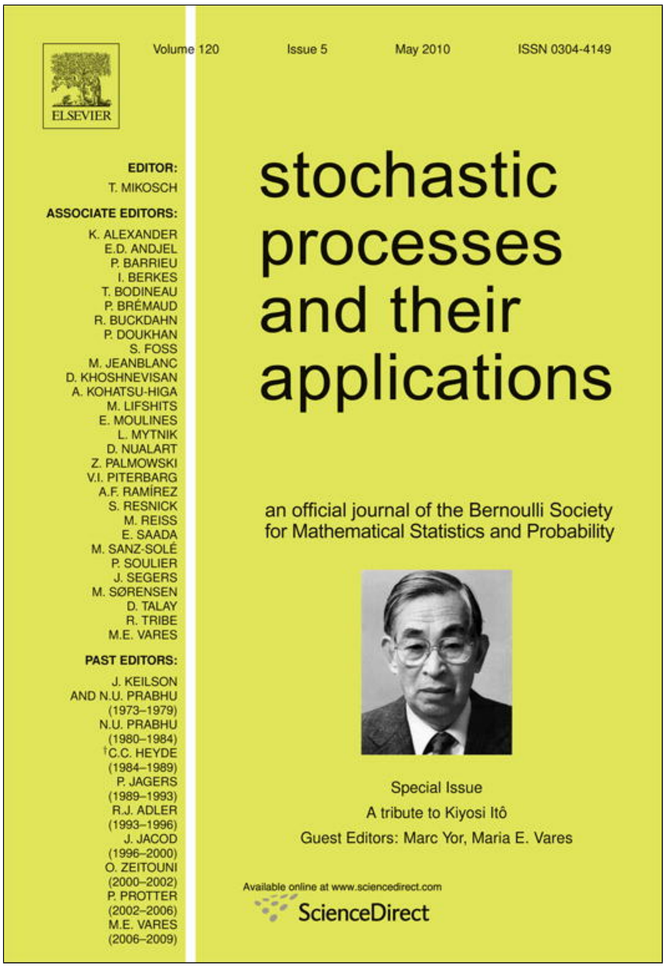

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From one dimensional diffusions to symmetric Markov processes

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Dedicated to the memory of Itô Kiyosi Sensei

Abstract

For an absorbing diffusion X^0 on a one dimensional regular interval I with no killing inside, the Dirichlet form of X^0 on $L^2(I; m)$ and its extended Dirichlet space are identified in terms of the canonical scale s of X^0, where m is the canonical measure of X^0. All possible symmetric extensions of X^0 will then be considered in relation to the active reflected Dirichlet space of X^0. Furthermore quite analogous considerations will be made for possible symmetric extensions of a specific diffusion in a higher dimension, namely, a time changed transient reflecting Brownian motion on a closed domain of $\mathbb{R}^d$, $d \geq 3$, possessing two branches of infinite cones.

MSC: primary 60J60; secondary 60J50; 60J55; 31C25

Keywords: Diffusion; Time change; Dirichlet form; Reflecting extension

1. Introduction

As a famous saying of William Feller goes, a one dimensional diffusion process X travels according to a road map indicated by its canonical scale s and with speed indicated by its canonical (speed) measure m. This was legitimated in the book of Kiyosi Itô and Henry McKean [18] by showing that a time change of X by means of a positive continuous additive functional (PCAF in abbreviation) with full support amounts to a replacement of the speed measure m while keeping the scale s invariant.

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This idea has been a driving force to develop the theory of symmetric Markov processes [21,11,13,4]. It is well understood now that a time change of a symmetric Markov process X on a state space E by means of a PCAF A corresponds precisely to the replacement of the symmetrizing measure m and the extended Dirichlet space $\mathcal{F}_e$, respectively, with the Revuz measure of A and the restriction of $\mathcal{F}_e$ to the support F of A. The road map is nothing but the Beurling–Deny formula [1] of the Dirichlet form $\mathcal{E}$ which is unchanged when $F = E$, but otherwise changed into a due restriction of $\mathcal{E}$ to F with an additional jump term specified by the Feller measures [10] that express the joint distribution of end points of excursions of X on $E \setminus F$ around F [7,4].

As is well known (see Itô [15, Theorem 5.15.1], Itô–McKean [18, Section 4.11]), a generic one dimensional diffusion process X has a symmetric resolvent density with respect to the speed measure m so that X is m-symmetric. To my knowledge however, the Dirichlet form of X on $L^2(m)$ and its extended Dirichlet space have not yet been identified precisely. The first aim of this paper is to identify them for the absorbing diffusion X^0 on a regular open interval with no killing inside.

Once the Dirichlet form of X^0 is identified, it provides us with two ways of quick recovery of X^0 from the pair (s, m) and also a transparent way to introduce and construct possible symmetric extensions of X^0. Among symmetric extensions, there is a maximal one corresponding to the general concept of the reflected Dirichlet space [21,2,4]. We shall discuss the possibility of symmetric extensions of X^0 in relation to its reflected Dirichlet space.

In the final section, we shall make an analogous consideration on the possibility of symmetric extensions of time changed transient reflecting Brownian motion on a special domain in $\mathbb{R}^d,\ d \geq 3$, possessing two branches of infinite closed cones that has appeared in a recent joint paper with Chen [5]. Here the construction of the extensions can be carried out by means of the Poisson point processes of excursions due to Kiyosi Itô [16,14,6]. The consideration will be extended in a more general context elsewhere.

2. Absorbing diffusion X^0 on a regular interval

Let $I = (r_1, r_2)$ be an open interval of $\mathbb{R}$ and $X^0 = (X_t^0, \mathbf{P}_x^0, \zeta^0)$ be a Markov process with state space I satisfying the following conditions:

(1) X^0 is a Hunt process, namely, a normal strong Markov process satisfying the quasi-left-continuity on $[0, \infty)$.
(2) X^0 is a diffusion process, namely, the sample path X_t^0 is continuous in $t \in [0, \zeta^0)$ almost surely.
(3) X^0 admits no killing inside I: $\mathbf{P}_x^0(\zeta^0 < \infty, X_{\zeta^0-}^0 \in I) = 0,\ x \in I$.
(4) Each point a of I is regular: $\mathbf{E}_a^0[\mathrm{e}^{-\sigma_{a+}}] = 1$ and $\mathbf{E}_a^0[\mathrm{e}^{-\sigma_{a-}}] = 1$, where $\mathbf{E}_a^0[\mathrm{e}^{-\sigma_{a\pm}}] = \lim_{b\to a\pm} \mathbf{E}_a^0[\mathrm{e}^{-\sigma_b}]$ and σ_b denotes the hitting time of the point b.

We call X^0 an *absorbing diffusion* on I because the sample path X_t^0 is killed upon approaching to the point at infinity of I due to the condition (1).

Denote by $\{R_\lambda; \lambda > 0\}$ the resolvent of X^0 and by $\mathcal{B}_b(I)$ (resp. $C_b(I)$) the space of all bounded Borel measurable (resp. continuous) functions on I. We refer to Itô [17, Section 6] or Itô–McKean [18, Chapter 3, Section 4.2] for the following facts: $R_\lambda(\mathcal{B}_b(I)) \subset C_b(I)$ and the generator $\mathcal{G}$ of X^0 is well defined by

$$\begin{aligned} &\mathcal{D}(\mathcal{G}) = R_\lambda(C_b(I)), \\ &(\mathcal{G}u)(x) = \lambda u(x) - f(x), \quad x \in I,\ \text{for } u = R_\lambda f,\ f \in C_b(I), \end{aligned} \tag{2.1}$$

independently of $\lambda > 0$. Furthermore there exist a strictly increasing continuous function s on I and a positive Radon measure m on I of full support such that

$$\mathcal{G}u = D_m D_s u, \quad u \in \mathcal{D}(\mathcal{G}), \tag{2.2}$$

in the following sense: if $u \in \mathcal{D}(\mathcal{G})$, then u is absolutely continuous with respect to s, a version $D_s u$ of the Radon Nikodym derivative is of bounded variation and absolutely continuous with respect to $\mathrm{d}m$, and a version $D_m D_s u$ of the Radon Nikodym derivative is continuous and coincides with $\mathcal{G}u$. s and m are called a *canonical scale* and a *canonical measure* of X^0, respectively. The pair (s, m) is unique up to a multiplicative positive constant in the sense that, if $(\widetilde{s}, \widetilde{m})$ is another such pair, then $\mathrm{d}\widetilde{s} = c\mathrm{d}s$, $\mathrm{d}\widetilde{m} = c^{-1}\mathrm{d}m$ for some constant $c > 0$.

More specifically, for any $J = (a, b)$, $r_1 < a < b < r_2$, it is shown in [17, Section 6] that

$$E_x^0[\sigma_a \wedge \sigma_b] < \infty, \qquad P_x^0(\sigma_a < \sigma_b) > 0, \qquad P_x^0(\sigma_a > \sigma_b) > 0, \quad x \in J. \tag{2.3}$$

If we let

$$s_J(x) = P_x^0(\sigma_a > \sigma_b), \qquad m_J(x) = -\frac{\mathrm{d}E_x^0(\sigma_a \wedge \sigma_b)}{\mathrm{d}s_J(x)}, \quad x \in J, \tag{2.4}$$

then (2.2) holds on J for any $u \in \mathcal{D}(\mathcal{G})$ with s_J, m_J in place of s, m. For different intervals J, s_J differs only by a linear transformation on their intersection and so s and m can be defined consistently on I to satisfy (2.2).

Lemma 2.1. *Fix $\lambda > 0$. For an interval $J = (j_1, j_2)$ with $r_1 < j_1 < j_2 < r_2$, let*

$$\varphi_i(x) = E_x^0\left[\mathrm{e}^{-\lambda\tau_J} : X_{\tau_J} = j_i\right], \quad x \in J,\ i = 1, 2,$$

where $\tau_J = \sigma_{j_1} \wedge \sigma_{j_2}$. Then

$$(\lambda - D_m D_s)\varphi_i(x) = 0, \quad x \in J,\ i = 1, 2, \tag{2.5}$$

and

$$\varphi_1(j_1+) = 1, \qquad \varphi_1(j_2-) = 0; \qquad \varphi_2(j_1+) = 0, \qquad \varphi_2(j_2-) = 1. \tag{2.6}$$

Proof. This lemma was proved in Theorem 5.9.2 of Itô's book [15] which was first published as early as in 1957 so that the definition of a generator $\mathcal{G}$ employed in it was different from the later one (2.1) although they are equivalent to each other (cf. [18, Section 3.8]). For the reader's convenience, we supply here an alternative and simpler proof of this lemma.

(2.5) can be shown as above by considering the stopped process of X^0 on $[j_1, j_2]$ in place of X^0. To show (2.6), put $\psi(x) = E_x^0[\mathrm{e}^{-\lambda\sigma_{j_2}}]$, $x \in I$, and let f be the indicator function of $[j_2, \infty)$. Then $R_\lambda f \in C_b(I)$ and $R_\lambda f(x) = \psi(x) R_\lambda f(j_2)$, $x < j_2$, so that $R_\lambda f(j_2) = \psi(j_2-) R_\lambda f(j_2)$. Since $R_\lambda f(j_2) > 0$ by (2.3), we have $\psi(j_2-) = 1$, which in turn implies that, for any $\epsilon > 0$, $P_{j_2-}^0(\sigma_{j_2} < \epsilon) = 1$. Now, for any $\epsilon > 0$ and $x \in J$,

$$\begin{aligned}\varphi_2(x) &\geq E_x^0\left[\mathrm{e}^{-\lambda\sigma_{j_2}}; \sigma_{j_2} < \epsilon,\ \sigma_{j_1} \geq \epsilon\right] \geq E_x^0\left[\mathrm{e}^{-\lambda\sigma_{j_2}}; \sigma_{j_2} < \epsilon\right] - E_x^0\left[\mathrm{e}^{-\lambda\sigma_{j_2}}; \sigma_{j_1} < \epsilon\right] \\ &\geq \mathrm{e}^{-\lambda\epsilon} P_x^0(\sigma_{j_2} < \epsilon) - E_{x_0}^0\left[\mathrm{e}^{-\lambda\sigma_{j_2}}; \sigma_{j_1} < \epsilon\right], \quad \text{for } j_1 < x_0 < x < j_2.\end{aligned}$$

By letting $x \uparrow j_2$ and then $\epsilon \downarrow 0$, we obtain the last identity of (2.6). Using (2.3), we get

$$\varphi_1(x) \leq 1 - P_x^0(\sigma_{j_1} > \sigma_{j_2}) \leq 1 - P_x^0(\sigma_{j_2} < \epsilon) + P_{x_0}^0(\sigma_{j_1} < \epsilon),$$

which leads us to the second identity of (2.6) similarly. The first and third ones can be proved analogously. □

In accordance with Itô–McKean [18], the left boundary r_1 is called *exit* (resp. *entrance*) if

$$\int_{(r_1,c)} m((x,c))s(\mathrm{d}x) < \infty \quad \left(\text{resp. } \int_{(r_1,c)} (s(c)-s(x))m(\mathrm{d}x) < \infty\right) \quad \text{for } c \in I. \tag{2.7}$$

An analogous definition is in force for the right boundary r_2.

We still fix a $\lambda > 0$. Let u_1 (resp. u_2) be a positive strictly increasing (resp. decreasing) solution of the equation

$$(\lambda - D_m D_s)u(x) = 0, \quad x \in I$$

such that their Wronskian equals 1. The functions $u_i,\ i = 1, 2$, were constructed and their detailed boundary behaviors were studied in Itô [15, Section 5.13]. If the boundary r_i of I is exit and entrance simultaneously, there are many choices of u_i but otherwise they are unique up to multiplicative positive constants. When the left (resp. right) boundary r_1 (resp. r_2) is exit and entrance, then we make a special choice of u_1 (resp. u_2) such that $u_1(r_1+) = 0$ (resp. $u_2(r_2-) = 0$). Define

$$K_\lambda(x,y) = \begin{cases} u_1(x)u_2(y), & r_1 < x \le y < r_2, \\ u_2(x)u_1(y), & r_1 < y \le x < r_2. \end{cases}$$

The next theorem is taken from Itô [15, Theorem 5.15.1]. For a function f on I, the right (resp. left) limit of f at r_1 (resp. r_2) will be denoted by $f(r_1)$ (resp. $f(r_2)$) if it exists.

Theorem 2.2. *It holds for $x \in I$ that*

$$R_\lambda(x,B) = \int_B K_\lambda(x,y)m(\mathrm{d}y), \quad B \in \mathcal{B}(I) \tag{2.8}$$

$$E_x^0\left[\mathrm{e}^{-\lambda\zeta}; X_{\zeta-} = r_1\right] = \frac{u_2(x)}{u_2(r_1)}, \qquad E_x^0\left[\mathrm{e}^{-\lambda\zeta}; X_{\zeta-} = r_2\right] = \frac{u_1(x)}{u_1(r_2)}. \tag{2.9}$$

Observe that, as $J \uparrow I$, $\tau_J \uparrow \zeta$ almost surely. Therefore this theorem is readily derived just as in [15] by using Lemma 2.1 together with the stated construction of functions u_i.

We say that the boundary r_i is *approachable in finite time* if

$$P_x^0(\zeta < \infty, X_{\zeta-} = r_i) > 0 \quad x \in I. \tag{2.10}$$

From (2.9) and [15, Section 5.13], we can conclude that r_i is approachable in finite time if and only if it is exit.

Since $K_\lambda(x,y)$ is symmetric in $x, y \in I$, (2.8) implies that X^0 is symmetric with respect to the canonical measure m in the sense that its transition function $\{P_t^0; t \ge 0\}$ satisfies

$$\int_I P_t^0 f(x)g(x)m(\mathrm{d}x) = \int_I f(x)P_t^0 g(x)m(\mathrm{d}x), \quad f, g \in \mathcal{B}_+(I).$$

In the next section, we shall identify the Dirichlet form of X^0 on $L^2(I;m)$.

3. Identification of the Dirichlet form

Define

$$\mathcal{E}^{(s)}(u,v) = \int_I D_s u(x) D_s v(x) \mathrm{d}s(x) \tag{3.1}$$

$$\mathcal{F}^{(s)} = \left\{ u : u \text{ is absolutely continuous in } s \text{ and } \mathcal{E}^{(s)}(u,u) < \infty \right\}. \tag{3.2}$$

From the elementary identity $u(b) - u(a) = \int_a^b D_s u(x) \mathrm{d}s(x)$, $a, b \in I$, we get

$$(u(b) - u(a))^2 \le |s(b) - s(a)| \mathcal{E}^{(s)}(u,u), \quad a, b \in I, \; u \in \mathcal{F}^{(s)}. \tag{3.3}$$

We call the boundary r_i *approachable* if $|s(r_i)| < \infty$, $i = 1, 2$. If r_i is approachable, then any $u \in \mathcal{F}^{(s)}$ admits a finite limit $u(r_i)$ by (3.3). Let us introduce the space

$$\mathcal{F}_0^{(s)} = \left\{ u \in \mathcal{F}^{(s)} : u(r_i) = 0 \text{ whenever } r_i \text{ is approachable} \right\}. \tag{3.4}$$

When r_i is approachable, we have

$$u(b)^2 \le |s(b) - s(r_i)| \mathcal{E}^{(s)}(u,u), \quad b \in I, \; u \in \mathcal{F}_0^{(s)}. \tag{3.5}$$

(3.3) and (3.5) in particular mean that, if $\{u_n\} \subset \mathcal{F}_0^{(s)}$ is $\mathcal{E}^{(s)}$-Cauchy and convergent at one point $a \in I$, then it is convergent to a function of $\mathcal{F}_0^{(s)}$ uniformly on each compact subinterval of I. Therefore we are led to the first assertion of the next lemma just as in [13, Example 1.2.2] and the second one by the Banach–Saks theorem.

Lemma 3.1. (i) *If $\{u_n\} \subset \mathcal{F}_0^{(s)}$ is $\mathcal{E}^{(s)}$-Cauchy and convergent to a function u m-a.e. as $n \to \infty$, then $u \in \mathcal{F}_0^{(s)}$ and* $\lim_{n\to\infty} \mathcal{E}^{(s)}(u_n - u, u_n - u) = 0$.
(ii) *Consider the contractive real functions $\varphi_\ell(t) = t - (-1/\ell) \vee t \wedge (1/\ell)$, $t \in \mathbb{R}$, $\ell \in \mathbb{N}$. For any $u \in \mathcal{F}_0^{(s)}$, the Cesàro mean sequence $\{u_n\}$ of a certain subsequence of $\{\varphi_\ell(u)\}$ is $\mathcal{E}^{(s)}$ convergent to u.*

$C_c(I)$ will denote the space of continuous functions on I with compact support. $\sqrt{\mathcal{E}^{(s)}(u,u)}$ will be designated by $\|u\|_{\mathcal{E}^{(s)}}$ occasionally.

Theorem 3.2.

$$(\mathcal{E}, \mathcal{F}) = (\mathcal{E}^{(s)}, \mathcal{F}_0^{(s)} \cap L^2(I;m)) \tag{3.6}$$

is a regular, strongly local, irreducible Dirichlet form on $L^2(I;m)$.

Let $(\mathcal{F}_e, \mathcal{E})$ be its extended Dirichlet space. Then

$$\mathcal{F}_e = \mathcal{F}_0^{(s)}, \qquad \mathcal{E} = \mathcal{E}^{(s)}. \tag{3.7}$$

Proof. By Lemma 3.1(i), (3.6) is a closed symmetric form on $L^2(I;m)$, and just as in [13, Example 1.2.2], it can be shown to be Markovian so that it is a Dirichlet form on $L^2(I;m)$. Obviously it is strongly local. Suppose there is a Borel set $A \subset I$ such that the indicator function 1_A equals some function $u \in \mathcal{F}$ m-a.e. Since u is continuous and takes values 0 or 1 only, $u^{-1}(1)$

is a closed and open subset of I, and consequently either A or A^c is m-negligible, yielding the irreducibility of (3.6).

Lemma 3.1(i) also implies the inclusion $\mathcal{F}_e \subset \mathcal{F}_0^{(s)}$. To prove the converse inclusion, take any $u \in \mathcal{F}_0^{(s)}$. We may assume without loss of generality that u is bounded, that is, $|u| \le M$ for some constant M.

We consider a sequence of functions $\psi_n \in C_c^1(\mathbb{R}_+)$ such that

$$\begin{cases} \psi_n(x) = 1 \quad \text{for } 0 \le x < n; \qquad \psi_n(x) = 0 \quad \text{for } x > 2n+1; \\ |\psi_n'(x)| \le \dfrac{1}{n}, \quad n \le x \le 2n+1; \qquad 0 \le \psi_n(x) \le 1, \quad x \in \mathbb{R}_+. \end{cases}$$

Put $w_n(x) = u_n(x) \cdot \psi_n(|s(x)|)$ for $x \in I$, where $u_n,\ n \ge 1$, are the functions constructed in Lemma 3.1(ii) for u. Then, $w_n \in \mathcal{F}_0^{(s)} \cap C_c(I)$ because u_n vanishes on a neighborhood of r_i if r_i is approachable, while so does $\psi_n(|s(x)|)$ otherwise. Further, since $u(x) - w_n(x) = u(x)(1 - \psi_n(|s(x)|)) + (u(x) - u_n(x))\psi_n(|s(x)|)$ and $|u(x) - u_n(x)| \le |u(x)|$,

$$\begin{aligned} \|u - w_n\|^2_{\mathcal{E}^{(s)}} &\le 4 \int_I (D_s u(x))^2 (1 - \psi_n(|s(x)|))^2 \mathrm{d}s(x) \\ &\quad + 8 \int_I u(x)^2 \psi_n'(|s(x)|)^2 \mathrm{d}s(x) + 4 \int_I (D_s u(x) - D_s u_n(x))^2 \psi_n(|s(x)|)^2 \mathrm{d}s(x) \\ &\le 4 \int_{|s(x)| \ge n} (D_s u(x))^2 \mathrm{d}s(x) + 8M^2 \int_{n \le |s(x)| < 2n+1} \psi_n'(|s(x)|)^2 \mathrm{d}s(x) + 4\|u - u_n\|^2_{\mathcal{E}^{(s)}} \\ &\le 4 \int_{|s(x)| \ge n} (D_s u(x))^2 \mathrm{d}s(x) + 16M^2 \frac{n+1}{n^2} + 4\|u - u_n\|^2_{\mathcal{E}^{(s)}} \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

This shows that $\{w_n\} \subset \mathcal{F}$ is $\mathcal{E}^{(s)}$-Cauchy. Since w_n converges to u pointwise, we get $u \in \mathcal{F}_e$.

For any bounded $u \in \mathcal{F}$, the same functions $\{u_n, n \ge 1\}$ as above are in $\mathcal{F} \cap C_c(I)$ and $\mathcal{E}_1$ convergent to u as $n \to \infty$. Obviously $\mathcal{F} \cap C_c(I)$ is uniformly dense in $C_c(I)$. Thus $(\mathcal{E}, \mathcal{F})$ is regular. □

Let us call the boundary r_1 *regular* if r_1 is approachable and $m((r_1, c)) < \infty$ for $c \in I$. In view of (2.7), we see that r_1 is regular if and only if it is exit and entrance. An analogous definition is in force for the right boundary r_2.

The inner product in $L^2(I; m)$ is denoted by (u, v) and we put $\mathcal{E}_\lambda(u, v) = \mathcal{E}(u, v) + \lambda(u, v),\ \lambda > 0$.

Theorem 3.3. *(3.6) is the Dirichlet form of X^0 on $L^2(I; m)$.*

Proof. Let $(\mathcal{E}, \mathcal{F})$ be defined by (3.6). It suffices to show for any $\lambda > 0$ that

$$R_\lambda(C_c(I)) \subset \mathcal{F}, \tag{3.8}$$

$$\mathcal{E}_\lambda(R_\lambda f, v) = (f, v), \quad f \in C_c(I),\ v \in \mathcal{F} \cap C_c(I). \tag{3.9}$$

$(\mathcal{E}, \mathcal{F})$ is regular by Theorem 3.2 so that $\mathcal{F} \cap C_c(I)$ is $\mathcal{E}_\lambda$-dense in $\mathcal{F}$. Therefore, if (3.8) and (3.9) hold true, then (3.9) holds for any $v \in \mathcal{F}$, which means that the resolvent $\{R_\lambda; \lambda > 0\}$ of X^0 is associated with $(\mathcal{E}, \mathcal{F})$.

Take $f \in C_c(I)$ and set $w = R_\lambda f$. By Theorem 2.2, $w(r_2) = u_2(r_2) \int_I f(y) u_1(y) m(\mathrm{d}y)$ which vanishes if r_2 is regular. Similarly $w(r_1) = 0$ if r_1 is regular. By (2.1) and (2.2), we have for any open subinterval J of I with a compact closure

$$-\int_J D_m D_s w \cdot w \mathrm{d}m = \int_J (f - \lambda w) w \mathrm{d}m.$$

Integrating by parts and letting $J \uparrow I$, we get

$$\mathcal{E}_{\lambda}^{(s)}(w, w) = \int_I f w \mathrm{d}m + D_s w(r_2) w(r_2) - D_s w(r_1) w(r_1).$$

Since $D_s w(r_2) = D_s u_2(r_2) \int_I f(y) u_1(y) m(\mathrm{d}y)$, the second term on the right-hand side is finite on account of [15, Theorem 5.13.4], and actually it vanishes regardless the type of the boundary r_2 due to the present choice of the function u_2. Similarly, the third term on the right-hand side vanishes.

We have shown that $w \in \mathcal{F}^{(s)} \cap L^2(I; m)$. We have also seen that $w(r_i) = 0$ provided that r_i is approachable and m is finite in a neighborhood of r_i, namely, r_i is regular. If r_i is approachable but m is divergent in a neighborhood of r_i, then w admits a finite limit $w(r_i)$ which must vanish because $w \in L^2(I; m)$. We get (3.8).

The same computation as above yields (3.9) but more easily because v is of compact support. □

Corollary 3.4. *If either r_1 or r_2 is approachable, then X^0 is transient. Otherwise X^0 is recurrent.*

Proof. By the above two theorems, the extended Dirichlet space $(\mathcal{F}_e, \mathcal{E})$ of X is given by (3.7). If either r_1 or r_2 is approachable, then $u \in \mathcal{F}_e$, $\mathcal{E}(u, u) = 0$ implies that u vanishes identically on I. Otherwise, $1 \in \mathcal{F}_e$ and $\mathcal{E}(1, 1) = 0$. The assertions then follow from [13, Chapter 1]. □

4. Two ways of constructing X^0 from (s, m)

In the preceding two sections, we have started with a Markov process X^0 on an interval $I = (r_1, r_2)$ satisfying conditions (1), (2), (3), (4) and found that X^0 is symmetric with respect to its canonical measure m and its Dirichlet form on $L^2(I; m)$ is given by (3.6) in terms of its canonical scale s.

In this section, we assume conversely that we are given an arbitrary strictly increasing continuous function s on I and an arbitrary positive Radon measure m on I of full support. (s, m) then defines a Dirichlet form (3.6) on $L^2(I; m)$.

Theorem 4.1. *There exists a unique m-symmetric Markov process $X^0 = (X_t^0, \mathbf{P}_x^0, \zeta^0)$ on I associated with the Dirichlet form (3.6) and X^0 satisfies conditions (1), (2), (3), (4).*

Proof. The Dirichlet form (3.6) is regular, strongly local and irreducible by Theorem 3.2. Further we see from (3.3) that, for each compact set $K \subset I$, there exists a positive constant C_K with $\sup_{b \in K} u(b)^2 \le C_K \mathcal{E}_1(u, u)$, $u \in \mathcal{F}$, which particularly means that each one point of I has a positive 1-capacity relative to the Dirichlet form (3.6). Therefore, by using general theorems in [13], we readily conclude the existence of an m-symmetric Markov process X^0 on I uniquely associated with the Dirichlet form (3.6) that satisfies properties (1), (2), (3) as well as $\mathbf{P}_a^0(\sigma_b < \infty) > 0$ for any $a, b \in I$, from which property (4) for X^0 also follows. □

We can also recover Itô–McKean's construction of X^0 from the pair (s, m) in the following manner. Let $\widetilde{I} = (s(r_1), s(r_2)) \subset \mathbb{R}$. s maps I homeomorphically onto $\widetilde{I}$. Let $\widetilde{m}$ be the image measure of m by s. We consider the absorbing Brownian motion $Z = (Z_t, \mathbf{Q}_x, \eta)$ on $\widetilde{I}$ and its time changed process $\widetilde{X}^0 = (\widetilde{X}_t^0, \widetilde{\mathbf{P}}_x^0, \widetilde{\zeta}^0)$ by means of the PCAF $A_t = \int_{\widetilde{I}} \ell(t, x) \widetilde{m}(\mathrm{d}x)$ where $\ell(t, x)$ is the local time of Z at x, or more specifically, the PCAF of Z with Revuz measure δ_x relative to the symmetrizing measure $2\mathrm{d}x$. Thus $\widetilde{X}_t^0 = Z_{\tau_t}$, $\tau_t = A_t^{-1}$, $\widetilde{P}_x^0 = \mathbf{Q}_x$, $\widetilde{\zeta}^0 = A_\eta$.

Theorem 4.2. *The m-symmetric Markov process $X^0 = (X_t^0, \mathbf{P}_x^0, \zeta^0)$ associated with (3.6) can be obtained as the inverse image of the time changed absorbing Brownian motion $\widetilde{X}^0$ on $\widetilde{I}$ by the map s:*

$$X_t^0 = s^{-1}\widetilde{X}_t^0, \quad t \geq 0, \qquad \mathbf{P}_x^0 = \widetilde{\mathbf{P}}_{s(x)}^0, \quad x \in I, \ \zeta^0 = \widetilde{\zeta}^0. \tag{4.1}$$

Proof. As the pair of canonical scale and canonical measure for the absorbing Brownian motion Z on $\widetilde{I}$, we make a special choice $(x, 2\mathrm{d}x)$. On account of Theorem 3.3, the Dirichlet form $(\mathcal{E}^Z, \mathcal{F}^Z)$ of Z on $L^2(\widetilde{I}; 2\mathrm{d}x)$ and its extended Dirichlet space $\mathcal{F}_e^Z$ are then given by

$$\mathcal{F}_e^Z = H_{e,0}(\widetilde{I}), \qquad \mathcal{F}^Z = H_{e,0}(\widetilde{I}) \cap L^2(\widetilde{I}; 2\mathrm{d}x), \qquad \mathcal{E}^Z(u, v) = \int_{\widetilde{I}} u'(x)v'(x)\mathrm{d}x,$$

where

$$H_{e,0}(\widetilde{I}) = \Big\{ u : \text{ absolutely continuous on } \widetilde{I},$$
$$\int_{\widetilde{I}} u'(x)^2 \mathrm{d}x < \infty, \ u(s(r_i)) = 0, \text{ if } |s(r_i)| < \infty \Big\}.$$

Since the above PCAF A of Z possesses $\widetilde{m}$ as its Revuz measure relative to $2\mathrm{d}x$, we can use a time change theorem in [13] to conclude that $\widetilde{X}^0$ is $\widetilde{m}$-symmetric, and its Dirichlet form $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ on $L^2(\widetilde{I}; \widetilde{m})$ and extended Dirichlet space $\widetilde{\mathcal{F}}_e$ are given by

$$\widetilde{\mathcal{F}}_e = H_{e,0}(\widetilde{I}), \qquad \widetilde{\mathcal{F}} = H_{e,0}(\widetilde{I}) \cap L^2(\widetilde{I}; \widetilde{m}), \qquad \widetilde{\mathcal{E}}(u, v) = \int_{\widetilde{I}} u'(x)v'(x)\mathrm{d}x.$$

Theorem 4.2 is then an easy consequence of the next lemma and Theorem 3.3. □

Lemma 4.3. *Let $(\widetilde{E}, \widetilde{m})$ be a σ-finite measure space, $\widetilde{X} = (\widetilde{X}_t, \widetilde{\mathbf{P}}_{\widetilde{x}})$ an $\widetilde{m}$-symmetric Markov process on $\widetilde{E}$, $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ the Dirichlet form of $\widetilde{X}$ on $L^2(\widetilde{E}; \widetilde{m})$ and $\widetilde{\mathcal{F}}_e$ its extended Dirichlet space. Let γ be a one-to-one transformation from $\widetilde{E}$ onto a space E and m the image measure of $\widetilde{m}$: $m(B) = \widetilde{m}(\gamma^{-1}(B))$. We put*

$$X_t = \gamma(\widetilde{X}_t), \quad t \geq 0, \qquad \mathbf{P}_x = \widetilde{\mathbf{P}}_{\gamma^{-1}x}, \quad x \in E.$$

Then $X = (X_t, \mathbf{P}_x)$ is an m-symmetric Markov process on E. The Dirichlet form $(\mathcal{E}, \mathcal{F})$ of X on $L^2(E; m)$ and its extended Dirichlet space $\mathcal{F}_e$ are given respectively by

$$\mathcal{F} = \left\{ u \in L^2(E; m) : u \circ \gamma \in \widetilde{\mathcal{F}} \right\} \qquad \mathcal{E}(u, v) = \widetilde{\mathcal{E}}(u \circ \gamma, v \circ \gamma), \quad u, v \in \mathcal{F}, \tag{4.2}$$

$$\mathcal{F}_e = \left\{ u : m\text{-measurable } u \circ \gamma \in \widetilde{\mathcal{F}}_e \right\}. \tag{4.3}$$

(4.2) has been shown in [9, Lemma 3.1]. (4.2) can be restated as

$$\mathcal{F} = \left\{ u : \ m\text{-measurable } u \circ \gamma \in L^2(\widetilde{E}; \widetilde{m}) \cap \widetilde{\mathcal{F}}_e \right\},$$

from which follows (4.3).

As $\widetilde{X}^0$ is a time change of the absorbing Brownian motion Z on $\widetilde{I} = (s(r_1), s(r_2))$, $\lim_{t \to \widetilde{\zeta}^0} \widetilde{X}_t^0 = s(r_i)$ with positive probability if and only if $s(r_i)$ is finite. Therefore we get from Theorem 4.2.

Corollary 4.4. *The boundary r_i of I is approachable if and only if*

$$\mathbf{P}_x\left(\lim_{t\to\zeta^0} X_t^0 = r_i\right) > 0, \quad x \in I.$$

This legitimates our usage of the term 'approachable' for the boundary r_i of I.

5. Reflected Dirichlet space and symmetric extensions

First we present some general results on the reflected Dirichlet space and symmetric extensions of a general regular irreducible Dirichlet form.

Let E be a locally compact separable metric space, m a positive Radon measure on E with full support, $(\mathcal{E}, \mathcal{F})$ a regular irreducible Dirichlet form on $L^2(E; m)$ and $X = (X_t, \mathbf{P}_x, \zeta)$ an associated Hunt process on E. Denote by $\mathcal{F}_e$ the extended Dirichlet space of $(\mathcal{E}, \mathcal{F})$: $u \in \mathcal{F}_e$ iff there exists a sequence $\{u_n\} \subset \mathcal{F}$ such that it is $\mathcal{E}$-Cauchy and $\lim_{n\to\infty} u_n = u$, m-a.e. $\mathcal{E}$ then extends to $\mathcal{F}_e \times \mathcal{F}_e$ and

$$\mathcal{F} = \mathcal{F}_e \cap L^2(E; m). \tag{5.1}$$

Each element of $\mathcal{F}_e$ will be represented by its $\mathcal{E}$-quasicontinuous m-version.

For any $u \in \mathcal{F}_e$, the unique decomposition

$$u(X_t) - u(X_0) = M_t^{[u]} + N_t^{[u]}, \quad t \geq 0$$

holds true where $M^{[u]}$ is a martingale additive functional (MAF in abbreviation) of finite energy and $N^{[u]}$ is a continuous additive functional of zero energy of X. $M^{[u],c}$ denotes the continuous part of the MAF $M^{[u]}$. The predictable quadratic variation $\langle M^{[u],c}\rangle$ is a PCAF of X and so admits its Revuz measure denoted by $\mu^c_{\langle u\rangle}$.

Let $(N(x, \mathrm{d}y), H)$ be a Lévy system of X. The *jumping measure* J and the *killing measure* κ of X are then defined by

$$J(\mathrm{d}x, \mathrm{d}x) = N(x, \mathrm{d}y)\mu_H(\mathrm{d}x), \qquad \kappa(\mathrm{d}x) = N(x, \{\partial\})\mu_H(\mathrm{d}x),$$

respectively, where μ_H denotes the Revuz measure of the PCAF H and ∂ is the cemetery adjoined to E.

We then have the following *Beurling–Deny formula* [1,13,4]: For any $u \in \mathcal{F}_e, \mathcal{E}(u, u)$ is represented as

$$\mathcal{E}(u, u) = \frac{1}{2}\mu^c_{\langle u\rangle}(E) + \frac{1}{2}\int_{E\times E\backslash d}(u(x) - u(y))^2 J(\mathrm{d}x, \mathrm{d}y) + \int_E u(x)^2\kappa(\mathrm{d}x).$$

Denote by $\mathcal{F}_{\mathrm{loc}}$ the space of functions u on E such that, for any relatively compact open set $G \subset E$, there exists $v \in \mathcal{F}$ with $v = u$ m-a.e. on G. Any $u \in \mathcal{F}_{\mathrm{loc}}$ admits an $\mathcal{E}$-quasicontinuous m-version and the measure $\mu^c_{\langle u\rangle}$ is well defined for $u \in \mathcal{F}_{\mathrm{loc}}$. Therefore

$$\widehat{\mathcal{E}}(u, u) = \frac{1}{2}\mu^c_{\langle u\rangle}(E) + \frac{1}{2}\int_{E\times E\backslash d}(u(x) - u(y))^2 J(\mathrm{d}x, \mathrm{d}y) + \int_E u(x)^2\kappa(\mathrm{d}x)(\leq \infty)$$

is well defined for any $u \in \mathcal{F}_{\mathrm{loc}}$. For a function u on E, we let $\tau_k u = ((-k) \vee u) \wedge k$ for $k \in \mathbb{N}$.

The *reflected Dirichlet space* of X is then defined by

$$\begin{cases} \mathcal{F}^{\mathrm{ref}} = \left\{ u : |u| < \infty \ m\text{-a.e. } \tau_k u \in \mathcal{F}_{\mathrm{loc}}, \ \sup\limits_{k\ge 1} \widehat{\mathcal{E}}(\tau_k u, \tau_k u) < \infty \right\} \\ \mathcal{E}^{\mathrm{ref}}(u,u) = \widehat{\mathcal{E}}(u,u), \quad u \in \mathcal{F}^{\mathrm{ref}}, \end{cases} \tag{5.2}$$

while the *active reflected Dirichlet space* of X is defined by

$$\mathcal{F}_a^{\mathrm{ref}} = \mathcal{F}^{\mathrm{ref}} \cap L^2(E; m). \tag{5.3}$$

A Dirichlet form $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ on $L^2(E; m)$ is said to be a *Silverstein extension* of $(\mathcal{E}, \mathcal{F})$ if $\mathcal{F}_b$ is an ideal of $\widetilde{\mathcal{F}}_b$ (that is, $\mathcal{F}_b \subset \widetilde{\mathcal{F}}_b$ and $fg \in \mathcal{F}_b$ for every $f \in \mathcal{F}_b$ and $g \in \widetilde{\mathcal{F}}_b$) and $\widetilde{\mathcal{E}} = \mathcal{E}$ on $\mathcal{F}_b \times \mathcal{F}_b$.

We introduce a semi-order $\prec$ among Dirichlet forms on $L^2(E; m)$ as follows: for two Dirichlet forms $(\mathcal{E}^{(i)}, \mathcal{F}^{(i)})$, $i = 1, 2$, on $L^2(E; m)$, we write $(\mathcal{E}^{(1)}, \mathcal{F}^{(1)}) \prec (\mathcal{E}^{(2)}, \mathcal{F}^{(2)})$ if

$$\mathcal{F}^{(1)} \subset \mathcal{F}^{(2)} \quad \text{and} \quad \mathcal{E}^{(1)}(u,u) \ge \mathcal{E}^{(2)}(u,u) \quad \text{for every } u \in \mathcal{F}^{(1)}.$$

The next theorem is taken from [4]. See also [2,20].

Theorem 5.1. (i) *The active reflected Dirichlet form $(\mathcal{E}^{\mathrm{ref}}, \mathcal{F}_a^{\mathrm{ref}})$ of X is a Dirichlet form on $L^2(E; m)$ and a Silverstein extension of $(\mathcal{E}, \mathcal{F})$.*
(ii) *$(\mathcal{E}^{\mathrm{ref}}, \mathcal{F}_a^{\mathrm{ref}})$ is maximal among all Silverstein extensions of $(\mathcal{E}, \mathcal{F})$: if $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ is a Silverstein extension of $(\mathcal{E}, \mathcal{F})$, then $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}}) \prec (\mathcal{E}^{\mathrm{ref}}, \mathcal{F}_a^{\mathrm{ref}})$.*
(iii) *If X is recurrent, then $(\mathcal{E}^{\mathrm{ref}}, \mathcal{F}^{\mathrm{ref}}) = (\mathcal{E}, \mathcal{F}_e)$ and consequently $(\mathcal{E}^{\mathrm{ref}}, \mathcal{F}_a^{\mathrm{ref}}) = (\mathcal{E}, \mathcal{F})$.*
If X is conservative, namely, $\mathbf{P}_x(\zeta < \infty) = 0$, $x \in E$, then $(\mathcal{E}^{\mathrm{ref}}, \mathcal{F}_a^{\mathrm{ref}}) = (\mathcal{E}, \mathcal{F})$.

Clearly $\mathcal{F} \subset \mathcal{F}_a^{\mathrm{ref}}$ and $\mathcal{E}^{\mathrm{ref}}(u,u) = \mathcal{E}(u,u)$ for $u \in \mathcal{F}$.

We say that X *admits a reflecting extension* if

$$\mathcal{F} \ne \mathcal{F}_a^{\mathrm{ref}}. \tag{5.4}$$

In view of Theorem 5.1, the condition (5.4) is equivalent to the existence of at least one proper Silverstein extension of $(\mathcal{E}, \mathcal{F})$. Furthermore, for (5.4) to be fulfilled, it is necessary that X is transient, and even more strongly, non-conservative.

We now return to the diffusion X^0 on a interval $I = (r_1, r_2)$ satisfying (1), (2), (3), (4). X^0 is symmetric with respect to its canonical measure m and its Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(I; m)$ is given by (3.6) in terms of the canonical scale s by virtue of Theorem 3.3. Recall that the boundary r_i is called approachable if $s(r_i)$ is finite and regular if r_i is approachable and m is finite near r_i.

Theorem 5.2. *X^0 admits a reflecting extension if and only if either r_1 or r_2 is regular.*

Proof. We can readily see from (3.6) and (5.3) that

$$\mathcal{F}_a^{\mathrm{ref}} = \mathcal{F}^{(s)} \cap L^2(I; m). \tag{5.5}$$

Therefore the 'if' part is obvious. Suppose both r_1 and r_2 are non-regular. If both of them are non-approachable, then $\mathcal{F}_e = \mathcal{F}^{(s)}$ and (5.4) fails. If r_i is approachable, then any u in the right-hand side of (5.5) admits a finite limit at r_i, which must vanish because $u \in L^2(I; m)$ and m is infinite near r_i. This means $u \in \mathcal{F}$. $\square$

Suppose either r_1 or r_2 is regular. Denote by I^* the interval obtained by adding the point r_i to I if r_i is regular. Denote by m^* the extension of m from I to I^* obtained by setting $m^*(\{r_i\}) = 0$ when r_i is regular. We identify $L^2(I; m)$ with $L^2(I^*; m^*)$. Then the active reflected Dirichlet form

$$(\mathcal{E}^{\text{ref}}, \mathcal{F}_a^{\text{ref}}) = (\mathcal{E}^{(s)}, \mathcal{F}^{(s)} \cap L^2(I; m)) \tag{5.6}$$

can be regarded as a regular, strongly local irreducible Dirichlet form on $L^2(I^*; m^*)$ and accordingly admits an associated diffusion process $X^r = (X_t^r, \mathbf{P}_x^r, \zeta^r)$ on I^* with no killing inside I^*. [4, Section 2.2.3] makes a direct consideration of the Dirichlet form (5.6) without referring to (3.6).

The Dirichlet form (3.6) is the part of the Dirichlet form (5.6) on I: $\mathcal{F} = \{u \in \mathcal{F}_a^{\text{ref}} : u = 0 \text{ on } I^* \setminus I\}$. Accordingly, X^0 is the part process of X^r on I, namely, X^0 is obtained from X^r by killing upon hitting $I^* \setminus I$. In other words, X^r is a symmetric extension of X^0 from I to I^* with no sojourn nor killing on $I^* \setminus I$. We call X^r the *reflecting extension* of X^0. See also [19].

Suppose next that both r_1 and r_2 are regular. Then, besides the reflecting extension of X^0, there are three other symmetric extensions of X^0 with no sojourn nor killing at the added boundary points; the extension to $[r_1, r_2)$ reflecting only at r_1, the extension to $(r_1, r_2]$ reflecting only at r_2 and an extension to the one point compactification $\dot{I} = I \cup \Delta$ of I. The last one can be described as follows. Extend m to $\dot{m}$ on $\dot{I}$ by setting $\dot{m}(\{\Delta\}) = 0$. Then

$$(\dot{\mathcal{E}}, \dot{\mathcal{F}}) = \left(\mathcal{E}^{(s)},\ \{u \in \mathcal{F}^{(s)} : u(r_1) = u(r_2)\}\right) \tag{5.7}$$

is a regular strongly local irreducible Dirichlet form on $L^2(\dot{I}, \dot{m})$ and the associated diffusion $\dot{X}$ on $\dot{I}$ is a symmetric extension of X^0 with no sojourn nor killing at Δ. The diffusion $\dot{X}$ can be also constructed from X^0 probabilistically as in [14] by piecing together excursions of X^0 around Δ evolving as a Poisson point process with a characteristic measure uniquely determined by the pair (s, m). Similarly, the reflecting extension X^r can be constructed from X^0 by a repeated usage of certain Poisson point processes of excursions as in [6].

Suppose r_2 is non-regular and r_1 is exit in the sense of (2.7). Then r_1 is approachable in finite time in the sense of (2.10) and X^0 is non-conservative. Nevertheless, Theorem 5.2 says that X^0 does not admit any symmetric extension when $m(r_1, c) = \infty,\ c \in I$. In this case, r_1 is non-entrance in the sense of (2.7).

6. Domain with two branches of infinite cones

Let D be a domain of $\mathbb{R}^d$ with $d \geq 1$ and $L^2(D)$ be the L^2-space of functions on D based on the Lebesgue measure $\mathrm{d}x$. We consider the space

$$\mathrm{BL}(D) = \left\{u \in L^2_{\text{loc}}(D) : \frac{\partial u}{\partial x_i} \in L^2(D),\ 1 \leq i \leq d\right\}, \tag{6.1}$$

where the derivatives are taken in Schwartz distribution sense. Members in $\mathrm{BL}(D)$ are called *BL* (Beppo Levi) *functions* on D (cf. [8]). The *Sobolev space of order* (1, 2) is defined by

$$W^{1,2}(D) = \mathrm{BL}(D) \cap L^2(D).$$

Let $\mathbf{D}(u, v) = \int_D \nabla u(x) \cdot \nabla v(x) \mathrm{d}x$. Then

$$(\mathcal{E}, \mathcal{F}) = \left(\frac{1}{2}\mathbf{D}, W^{1,2}(D)\right) \tag{6.2}$$

is a strongly local irreducible Dirichlet form on $L^2(D)$. The extended Dirichlet space of (6.2) will be denoted by $W_e^{1,2}(D)$.

A domain $D \subset \mathbb{R}^d$ is called recurrent (resp. transient) if the Dirichlet form (6.2) is recurrent (resp. transient). D is recurrent if either $d \le 2$ or the Lebesgue measure of D is finite [12]. If $d \ge 3$, any domain containing an infinite cone is transient [4, Section 3.5]. But an infinite cylinder is recurrent for any d. In what follows, we assume that D is transient (in particular $d \ge 3$ and D is of infinite Lebesgue measure) and D is of continuous boundary in the sense that D locally lies above the graph of a continuous function.

The Dirichlet form (6.2) can then be regarded as a regular Dirichlet form on $L^2(\overline{D}; 1_D(x)\mathrm{d}x)$ and the associated diffusion process $Z = (Z_t, \mathbf{Q}_x)$ on $\overline{D}$ is the *reflecting Brownian motion* by definition. $Z = (Z_t, \mathbf{Q}_x)$ is always conservative due to Takeda's test [13]. But we see from [5, (3.2)] that Z escapes to infinity as $t \to \infty$:

$$\mathbf{Q}_x\left(\lim_{t\to\infty} Z_t = \partial\right) = 1 \quad x \in \overline{D}, \tag{6.3}$$

where ∂ is the point at infinity of $\overline{D}$. Any function in the space $\mathrm{BL}(D)$ is represented by its quasicontinuous version. As is verified in [5], any $u \in W_e^{1,2}(D)$ then satisfies

$$\mathbf{Q}_x\left(\lim_{t\to\infty} u(Z_t) = 0\right) = 1 \quad \text{for q.e. } x \in \overline{D}. \tag{6.4}$$

It is proved in [5, Theorem 3.1] that the reflected Dirichlet space of the Dirichlet form (6.2) is identical with the space $(\frac{1}{2}\mathbf{D}, \mathrm{BL}(D))$ of BL functions. The extended Sobolev space $W_e^{1,2}(D)$ is a subspace of $\mathrm{BL}(D)$, and indeed a proper subspace because the former does not contain non-zero constant function due to the transience assumption while the latter does.

Let $\mathbb{H}^*(D)$ be the space of functions in $\mathrm{BL}(D)$ which are $\mathbf{D}$-orthogonal to $W_e^{1,2}(D)$. Then $\mathbb{H}^*(D) \subset \mathbb{H}(D)$, where $\mathbb{H}(D)$ denotes the space of harmonic functions on D with finite Dirichlet integral. Moreover, by virtue of [3] (see also [4, Section 6.7]), any $u \in \mathbb{H}^*(D)$ is *harmonic* with respect to the reflecting Brownian motion Z on $\overline{D}$ in the sense that, for any relatively compact open subset D_1 of $\overline{D}$,

$$u(x) = \mathbf{E}^{\mathbf{Q}_x}\left[u(Z_{\tau_{D_1}})\right] \quad \text{for q.e. } x \in D_1, \tag{6.5}$$

where τ_{D_1} denotes the first exit time from D_1.

We will be concerned with the condition that

$$\mathbb{H}^*(D) \text{ consists of constant functions on } D. \tag{6.6}$$

It is known that (6.6) holds true when $D = \mathbb{R}^d$, $d \ge 3$, (cf. [12]). This property remains valid for an unbounded uniform domain. A domain D is called a *uniform domain* if there exists $C > 1$ such that for every $x, y \in D$, there is a rectifiable curve γ in D connecting x and y with length $(\gamma) \le C|x - y|$ and moreover

$$\min\{|x - z|,\ |z - y|\} \le C\mathrm{dist}(z, D^c) \quad \text{for every } z \in \gamma.$$

An infinite cone is a special unbounded uniform domain. It is shown in [5] that a domain containing an unbounded uniform domain is transient. Furthermore Proposition 3.6 of [5] states that

$$D \setminus \overline{B_r(0)} \text{ is an unbounded uniform domain for some } r > 0 \Longrightarrow (6.6) \text{ holds.} \tag{6.7}$$

Here $B_r(0)$ denotes the ball with center 0 and radius r.

Since the reflected Dirichlet space of (6.2) equals $\mathrm{BL}(D)$, its active reflected Dirichlet space coincides with $\mathrm{BL}(D) \cap L^2(D) = W^{1,2}(D)$ so that condition (5.4) fails and the reflecting

Brownian motion Z does not admit a reflecting extension. But, if we make a time change of Z, then the situation may change radically.

Now let m be a positive Radon measure on $\overline{D}$ charging no polar set possessing full quasi-support with respect to the Dirichlet form (6.2). For instance, $m(\mathrm{d}x) = f(x)\mathrm{d}x$ for a strictly positive $f \in L^1_{\mathrm{loc}}(D)$ has these properties. Let $X = (X_t, \mathbf{P}_x, \zeta)$ be the time changed process of the reflecting Brownian motion Z on $\overline{D}$ by means of its PCAF A with Revuz measure m:

$$X_t = Z_{\tau_t}, \qquad \tau_t = A_t^{-1}, \qquad \mathbf{P}_x = \mathbf{Q}_x, \qquad \zeta = A_\infty.$$

X is then m-symmetric, and the Dirichlet form of X on $L^2(\overline{D}; m)$ and its active reflected Dirichlet space are given by

$$\left(\frac{1}{2}\mathbf{D},\, W_e^{1,2}(D) \cap L^2(\overline{D}; m)\right) \qquad \left(\frac{1}{2}\mathbf{D},\, \mathrm{BL}(D) \cap L^2(\overline{D}; m)\right), \tag{6.8}$$

respectively, because not only the extended Dirichlet space but also the reflected Dirichlet space are invariant under the time change [4, Section 6.4].

We see that X admits a reflecting extension if $m(\overline{D}) < \infty$, because then the two Dirichlet forms in (6.8) differ; the former does not contain a non-zero constant function while the latter does. Furthermore

Proposition 6.1. *Assume that a domain satisfies condition (6.6). Then X admits a reflecting extension if and only if $m(\overline{D}) < \infty$.*

Proof. If $m(\overline{D}) = \infty$, then $1 \notin L^2(D; m)$ and the two Dirichlet forms in (6.8) coincide under (6.6). $\square$

(6.7) gives a sufficient condition for the validity of (6.6). If an unbounded domain D is not a uniform domain, the dimension of the space $\mathbb{H}^*(D)$ may exceed 2 as we shall see in the following example. Let

$$D = B_1(0) \cup \left\{ x = (x_1, x_2, \ldots, x_d) : x_d^2 > \sum_{k=1}^{d-1} x_k^2 \right\}. \tag{6.9}$$

D contains the upper cone C_+ and lower cone C_- where

$$C_+ = B_1(0)^c \cap \left\{ x_d > \left(\sum_{k=1}^{d-1} x_k^2 \right)^{1/2} \right\} \qquad C_- = B_1(0)^c \cap \left\{ x_d < -\left(\sum_{k=1}^{d-1} x_k^2 \right)^{1/2} \right\}.$$

so that D is transient as it contains an infinite cone C_+ but D cannot be a uniform domain because it has a bottle neck $B_1(0)$.

The point at infinity of $\overline{D}$ at the upper end (lower end) is denoted by ∂_+ (∂_-). Let

$$\varphi^+(x) = \mathbf{Q}_x\left(\lim_{t\to\infty} Z_t = \partial_+\right) \qquad \varphi^-(x) = \mathbf{Q}_x\left(\lim_{t\to\infty} Z_t = \partial_-\right).$$

We then see from (6.3) that

$$\varphi^+(x) + \varphi^-(x) = 1, \quad x \in \overline{D}. \tag{6.10}$$

Furthermore

$$\varphi^+(x) > 0, \qquad \varphi^-(x) > 0, \quad \text{for q.e. } x \in \overline{D}, \tag{6.11}$$

because $\varphi^+(x)$ is either strictly positive q.e. or identically zero q.e. due to the irreducibility so that the above follows from the symmetry of the domain D.

Theorem 6.2. (i) $\mathbb{H}^*(D) = \{c_+\varphi^+ + c_-\varphi^- : c_+,\ c_- \in \mathbb{R}\}$.
(ii) *The time changed reflecting Brownian motion X on $\overline{D}$ admits a reflecting extension if and only if either $m(\overline{C_+})$ or $m(\overline{C_-})$ is finite.*
(iii) *Suppose $m(\overline{D}) < \infty$. Then $\mathbf{P}_x(\zeta < \infty) = 1$ for q.e. $x \in \overline{D}$ and X admits four different kinds of symmetric extensions analogously to the case of the one dimensional diffusion with two regular boundaries as is stated in Section 5. In particular, the two points extension of X from $\overline{D}$ to $\overline{D}\cup\{\partial_+\}\cup\{\partial_-\}$ associated with the active reflected Dirichlet form $(\frac{1}{2}\mathbf{D}, \mathrm{BL}(D)\cap L^2(\overline{D}; m))$ can be constructed by repeating the one point extensions using Poisson point processes of excursions as in* [6,14].

Proof. (i) We first show that, for any $u \in \mathrm{BL}(D)$, there exist constants c_+, c_- such that

$$\begin{cases} \mathbf{Q}_x\left(Z_{\infty-} = \partial_+, \lim_{t\to\infty} u(Z_t) = c_+\right) = \mathbf{Q}_x(Z_{\infty-} = \partial_+) \\ \mathbf{Q}_x\left(Z_{\infty-} = \partial_-, \lim_{t\to\infty} u(Z_t) = c_-\right) = \mathbf{Q}_x(Z_{\infty-} = \partial_-). \end{cases} \tag{6.12}$$

To this end, define, for $n \geq 1$,

$$D_n = B_1(0) \cup C_+ \cup (C_- \cap B_n(0)), \qquad \Gamma_n = C_- \cap \{|x| = n\}.$$

Then $D_n \setminus \overline{B_n(0)}$ is a uniform domain and D_n increases to D as $n \to \infty$. For any $u \in \mathrm{BL}(D), u|_{D_n} \in \mathrm{BL}(D_n)$ which is a sum of a function in $W_e^{1,2}(D_n)$ and some constant c_+ in view of (6.7). Let $Z^n = (Z_t^n, \mathbf{Q}_x^n)$ be the reflecting Brownian motion on $\overline{D}_n$, namely, a diffusion associated with the Dirichlet form $(\frac{1}{2}\mathbf{D}, W^{1,2}(D_n))$ on $L^2(D_n)$. On account of (6.3) and (6.4), we have

$$\mathbf{Q}_x^n\left(\lim_{t\to\infty} u(Z_t^n) = c_+,\ Z_{\infty-} = \partial_+\right) = 1, \quad x \in \overline{D}_n,$$

and we see that c_+ is independent of n.

Since the part processes of Z and Z^n on $\overline{D}_n \setminus \Gamma_n$ are identical in law, we further obtain

$$\begin{aligned} &\mathbf{Q}_x\left(Z_{\infty-} = \partial_+,\ \lim_{t\to\infty} u(Z_t) = c_+\right) \\ &\quad = \lim_{n\to\infty} \mathbf{Q}_x\left(\sigma_{\Gamma_n} = \infty,\ Z_{\infty-} = \partial_+,\ \lim_{t\to\infty} u(Z_t) = c_+\right) \\ &\quad = \lim_{n\to\infty} \mathbf{Q}_x^n\left(\sigma_{\Gamma_n} = \infty,\ Z_{\infty-}^n = \partial_+, \lim_{t\to\infty} u(Z_t^n) = c_+\right) \\ &\quad = \lim_{n\to\infty} \mathbf{Q}_x^n(\sigma_{\Gamma_n} = \infty) = \lim_{n\to\infty} \mathbf{Q}_x(\sigma_{\Gamma_n} = \infty) = \mathbf{Q}_x(Z_{\infty-} = \partial_+), \end{aligned}$$

completing the proof of the first identity of (6.12). The second one can be shown similarly.

Now take any $u \in \mathbb{H}^*(D)$. We may assume that u is bounded. Let τ_n be the first exit time of Z from the set $\overline{D} \cap B_n(0)$. By virtue of (6.5), $(\{u(Z_{\tau_n})\},\ \mathbf{Q}_x)$ is an uniformly integrable martingale so that the limit $\Phi = \lim_{n\to\infty} u(Z_{\tau_n})$ exists $\mathbf{Q}_x$-a.s. and $u(x) = \mathbf{E}^{\mathbf{Q}_x}[\Phi],\ x \in \overline{D}$. Let $c_\pm$ be constants corresponding to u by (6.10). Then

$$\Phi = \Phi \mathbf{1}_{\{Z_{\infty-}=\partial_+\}} + \Phi \mathbf{1}_{\{Z_{\infty-}=\partial_-\}} = c_+ \mathbf{1}_{\{Z_{\infty-}=\partial_+\}} + c_- \mathbf{1}_{Z_{\{\infty-=\partial_-\}}},$$

and we arrive at the inclusion $\subset$ in (i) by taking $\mathbf{Q}_x$-expectation.

Conversely, for any $c_+, \ c_- \in \mathbb{R}$, there exists $u \in \mathrm{BL}(D)$ with $\lim_{x\to\partial_\pm} u(x) = c_\pm$. Then $v = u - \mathbf{P}_{W_e^{1,2}(D)}u$ is a function in $\mathbb{H}^*(D)$ with $v = c_+\varphi^+ + c_-\varphi^-$.
(ii) This is obvious from (i) and (6.8).
(iii) We only give a proof of the first assertion. Since the time changed process X is transient together with Z and $1 \in L^1(D; m)$ when m is finite, we get $\mathbf{E}_x[\zeta] = R1(x) < \infty$ for q.e. $x \in \overline{D}$ where R denotes the 0-order resolvent of X. $\square$

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