6) is a PCAF of Z^* (admitting exceptional λ_0 -polar set) with full quasi-support E_0^* possessing m as Revuz measure. We first show that the equivalence (6.7) holds for q.e. $w \in \widehat{E}_0^*$, namely, except for a $\widehat{\lambda}_0$ -polar set.

By a general theory (see, e.g., [11, Theorem 5.2.1]), the time-changed process $Y = (Z_{\tau_t}^*, A_{\zeta^*}^*, \mathbf{P}_z^*)$ of Z^* is an m -symmetric diffusion process on $E_0^* \setminus \mathcal{N}$ for some properly exceptional set $\mathcal{N} \subset E_0$. Clearly, $\phi(Y)$ is a well-defined continuous strong Markov process on $\widehat{E}_0^* \setminus \phi(\mathcal{N})$. By the change of variable formula

$$\int_{E_0} u(\phi(z)) m(dz) = \int_{\widehat{E}_0} u(w) \widehat{\lambda}_0(dw), \quad (6.8)$$

we see as in [10, Section 5.3 (2)] that $\phi(Y)$ is $\widehat{\lambda}_0$ -symmetric.

On the other hand, the part $Z^0 = (Z_t^0, \zeta^0, \mathbf{P}_z^0)$ of Z^* on E_0 is the absorbing Brownian motion, while the part of Y on $E_0 \setminus \mathcal{N}$ is the time change of $Z^0|_{E_0 \setminus \mathcal{N}}$ by its PCAF

$$A_t^0 = \int_0^{t \wedge \tau_{E_0}} |\phi'(Z_s^*)|^2 ds = \int_0^{t \wedge \zeta^0} |\phi'(Z_s^0)|^2 ds.$$

Consequently by the conformal invariance of the absorbing Brownian motion (see, e.g., [11, Section 5.3 (1)]), $\phi(Y)$ is a $\widehat{\lambda}_0$ -symmetric extension of the absorbing Brownian motion in $\widehat{E}_0 \setminus \phi(\mathcal{N})$. Thus by the uniqueness of BMD, (6.7) holds for $w \in \widehat{E}_0^* \setminus \phi(\mathcal{N})$.

(ii) Our next task is to show that A_t defined by (6.6) is a PCAF of Z^* in the strict sense. Then (6.7) holds for every $w \in \widehat{E}_0^*$ without exceptional set, completing the proof of Theorem 6.4.

Take a sufficiently large $\ell > 0$ such that the set $E_\ell = E \cap \{z : |z| < \ell\}$ contains K and put $E_{\ell,0}^* = (E_\ell \setminus K) \cup \{a^*\}$. Since the resolvent of Z^* is absolutely continuous with respect to λ_0 by (4.5) and the Dirichlet form of Z^* on $L^2(E_0^*; \lambda_0)$ is regular in view of [12, Theorem 2.3], Z^* is a Hunt process so that

$$\mathbf{P}_z^*(\lim_{\ell \rightarrow \infty} \tau_{E_{\ell,0}^*} = \zeta^*) = 1, \quad z \in E_0^*. \quad (6.9)$$

Let $Z^{\ell,*} = (Z_t^{\ell,*}, \mathbf{P}_z^{\ell,*})$ be the part of Z^* on $E_{\ell,0}^*$ that can be also regarded as the BMD on $E_{\ell,0}^*$ obtained from the absorbing Brownian motion $Z^{\ell,0} = (Z_t^{\ell,0}, \mathbf{P}_z^{\ell,0})$ on E_ℓ by rendering the set K into a single point a^* . Therefore, on account of [12, Lemma 5.1], the Green function (the 0-order resolvent density) $G^{\ell,*}$ of $Z^{\ell,*}$ admits an expression

$$G^{\ell,*}(z, \zeta) = G^{\ell,0}(z, \zeta) + 2p^{-1}\varphi(z)\varphi(\zeta), \quad z \in E_{\ell,0}^*, \quad z \in E_\ell \setminus K, \quad (6.10)$$

where $G^{\ell,0}$ is the Green function of the absorbing Brownian motion on $E_\ell \setminus K$, $\varphi(z) = \mathbf{P}_z^{\ell,*}(\sigma_K < \infty)$, $z \in E_{\ell,0}^*$, and p is the period of φ around K . In particular, $G^{\ell,*}(z, \zeta)$ has the same singularity as $G^{\ell,0}(z, \zeta)$ for $z \in E_\ell \setminus K$ and $G^{\ell,*}(a^*, \zeta)$ is bounded in $\zeta \in E_\ell \setminus K$. Furthermore, by the assumption that ϕ sends ∞ to ∞ , $\phi(E_\ell \setminus K)$ is finite and $m(E_\ell \setminus K) = \widehat{\lambda}_0(\phi(E_\ell \setminus K)) < \infty$ by (6.8). Accordingly,

$$\mathbf{E}_z^* \left[\int_0^{\tau_{E_{\ell,0}^*}^*} |\phi'(Z_s^*)|^2 ds \right] = \int_{E_\ell \setminus K} G^{\ell,*}(z, \zeta) m(d\zeta) < \infty \quad \text{for every } z \in E_{\ell,0}^*. \quad (6.11)$$

It follows from (6.9) and (6.11) that A_t defined by (6.6) is a PCAF of Z^* in the strict sense. \square

Remark 6.5. (i) Theorem 6.4 for the case that $E = \mathbb{C}$ is given in [11, Theorem 7.6.3]. However the proof that A_t is a PCAF of Z^* in the strict sense and so the time-change property holds for every starting point rather than q.e. starting point (that is, step (ii) in the above proof) was missing there. Theorem 6.4 extends with essentially the same proof to the BMD allowing finitely many disjoint holes $\{K_1, \dots, K_N\}$ in E , as given in [11, Theorem 7.8.1, Remark 7.8.2] (modulo the PCAF in the strict sense issue).

(ii) We would like to take this opportunity to make a correction of a statement in [11, Section 5.3 (2)] concerning a conformal invariance of the reflecting Brownian motion (RBM) as follows:

Let ϕ be a conformal map from \mathbb{D} onto a Jordan domain $D \subset \mathbb{C}$, $Z = (Z_t, \mathbf{P}_z)$ be the RBM on $\overline{\mathbb{D}}$, $A_t = \int_0^t |\phi'(Z_s)|^2 ds$ and $\tau_t = \inf\{s : A_s > t\}$. Then $(\phi(Z_{\tau_t}), \mathbf{P}_{\phi^{-1}(w)})$ is equivalent in law to the RBM on \overline{D} for every $w \in \overline{D} \setminus \mathcal{N}$, where \mathcal{N} is a certain λ -polar subset of ∂D . Here a domain $D \subset \mathbb{C}$ is called Jordan if it is simply connected and its boundary ∂D is a Jordan curve.

As the resolvent density $G_\alpha(z, z')$ of Z has a singularity on $\partial\mathbb{D}$, a proof like the second step (ii) in the above does not work to get the equivalence for every $w \in \partial D$ as is claimed after (5.3.4) of [11]. However, the proof does carry over if $\phi'(z)$ is bounded on \mathbb{D} and D is of finite area. Consequently, the equivalence holds for every $w \in \overline{D}$ in this case. Moreover, we will show below that if D is a locally non-trap domain in the sense of [6], then the conformal invariance for RBM holds also for every $w \in \overline{D}$.

A planar domain D of finite area is called a *non-trap* domain if $\sup_{w \in D} \tilde{\mathbf{E}}_w[\sigma_B] < \infty$, where $B \subset D$ is a closed ball and σ_B is the first hitting time of B by the RBM (Z_t, \mathbf{P}_w) on \overline{D} . The definition of non-trap domain is independent of the choice of the closed ball B in D . We refer the reader to [6] for the complete characterization of planar non-trap domains (Theorem 2.2 there) and various sufficient conditions for D to be non-trap. For example, by Propositions 1.2 and 1.3 as well as Corollaries 2.7 and 2.8 of [6], every locally uniform domain (also called (ε, δ) -domains) and every John domains that are of finite area are non-trap. In particular, the von Koch snowflake domain is non-trap.

Since D is a Jordan domain, the conformal mapping ϕ extends to be a homeomorphism from $\overline{\mathbb{D}}$ onto \overline{D} . So $\phi(Z)$ gives rise a continuous strong Markov process on \overline{D} that starts from every point in \overline{D} . For a possibly unbounded Jordan domain D , we fix $z_0 \in D$ and, for $k \geq 1$, let D_k be the connected component of $D \cap B(z_0, k)$ that contains z_0 . We now show that if D is a Jordan domain in \mathbb{C} and if there is $k_0 \geq 1$ so that D_k is non-trap for every $k \geq k_0$, then the PCAF $A_t = \int_0^t |\phi'(Z_s)|^2 ds$ is a PCAF of Z in the strict sense, yielding the desired equivalence for every $w \in \overline{D}$.

Let B be a closed ball inside D_{k_0} and $F = \phi^{-1}(B) \subset \mathbb{D}$. For $k \geq k_0$, let $U_k := \phi^{-1}(\overline{D}_k) \subset \overline{\mathbb{D}}$. Clearly $\cup_{k \geq k_0} U_k = \overline{\mathbb{D}}$. For $k \geq k_0$, since each D_k is non-trap domain of finite area, by the conformal invariance of RBM with starting point away from the boundary, we have

$$\sup_{z \in \mathbb{D} \setminus F} \mathbf{E}_z \left[A_{\sigma_F \wedge \tau_{U_k}} \right] = \sup_{w \in D \setminus B} \tilde{\mathbf{E}}_w [\sigma_B \wedge \tau_{D_k}] =: C_{1,k} < \infty.$$

On the other hand, there is $r \in (0, 1)$ so that $\mathbb{D}_r := B(0, r)$ contains F . Clearly,

$$c_0 := \sup_{z \in F, w \in \mathbb{D} \setminus \mathbb{D}_r} G_1(z, w) < \infty,$$

and since $\phi'(z)$ is continuous on \mathbb{D} , $c_1 := \sup_{z \in \mathbb{D}_r} |\phi'(z)|^2 < \infty$. Thus

$$\begin{aligned} \sup_{z \in F} \mathbf{E}_z \left[\int_0^{\tau_{U_k}} e^{-t} dA_t \right] &\leq \sup_{z \in F} \int_{U_k} G_1(z, z') |\phi'(z')|^2 \lambda(dz') \\ &\leq c_0 \int_{U_k \setminus \mathbb{D}_r} |\phi'(z')|^2 \lambda(dz') + c_1 \sup_{z \in F} \int_{\mathbb{D}_r} G_1(z, z') \lambda(dz') \\ &=: C_{2,k} < \infty. \end{aligned}$$

So we have $\mathbf{E}_z \left[\int_0^{\tau_{U_k}} e^{-t} dA_t \right] \leq C_{1,k} + C_{2,k} < \infty$ for any $z \in \overline{\mathbb{D}}$ and $k \geq k_0$. This proves that A_t is a PCAF of Z in the strict sense. \square

Example 6.6 (*Walsh's Brownian Motion with Darning*). We get back to the setting of Section 5. Recall that $X^0 = (X_t^0, \zeta_t^0, \mathbf{P}_x^0)$ is the diffusion process on the punctured plane E_0 that behaves like a one-dimensional absorbing Brownian motion on the ray R_θ that connects the starting position $x = (r, \theta)$ with the origin $\mathbf{0}$. The diffusion process is symmetric with respect to $m = \lambda \times \eta$ on $E_0 = (0, \infty) \times [0, 2\pi)$.

Let $\psi(\theta)$, $0 \leq \theta < 2\pi$, be an upper semi-continuous function on the circle that is bounded between two positive constants, and $K = \{(r, \theta) : 0 < r \leq \psi(\theta), \theta \in [0, 2\pi)\} \cup \{\mathbf{0}\}$, which is a compact set having $\mathbf{0}$ as its interior point.

Let $\tilde{X}^0 = (\tilde{X}_t^0, \tilde{\zeta}_t^0, \tilde{\mathbf{P}}_x^0)$ be the part process of X^0 on $\tilde{E}_0 = \mathbb{R}^2 \setminus K$. Then \tilde{X}^0 is still symmetric with respect to the restriction \tilde{m} of $m = \lambda \times \eta$ to \tilde{E}_0 and its Dirichlet form on $L^2(\tilde{E}_0, \tilde{m})$ is regular [11, Theorem 3.3.9]. \tilde{X}^0 is not irreducible and its transition function does not satisfy the absolute continuity with respect to \tilde{m} . Nevertheless \tilde{X}^0 satisfies

$$\tilde{\mathbf{P}}_x(\tilde{X}_{\tilde{\zeta}_0^-}^0 \in K, \tilde{\zeta}_0^0 < \infty) = \tilde{\mathbf{P}}_x(\tilde{\zeta}_0^0 < \infty) = 1, \quad x \in \tilde{E}_0, \quad (6.12)$$

namely, it fulfills the conditions (A^o.1) and (A^o.2). Further it satisfies condition (A^o.3) as well. Therefore, by regarding K as a single point $\mathbf{0}^*$ and extending \tilde{m} to $E_0^* = \tilde{E}_0 \cup \{\mathbf{0}^*\}$ by setting $\tilde{m}(\{\mathbf{0}^*\}) = 0$, we can construct a one-point extension X^* of \tilde{X}^0 to E_0^* by means of the \tilde{X}^0 -canonical entrance law for K relative to \tilde{m}

$$\tilde{\mu}_t(dx) = \frac{1}{(2\pi t^3)^{1/2}} (r - \psi(\theta)) e^{-\frac{(r-\psi(\theta))^2}{2t}} dr \eta(d\theta), \quad x = (r, \theta) \in \tilde{E}_0. \quad (6.13)$$

We may call X^* a *WBMD* (Walsh's Brownian motion with darning).

Let $(\tilde{\mathcal{E}}^0, \tilde{\mathcal{F}}^0)$ be the Dirichlet form of \tilde{X}^0 on $L^2(\tilde{E}_0, \tilde{m})$. Using the transition function of \tilde{X}^0 , we immediately obtain

$$C_c^\infty(\tilde{E}_0) \subset \tilde{\mathcal{F}}^0, \quad \tilde{\mathcal{E}}^0(f, g) = \frac{1}{2} \int_0^{2\pi} \int_{\psi(\theta)}^\infty f'_\theta g'_\theta dr \eta(d\theta). \quad (6.14)$$

For $(\tilde{\mathcal{E}}^0, \tilde{\mathcal{F}}^0)$, we can make statements analogous to Theorem 5.1 by replacing $\mathbf{D}(f, g)$ with $\int_{\psi(\theta)}^\infty f'_\theta g'_\theta dr$. We have also obvious counterparts of Theorems 5.2, 5.4 and 5.6 for the WBMD X .

Those expressions tell us that WBMD can be identified in distribution with WBM if K is a disk centered at $\mathbf{0}$, namely, $\psi(\theta)$ is constant. Otherwise they are rather different.

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