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Recurrent Dirichlet Forms and Markov Property of Associated Gaussian Fields

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Abstract For the extended Dirichlet space $\mathcal{F}_e$ of a general irreducible recurrent regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(E; m)$, we consider the family $\mathbb{G}(\mathcal{E}) = \{X_u; u \in \mathcal{F}_e\}$ of centered Gaussian random variables defined on a probability space $(\Omega, \mathcal{B}, \mathbb{P})$ indexed by the elements of $\mathcal{F}_e$ and possessing the Dirichlet form $\mathcal{E}$ as its covariance. We formulate the Markov property of the Gaussian field $\mathbb{G}(\mathcal{E})$ by associating with each set $A \subset E$ the sub-σ-field $\sigma(A)$ of $\mathcal{B}$ generated by X_u for every $u \in \mathcal{F}_e$ whose spectrum $s(u)$ is contained in A. Under a mild absolute continuity condition on the transition function of the Hunt process associated with $(\mathcal{E}, \mathcal{F})$, we prove the equivalence of the Markov property of $\mathbb{G}(\mathcal{E})$ and the local property of $(\mathcal{E}, \mathcal{F})$. One of the key ingredients in the proof is in that we construct potentials of finite signed measures of zero total mass and show that, for any Borel set B with $m(B) > 0$, any function $u \in \mathcal{F}_e$ with $s(u) \subset B$ can be approximated by a sequence of potentials of measures supported by B.

Keywords Recurrent potential · Dirichlet form · Spectrum · Gaussian field

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1 Introduction

Let E be a locally compact separable metric space and m an everywhere dense positive Radon measure on E. Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form on $L^2(E; m)$. In this paper, we consider the family $\mathbb{G}(\mathcal{E}) = \{X_u : u \in \mathcal{F}_e\}$ of centered Gaussian random variables defined on a probability space $(\Omega, \mathcal{B}, \mathbb{P})$ indexed by the functions of the extended Dirichlet space $\mathcal{F}_e$ and possessing the covariance $\mathbb{E}[X_u X_v] = \mathcal{E}(u, v), \ u, v \in \mathcal{F}_e$.

In order to formulate the Markov property of the Gaussian field $\mathbb{G}(\mathcal{E})$, we adopt the notion of the spectrum first introduced by A. Beurling and J. Deny [2]. For any $u \in \mathcal{F}_e$, the *spectrum* $s(u)$ is defined to be the complement of the largest open set G satisfying

$$\mathcal{E}(u, v) = 0, \qquad \forall v \in \mathcal{F} \cap \mathcal{C}_c(E), \quad \text{supp}[v] \subset G, \tag{1.1}$$

where $\mathcal{C}_c(E)$ is the family of continuous functions with compact support. See [4, p 166] and [8, p 99].

For $A \subset E$, let $\sigma(A)$ be the sub-σ-field of $\mathcal{B}$ defined by

$$\sigma(A) = \sigma\{X_u : u \in \mathcal{F}_e, \ s(u) \subset A\}. \tag{1.2}$$

Given a Borel set A of E, the Gaussian fields $\mathbb{G}(\mathcal{E})$ is said to possess the *Markov property with respect to* A if it holds that

$$\mathbb{E}\left[ZY|\sigma(\partial A)\right] = \mathbb{E}\left[Z|\sigma(\partial A)\right]\mathbb{E}\left[Y|\sigma(\partial A)\right] \tag{1.3}$$

for every bounded, $\sigma(\overline{E \setminus A})$-measurable function Y and $\sigma(\overline{A})$-measurable functon Z on Ω. If $\mathbb{G}(E)$ possesses the Markov property with respect to any open set (resp. any relatively compact open set) A, then $\mathbb{G}(\mathcal{E})$ is said to possess the *global Markov property* (resp.*local Markov property*).

When the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is transient, M. Röckner established the equivalence of the global Markov property of the Gaussian field $\mathbb{G}(\mathcal{E})$ and the local property of the form $(\mathcal{E}, \mathcal{F})$ [14, Theorem 7.4]. We now briefly recall a method of the proof in [14].

In the transient case, one can consider the family $\mathcal{M}_0 = \mathcal{S}_0^{(0)} - \mathcal{S}_0^{(0)}$ of signed Radon measures on E with finite (0-order) energy. Each measure $\mu \in \mathcal{M}_0$ admits a unique (kernel free) potential $U\mu$ sitting in the extended Dirichlet space $\mathcal{F}_e$ and satisfies the (generalized) *Poisson equation*

$$\mathcal{E}(U\mu, u) = \langle \mu, \widetilde{u} \rangle, \quad \text{for any } u \in \mathcal{F}_e, \tag{1.4}$$

where $\widetilde{u}$ is any quasi-continuous version of u. It follows that $s(U\mu) = \text{supp}(\mu)$.

The subcollection $\{X_{U\mu}, \mu \in \mathcal{M}_0\}$ of $\mathbb{G}(\mathcal{E})$ can be considered as a Gaussian field indexed by the space $\mathcal{M}_0$ of measures and accordingly it is denoted by $\mathbb{G}(\mathcal{M}_0)$. The above mentioned characterization of the Markov property can be well shown for the Gaussian field $\mathbb{G}(\mathcal{M}_0)$ by using the celebrated *balayage* operations on the space $\mathcal{M}_0$ because the swept out measure is concentrated on the boundary ∂A if and only if the form $\mathcal{E}$ is local. Since the field $\mathbb{G}(\mathcal{E})$ is obtained by completing $\mathbb{G}(\mathcal{M}_0)$ in $L^2(\Omega, \mathbb{P})$ (see Remark 2.1), the characterization is readily inherited by $\mathbb{G}(\mathcal{E})$.

We are concerned with extending the work [14] to recurrent Dirichlet forms $(\mathcal{E}, \mathcal{F})$. The first named author of the present paper has investigated in [7] the special cases where $\mathcal{E}$ is the half of the Dirichlet integral $\mathbf{D}$ and $\mathcal{F}$ are the Sobolev spaces H^1 over the complex plane $\mathbb{C}$, the upper half plane $\mathbb{H}$ and the real line $\mathbb{R}$. In the case of $\mathbb{C}$, we are a priori given the logarithmic kernel which leads to a natural choice of the space $\mathcal{M}_{00}(\mathbb{C})$ of finite signed measures of compact support, of finite logarithmic energy and with zero total mass. It is shown in [7] that the logarithmic potential $U\mu$ of any $\mu \in \mathcal{M}_{00}(\mathbb{C})$ is a quasi-continuous function belonging to the extended Dirichlet space of $H^1(\mathbb{C})$ and satisfies a counterpart of the Poisson equation (1.4). The Gaussian field $\mathbb{G}(\mathbb{C})$ indexed by $\mathcal{M}_{00}(\mathbb{C})$ is then shown to enjoy the local Markov property by using the balayage theorem for the logarithmic potentials presented in Port and Stone [12]. Exactly analogous considerations are made in [7] also for the cases of $\mathbb{H}$ and $\mathbb{R}$.

A primary purpose of this paper is to demonstrate the equivalence between the Markov property of $\mathbb{G}(\mathcal{E})$ and the local property of the Dirichlet form for a general regular irreducible recurrent Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(E; m)$ under a mild absolute continuity condition (2.6) on the transition function of the associated Hunt process $\mathbb{M}$ on E.

For each specific compact set $F \subset E$ called an *admissible set* (see Eq. 3.1), we shall make use of the time changed process of $\mathbb{M}$ by the positive continuous additive functional $C_t = \int_0^t I_F(X_s)ds$ and the subprocess of $\mathbb{M}$ generated by the multiplicative functional e^{-C_t} to construct a unique function $R\mu$ in $\mathcal{F}_e$ for each μ belonging to a certain class $\mathcal{M}_0$ of finite signed measures in such a way that μ and $R\mu$ satisfy a counterpart of the Poisson equation (1.4). The collection $\{R\mu : \mu \in \mathcal{M}_0\}$ will be called the *family of recurrent potentials relative to an admissible set F*.

When μ is absolutely continuous with respect to m with a density function f, such a potential Rf has been constructed and utilized by the second named author [11] of the present paper and in [8, Section 4.8] as well to obtain a Poincaré type inequality and thereby the Hilbertian structure of the quotient space of $\mathcal{F}_e$ by constant functions, that will be also derived in the present paper in Section 3.3. The first construction of this kind of potentials of functions for recurrent Markov processes goes back to the works by D. Revuz [13] and [10].

The class $\mathcal{M}_0$ of measures and the associated potentials $R\mu$ of $\mu \in \mathcal{M}_0$ depend on the choice of an admissible set F and we formulate a balayage theorem in Section 3.3 in this context. One of the key ingredients in the proof of the characterization of the Markov property of $\mathbb{G}(\mathcal{E})$ is to show that any $u \in \mathcal{F}_e$ whose spectrum $s(u)$ is contained in a closed set B of positive m-measure can be approximated by a sequence of potentials of measures supported by B.

The organization of this paper is as follows. In Section 2, we prepare preliminary results under the absolute continuity condition (2.6). In particular, some properties of the perturbed Dirichlet form by a non-negative function g and the corresponding canonical subprocess of $\mathbb{M}$ are presented. The construction of recurrent potentials of measures in $\mathcal{M}_0$ by means of a trace form and a perturbed form of $(\mathcal{E}, \mathcal{F})$ relative to an admissible set F will be carried out in Sections 3.1 and 3.2. Basic properties of the extended Dirichlet space $\mathcal{F}_e$ and recurrent potentials are presented in Section 3.3. The stated characterization of the Markov property of $\mathbb{G}(\mathcal{E})$ is proved in Section 4 using the balayage theorem in Section 3. In Section 5, the present recurrent potentials of measures are related to their logarithmic potentials in the special cases of $(\frac{1}{2}\mathbf{D}, H^1(\mathbb{C}))$ and $(\frac{1}{2}\mathbf{D}, H^1(\overline{\mathbb{H}}))$. Typical examples of regular strongly local Dirichlet forms in finite dimensions satisfying condition (2.6) are also exhibited.

Thus the present paper extends the work [14] from transient Dirichlet forms to recurrent ones. See references in [14] for related literatures earlier than 1985. Especially we like to mention the works by E.Nelson [9], Albeverio and Hoegh-Krohn [1] and Dynkin [5] that were closely related to the transient Dirichlet forms. The Hilbertian structure of quotient spaces of recurrent Dirichlet forms shown in [8] and in Theorem 3.7 below is being well utilized by M.Takeda [16].

2 Preliminaries and Perturbed Dirichlet Forms

Let E be a locally compact separable metric space and m an everywhere dense positive Radon measure of E. Given a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(E;m)$, let $\{T_t, t \geq 0\}$ and $\{G_\alpha, \alpha > 0\}$ be the associated semigroup and resolvent on $L^2(E;m)$, respectively. The Bochner integral $S_T f = \int_0^T T_t dt,\ T > 0,\ f \in L^2(E;m)$, induces a bounded operator on $L^1(E;m)$ and $Gf(x) = \lim_{T\uparrow\infty} S_T f(x) \leq \infty$ is well defined for non-negative $f \in L^1(E;m)$ up to m-equivalence.

The semigroup $\{T_t, t > 0\}$ or the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is called *transient* if $Gf < \infty$ m-a.e. for some positive, or equivalently, for all non-negative $f \in L^1(E;m)$ (see [8, Lemma 1.5.1].). It is called *recurrent* if $Gf(x)$ equals 0 or ∞ m-a.e. for any non-negative $f \in L^1(E;m)$. A measurable set A of E is called a T_t*-invariant set* if $T_t(1_A f) = 1_A T_t f$ m-a.e. for any $f \in L^2(E;m)$ and $t > 0$. $\{T_t, t > 0\}$ or $\mathcal{E}$ is called *irreducible* if any T_t-invariant set B satisfies $m(B) = 0$ or $m(E \setminus B) = 0$. If $\{T_t, t > 0\}$ is irreducible, then it is either transient or recurrent [8, Lemma 1.6.4].

Let $(\mathcal{F}_e, \mathcal{E})$ be the extended Dirichlet space of $(\mathcal{E}, \mathcal{F})$. $u \in \mathcal{F}_e$ if there exists an $\mathcal{E}$-Cauchy sequence $\{u_n\} \subset \mathcal{F}$ such that $\lim_{n\to\infty} u_n = u$ m-a.e. $\mathcal{E}$ is then extended from $\mathcal{F}$ to $\mathcal{F}_e$ by $\mathcal{E}(u,u) = \lim_{n\to\infty} \mathcal{E}(u_n, u_n)$. The above sequence $\{u_n\}$ is called an *approximating sequence* for $u \in \mathcal{F}_e$. Let $\mathrm{Cap}(A)$ be the $\mathcal{E}_1$-capacity of $A \subset E$. A function u on E is called *quasi-continuous* if, for any $\epsilon > 0$, there exists an open set G such that $\mathrm{Cap}(G) < \epsilon$ and $u|_{E\setminus G}$ is finite and continuous. 'q.e.' will means 'except for a set of zero capacity'. Any function $u \in \mathcal{F}_e$ has a quasi-continuous modification $\tilde{u}$ [8, Theorem 2.1.7].

For a function f and a measure μ on E, the integral $\int_E f(x)\mu(dx)$ will be denoted by $\langle f, \mu\rangle$ or $\langle \mu, f\rangle$ whenever the integral makes sense. For two functions $f,\ g$ and a measure μ on E, $\langle fg, \mu\rangle$ will be occasionally denoted by $(f, g)_\mu$. $(f, g)_m$ will be disignanted simply as (f, g).

A positive Radon measure μ on E is called a *measure of finite energy* and we write as $\mu \in \mathcal{S}_0$ if there exists a positive constant C such that

$$\langle |u|, \mu\rangle \leq C\sqrt{\mathcal{E}_1(u,u)}, \quad \text{for all } u \in \mathcal{F} \cap C_c(E).$$

For any $\mu \in \mathcal{S}_0$ and $\alpha > 0$, there exists uniquely $U_\alpha \mu \in \mathcal{F}$ satisfying

$$\mathcal{E}_\alpha(U_\alpha \mu, u) = \langle \mu, \widetilde{u}\rangle, \quad \text{for any } u \in \mathcal{F},$$

where $\mathcal{E}_\alpha(u, v) = \mathcal{E}(u, v) + \alpha(u, v),\ u, v \in \mathcal{F}$. $U_\alpha\mu$ is called the α-potential of μ.

The transience of the semigroup $\{T_t, t > 0\}$ is equivalent to the following condition of the Dirichlet form [8, Theorem 1.5.1]: there exists a bounded m-integrable function h strictly positive m-a.e. on E such that

$$(|u|, h) \leq \sqrt{\mathcal{E}(u,u)}, \quad \text{for any } u \in \mathcal{F}. \tag{2.1}$$

Then, this inequality is extended to any $u \in \mathcal{F}_e$ and $\mathcal{F}_e$ becomes a real Hilbert space with inner product $\mathcal{E}$. The function h in Eq. 2.1 is called a *reference function*.

Assume the transience. A positive Radon measure μ on E is called a *measure of finite 0-order energy* and we write as $\mu \in \mathcal{S}_0^{(0)}$ if there exists a positive constant C such that

$$\langle |u|, \mu\rangle \leq C\sqrt{\mathcal{E}(u,u)}, \quad \text{for all } u \in \mathcal{F} \cap C_c(E). \tag{2.2}$$

We let $\mathcal{M}_0 = \{\mu = \nu_1 - \nu_2 : \nu_i \in \mathcal{S}_0^{(0)},\ i = 1, 2\}$. Any $\mu \in \mathcal{M}_0$ then admits a unique $U\mu \in \mathcal{F}_e$ satisfying the Poisson equation (1.4).

Remark 2.1 In the transient case, Eq. 1.4 implies that the map $\mu \in \mathcal{M}_0 \mapsto U\mu \in \mathcal{F}_e$ is injective. Since $vh\cdot m \in \mathcal{M}_0$ for the reference function h and for any $v \in \mathcal{C}_c(E)$, the Poisson equation (1.4) also implies that $\{U\mu : \mu \in \mathcal{M}_0\}$ is dense in the Hilbert space $(\mathcal{F}_e, \mathcal{E})$.

For any bounded non-negative function $g \in L^1(E; m)$ such that $\langle m, g\rangle > 0$, let

$$\mathcal{E}^g(u, v) = \mathcal{E}(u, v) + (u, v)_{g\cdot m}. \tag{2.3}$$

Since $\mathcal{E}_1(u, u) \leq \mathcal{E}_1^g(u, u) \leq (\|g\|_\infty + 1)\mathcal{E}_1(u, u)$, $(\mathcal{E}^g, \mathcal{F})$ is a regular Dirichlet form on $L^2(E; m)$. Further, the capacity $\mathrm{Cap}^g(A)$ determined by $\mathcal{E}_1^g$ satisfies $\mathrm{Cap}(A) \leq \mathrm{Cap}^g(A) \leq (\|g\|_\infty + 1)\mathrm{Cap}(A)$. Hence the quasi-notions related to $\mathcal{E}$ and $\mathcal{E}^g$ coincide. We denote by $\{T_t^g, t > 0\}$ the semigroup corresponding to $(\mathcal{E}^g, \mathcal{F})$. Since any Borel set A is $(\mathcal{E}, \mathcal{F})$-invariant if and only if so it is for $(\mathcal{E}^g, \mathcal{F})$ in view of [8, Theorem 1.6.1], the irreducibility of $\{T_t, t > 0\}$ is equivalent to that of $\{T_t^g, t > 0\}$.

Let $\mathbb{M} = (X_t, \mathbb{P}_x)$ be the Hunt process associated with $(\mathcal{E}, \mathcal{F})$ and let $\{P_t, t > 0\}$ and $\{R_\alpha, \alpha > 0\}$ be its transition function and resolvent, respectively. For a function g on E as above, define the positive continuous additive functional (PCAF) $\{C_t(\omega), t > 0\}$ of $\mathbb{M}$ by

$$C_t = \int_0^t g(X_s)ds.$$

The regular Dirichlet form $(\mathcal{E}^g, \mathcal{F})$ on $L^2(E; m)$ is then associated with the canonical subprocess $\mathbb{M}^g$ of $\mathbb{M}$ with respect to the multiplicative functional e^{-C_t} [8, Section 6.1]. $\mathbb{M}^g$ has the transition function and resolvent given, respectively, by

$$P_t^g f(x) = \mathbb{E}_x\left[e^{-C_t} f(X_t)\right], \quad R_\alpha^g f(x) = \mathbb{E}_x\left[\int_0^\infty e^{-\alpha t - C_t} f(X_t)dt\right], \quad x \in E. \tag{2.4}$$

R_0^g will be denoted by R^g.

Lemma 2.2 *If $\{T_t, t > 0\}$ is irreducible and recurrent, then for any bounded non-negative function g such that $0 < \langle m, g\rangle < \infty$, $\{T_t^g, t > 0\}$ is transient and it holds that*

$$\mathbb{P}_x\left(C_\infty = \infty\right) = 1 \qquad q.e. \tag{2.5}$$

Proof For any $\alpha > 0$,

$$R_\alpha^g g(x) = \mathbb{E}_x\left[\int_0^\infty e^{-\alpha t - C_t} g(X_t)dt\right] \leq \mathbb{E}_x\left[\int_0^\infty e^{-C_t} dC_t\right] = 1 - \mathbb{E}_x\left[e^{-C_\infty}\right] \leq 1.$$

Hence

$$1 \geq R_\alpha^g g = R_\alpha(g - g \cdot R_\alpha^g g) \geq R_\alpha\left(g - g \cdot R^g g\right).$$

By letting $\alpha \to 0$, it follows that $1 \geq R^g g$ and $1 \geq R_0(g - g \cdot R^g g)$. In particular, $\{T_t^g, t > 0\}$ is transient. Further, since $g - g \cdot R^g g \geq 0$ and $\{T_t, t > 0\}$ is recurrent, it follows that $g(x)R^g g(x) = g(x)$ m-a.e. that is $\mathbb{P}_x\left(C_\infty = \infty\right) = 1$ m-a.e. on the support of g. Put $D = \{x \in E : \mathbb{P}_x(C_\infty = \infty) = 1\}$. Then D is an invariant set. In fact, if there exists a measurable set $B \subset D$ such that $m(B) > 0$ and $P_t 1_{E\setminus D}(x) > 0$ for all $x \in B$, then, we have a contradiction because, for $x \in B$,

$$\begin{aligned} 1 &= \mathbb{P}_x\left(C_\infty = \infty\right) = \mathbb{P}_x\left(C_\infty \circ \theta_t = \infty\right) = \mathbb{E}_x\left[\mathbb{P}_{X_t}(C_\infty = \infty)\right] \\ &= \mathbb{P}_x(X_t \in D) + \mathbb{E}_x[\mathbb{P}_{X_t}(C_\infty = \infty) : X_t \in E \setminus D] < 1. \end{aligned}$$

Hence, by the irreducibility, we obtain that $\mathbb{P}_x(C_\infty = \infty) = 1$ m-a.e. on E. Since $\mathbb{P}_x(C_\infty = \infty)$ is excessive, it is finely continuous q.e. and hence $\mathbb{P}_x(C_\infty = \infty) = 1$ q.e. (see [8, Lemma 4.1.5]). □

Throughout the rest of this paper except for Section 3.1, we assume that $(\mathcal{E}, \mathcal{F})$ is an irreducible recurrent regular Dirichlet form on $L^2(E; m)$. Furthermore we assume that $\mathbb{M}$ satisfies the *absolute continuity condition*:

$$P_t(x, \cdot) \text{ is absolutely continuous with respect to } m$$
$$\text{for each} \quad t > 0 \quad \text{and} \quad x \in E \setminus N, \tag{2.6}$$

where N is a certain fixed Borel properly exceptional set. This condition is much milder than the one admitting no exceptional set N. Indeed, by virtue of [8, Theorem 4.2.7], (2.6) is fulfilled whenever the form $\mathcal{E}$ satisfies a Sobolev type inequality

$$\|u\|^2_{L^q(E;m)} \le S \left[\mathcal{E}(u, u) + \|u\|^2_{L^2(E;m)} \right], \quad u \in \mathcal{F}, \tag{2.7}$$

for some $q > 2$ and a constant $S > 0$. The inequality (2.7) is valid for a considerably large family of the Dirichlet forms $\mathcal{E}$ with a finite dimensional underlying space E. See Section 5.

Since N is a properly exceptional Borel set, we can regard $\mathbb{M}$ as a Hunt process on $E \setminus N$ and we shall do so from now on. In particular, for the PCAF C_t of Lemma 2.2, $\mathbb{P}_x(C_\infty = \infty) = \int_{E\setminus N} P_t(x, dy)\mathbb{P}_y(C_\infty = \infty) = 1$ for any $x \in E \setminus N$. Thus, $\mathbb{M}$ has the strong property that, for any bounded non-negative measurable function g such that $\langle m, g\rangle > 0$,

$$\mathbb{P}_x\left(\int_0^\infty g(X_t)dt = \infty\right) = 1 \qquad \text{for all } x \in E \setminus N. \tag{2.8}$$

In other words, the Hunt process $\mathbb{M}$ on $E \setminus N$ is *Harris recurrent*.

In view of [8, Lemma 4.2.4], the condition (2.6) further yields that, for any $x, y \in E \setminus N$ and $\alpha > 0$, there exists a jointly Borel measurable function $r_\alpha(x, y)$ such that

$$R_\alpha f(x) = \int_{E\setminus N} r_\alpha(x, y) f(y) m(dy) \text{ for any bounded Borel function } f \text{ on } E, \tag{2.9}$$

$$r_\alpha(x, y) = r_\alpha(y, x), \quad r_\alpha(x, y) \text{ is } \alpha\text{-excessive relative to } \mathbb{M} \text{ in } x \text{ and in } y \tag{2.10}$$

$$r_\alpha(x, y) = r_\beta(x, y) + (\beta - \alpha) \int_{E\setminus N} r_\beta(x, z) r_\alpha(z, y) m(dz), \quad \alpha, \beta > 0. \tag{2.11}$$

Lemma 2.3 (i) *For all $x, y \in E \setminus N$ and $\alpha > 0$, $r_\alpha(x, y) > 0$.*
(ii) *For all $x, y \in E \setminus N$,* $\lim_{\alpha\to 0} r_\alpha(x, y) = \infty$.

Proof (i) For all non-negative measurable function g such that $\langle m, g\rangle > 0$, it follows from Eq. 2.8 that $R_\alpha g(x) > 0$ for any $x \in E \setminus N$ and $\alpha > 0$. This implies that, for any $x \in E \setminus N$ and $\alpha > 0$, $r_\alpha(x, z) > 0$ m-a.e. $z \in E$, Hence, for any $x, y \in E \setminus N$, we get from Eq. 2.11 with $\beta > \alpha$ and the symmetry (2.10)

$$r_\alpha(x, y) \ge (\beta - \alpha) \int_{E\setminus N} r_\alpha(x, z) r_\beta(y, z) m(dz) > 0.$$

(ii) Put $r_0(x, y) = \lim_{\alpha\to 0} r_\alpha(x, y)$ for $x, y \in E \setminus N$. Since this is a monotone increasing limit, for any non-negative measureble function f such that $\langle m, f\rangle > 0$,

$$\int_{E\setminus N} r_0(z, y) f(z) m(dz) = R_0 f(y) = \infty \qquad \forall y \in E \setminus N.$$

This implies, for any $y \in E \setminus N$, that $r_0(z, y) = \infty$ for m-a.e. z. Since $r_0(\cdot, y)$ is excessive,

$$\alpha \int_{E\setminus N} r_\alpha(x, z) r_0(z, y) m(dz) \le r_0(x, y),$$

for all $x, y \in E \setminus N$, from which it follows that $r_0(x, y) = \infty$. □

We fix a bounded non-negative function $g \in L^1(E; m)$ with $\langle m, g\rangle > 0$ and consider again the regular Dirichlet form $(\mathcal{E}^g, \mathcal{F})$ on $L^2(E; m)$ defined by Eq. 2.3. By Lemma 2.2, $\mathcal{E}^g$ is transient. The associated canonical subprocess $\mathbb{M}^g$ of $\mathbb{M}$ on $E \setminus N$ enjoys the following properties:

Lemma 2.4 (i) *The transition function P_t^g of $\mathbb{M}^g$ satisfies the absolute continuity condition* (2.6) *with the properly exceptional set N being the same as the one for $\mathbb{M}$.*

(ii) *For each $t > 0$, $P_t^g f(x)$ is Borel measurable in $x \in E \setminus N$ for any bounded Borel function f on E.*

Proof (i). P_t^g is dominated by P_t due to the expression (2.4).

(ii). In general, a canonical subprocess of a Hunt process is a Hunt process again but with a weaker measurability that the semi-group and resolvent send a Borel function only to a universally measurable functions ([8, Theorem A.2.11]). But in the present case, the resolvent R_α^g of $\mathbb{M}^g$ satisfies

$$R_\alpha^g f = R_\alpha f - R_\alpha(g \cdot R_\alpha^g f), \quad \alpha > 0,$$

which combined with Eq. 2.6 implies that R_α^g makes the space of bounded Borel functions invariant. Further, for any bounded continuous function f, we have $\lim_{\alpha\to\infty} \alpha R_\alpha^g P_t^g f(x) = P_t^g f(x),\ x \in E \setminus N$. Since $R_\alpha^g(x, \cdot)$ is absolutely continuous with respect to m by (i), we can also see that P_t^g makes the space of bounded Borel functions invariant. □

Accordingly, the resolvent R_α^g of $\mathbb{M}^g$ admits a jointly Borel measurable density function $r_\alpha^g(x, y),\ x, y \in E \setminus N$, satisfying those properties (2.9)–(2.11) with $r_\alpha^g(x, y),\ R_\alpha^g,\ \mathbb{M}^g$ in place of $r_\alpha(x, y),\ R_\alpha,\ \mathbb{M}$.

Define

$$r^g(x, y) = \lim_{\alpha\downarrow 0} r_\alpha^g(x, y), \quad x, y \in E \setminus N, \tag{2.12}$$

Then $r^g(x, y) = r^g(y, x)\ x, y \in E \setminus N$. For any non-negative function $f \in L^1(E; m)$,

$$R^g f(x) = \int_{E\setminus N} r^g(x, y) f(y) m(dy),\ x \in E \setminus N, \tag{2.13}$$

which is finite m-a.e. and consequently q.e. because $R^g f$ is excessive relative to $\mathbb{M}^g$ [3, Theorem A.2.13 (v)]. As the increasing limit of α-excessive functions, $r^g(x, y)$ is excessive in y relative to $\mathbb{M}^g$ for each $x \in E \setminus N$:

$$r^g(x, y) \geq 0 \quad \text{and} \quad \int_{E\setminus N} P_t^g(y, dz) r^g(x, z) \uparrow r^g(x, y),\ t \downarrow 0, \quad x, y \in E \setminus N.$$

Since the Dirichlet form $(\mathcal{E}^g, \mathcal{F})$ on $L^2(E; m)$ is transient, its extended Dirichlet space $\mathcal{F}_e^g$ is a real Hilbert space with inner product $\mathcal{E}^g$. In view of [8, Lemma 6.2.5] or [3, Proposition 5.1.9], it further holds that

$$\mathcal{F}_e^g = \mathcal{F}_e \cap L^2(E; g \cdot m). \tag{2.14}$$

Let us consider the family $\mathcal{S}_0^{g,(0)}$ of finite 0-order energy relative to $(\mathcal{E}^g, \mathcal{F})$; a positive Radon measure μ belongs to $\mathcal{S}_0^{g,(0)}$ if there exists a constant C satisfying

$$\langle |u|, \mu\rangle \leq C\sqrt{\mathcal{E}^g(u, u)} \tag{2.15}$$

for all $u \in \mathcal{F} \cap \mathcal{C}_c(E)$. Each $\mu \in \mathcal{S}_0^{g,(0)}$ admits a unique potential $U^g\mu \in \mathcal{F}_e^g$ satisfying the Poisson equation

$$\mathcal{E}^g(U^g\mu, u) = \langle \mu, \widetilde{u} \rangle, \quad u \in \mathcal{F}_e^g. \tag{2.16}$$

For a positive Radon measure μ on E, we define the function $R^g\mu$ by

$$R^g\mu(x) = \int_{E\setminus N} r^g(x, y)\mu(dy), \quad x \in E \setminus N, \tag{2.17}$$

$R^g\mu$ is excessive relative to $\mathbb{M}^g$.

Proposition 2.5 (i) *Take $\alpha > 0$ with $\alpha \ge ||g||_\infty$. Then*

$$r_\alpha(x, y) \le r^g(x, y) \quad \text{for any } x, y \in E \setminus N. \tag{2.18}$$

(ii) *A positive Radon measure μ on E belongs to $\mathcal{S}_0^{g,(0)}$ if and only if*

$$\langle \mu, R^g\mu \rangle < \infty. \tag{2.19}$$

In this case, $R^g\mu$ is a quasi-continuous modification of the potential $U^g\mu \in \mathcal{F}_e^g$.
(iii) *For any function $f \in L^2(E; g \cdot m)$, the 0-order resolvent $R^g(fg)$ of the function fg relative to $\mathbb{M}^g$ is a quasi-continuous function belonging to $\mathcal{F}_e^g$.*

Proof (i). In view of Eq. 2.4,

$$e^{-\alpha t} P_t f(x) \le P_t^g f(x), \quad R_\alpha f(x) \le R^g f(x), \quad x \in E \setminus N,$$

for any non-negative Borel f on E. Hence, for a fixed $x \in E \setminus N$, we have $r_\alpha(x, z) \le r^g(x, z)$ for a.e. $z \in E$. Then, for any $y \in E \setminus N$,

$$e^{-\alpha t} \int_{E\setminus N} P_t(y, dz) r_\alpha(x, z) \le e^{-\alpha t} \int_{E\setminus N} P_t(y, dz) r^g(x, z) \le \int_{E\setminus N} P_t^g(y, dz) r^g(x, z).$$

By letting $t \downarrow 0$, we arrive at Eq. 2.18.

(ii). If $\mu \in \mathcal{S}_0^{g,(0)}$, then μ belongs to the class $\mathcal{S}_0^g$ of measures of finite (1-order) energy relative to $(\mathcal{E}^g, \mathcal{F})$. In view of the solution to Exercise 4.2.2 in [8], the function

$$R_\alpha^g\mu(x) = \int_{E\setminus N} r_\alpha^g(x, y)\mu(dy), \quad x \in E \setminus N,$$

is a quasi-continuous version of the α-potential $U_\alpha^g\mu \in \mathcal{F}^g$ relative to $(\mathcal{E}^g, \mathcal{F})$ for every $\alpha > 0$. If we let $\alpha \downarrow 0$, then $R_\alpha^g\mu$ converges to $R^g\mu$ pointwise, while $U_\alpha^g\mu$ is $\mathcal{E}$-convergent to $U^g\mu \in \mathcal{F}_e$ by virtue of [8, Lemma 2.2.11]. Consequently, $R^g\mu$ is a quasi-continuous version of $U^g\mu$ by [3, Theorem 2.3.4]. Finally, Eq. 2.19 follows from Eq. 2.16.

Conversely, suppose that a positive Radon measure μ satisfies (2.19), Then, for any $\alpha > 0$, $\langle \mu, R_\alpha^g\mu \rangle < \infty$ and consequently, by [8, Exercise 4.2.2] again, $\mu \in \mathcal{S}_0^g$ and $R_\alpha^g\mu$ is a quasi-continuous version of $U_\alpha^g\mu$. As in the proof of [8, Lemma 2.2.11], we can see that $R_\alpha^g\mu$ is $\mathcal{E}^g$ convergent as $\alpha \downarrow 0$ and further $\alpha(R_\alpha^g\mu, R_\alpha^g\mu)$ is uniformly bounded in $\alpha > 0$. Hence, by letting $\alpha \downarrow 0$ in the equation $\mathcal{E}_\alpha^g(R_\alpha^g\mu, u) = \langle \mu, u \rangle$, $u \in \mathcal{F} \cap \mathcal{C}_c(E)$, we conclude that $\mu \in \mathcal{S}_0^{g,(0)}$ and $R^g\mu$ is a quasi-continuous version of $U^g\mu$.

(iii). This is because the measure $\mu = |f|g \cdot m$ satisfies a bound (2.15). □

3 Recurrent Potentials of Measures

Throughout this section except for Section 3.1, we keep the assumption that we are given a regular irreducible and recurrent Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(E; m)$ with the associated

Hunt process $\mathbb{M} = (X_t, \mathbb{P}_x)$ on E satisfying the absolute continuity condition (2.6). As in the preceding section, let $r_\alpha(x, y)$ be the resolvent density satisfying (2.9), (2.10) and (2.11). By virtue of Lemma 2.3, $r_\alpha(x, y) > 0$ for any $x, y \in E \setminus N$.

The following lemma is essentially contained in [13] and [8, Lemma 4.8.3].

Lemma 3.1 *For any Borel set $B \subset E \setminus N$ such that $0 < m(B) < \infty$, there exists a compact subset F of B such that*

$$\begin{cases} m(F) > 0, \quad \text{and for some} \quad c > 0 \text{ and } 1/2 < a < 1, \\ m(\{y \in F : r_1(x, y) > c\}) > a\, m(F) \quad \text{for every} \quad x \in F. \end{cases} \tag{3.1}$$

Proof Let $B_n(x) = \{y \in B : r_1(x, y) > 1/n\}$. Then $\cup_{n=1}^{\infty} B_n(x) = B$ and hence $m(B_n(x))$ increases to $m(B)$ as $n \to \infty$ for all $x \in E \setminus N$. Put $K(B, a, n) = \{x \in B : m(B_n(x)) > a\, m(B)\}$ for $a < 1$. Then, $\cup_{n=1}^{\infty} K(B, a, n) = B$ for any $0 < a < 1$. For any $1/2 < a_0 < 1$, take a number n such that $m(K(B, a_0, n)) > 2(1-a_0)m(B)$. Further, take a compact subset F of $K(B, a_0, n)$ so close as $m(F) > 2(1-a_0)m(B)$. Then, for any $a \in (1/2, 1)$ satisfying $1 - a > (1 - a_0)m(B)/m(F)$, we have for any $x \in F$,

$$\begin{aligned} m(\{y \in F : r_1(x, y) \le \frac{1}{n}\}) &\le m(\{y \in B : r_1(x, y) \le \frac{1}{n}\}) = m(B) - m(B_n(x)) \\ &\le (1 - a_0)m(B) = (1 - a_0)\frac{m(B)}{m(F)}m(F) \\ &\le (1 - a)m(F). \end{aligned}$$

Hence $K(F, a, n) = F$, that is the assertion of the lemma holds for $c = 1/n$. □

We call a compact set $F \subset E \setminus N$ *admissible* if it has the property (3.1). For a given admissible set F, we shall construct recurrent potentials of certain class of measures on E by using recurrent potentials $\check{R}\varphi$ of L^2-functions φ on a quasi-support $\widetilde{F}$ of F relative to the trace Dirichlet form $(\check{\mathcal{E}}, \check{\mathcal{F}})$ of $(\mathcal{E}, \mathcal{F})$ on $L^2(\widetilde{F}; 1_{\widetilde{F}} \cdot m)$. The admissibility of F is well inherited by the trace Dirichlet form making this procedure possible.

Before carrying out this procedure, let us prepare a construction of recurrent potentials of L^2-functions in a special case.

3.1 Construction of Recurrent Potentials of L^2-Functions in a Special Case

In this subsection, we work under the following specific setting. E is a Lusin space, namely, a Borel subset of a compact metric space, m is a finite measure on E and $\mathbb{M}$ is an m-symmetric recurrent Borel right process on E. Let $(\mathcal{E}, \mathcal{F})$ be the Dirichlet form of $\mathbb{M}$ on $L^2(E; m)$. Denote the 1-order resolvent of $\mathbb{M}$ by R_1.

We assume that there exists jointly Borel measurable function $r_1(x, y),\ x, y \in E$, such that $R_1 f(x) = \int_E r_1(x, y) f(y)m(dy)$ for all $x \in E$ and all non-negative Borel function f on E. We further assume that there exist positive constants c and a with $c > 0,\ 1/2 < a < 1$ such that

$$m(\{y \in E : r_1(x, y) > c\}) > a\, m(E) \qquad \text{for all } x \in E. \tag{3.2}$$

Clearly this condition is satisfied if $r_1(x, y) > c$ for all $x, y \in E$.

Let $m_1(dx) = m(dx)/m(E)$ be the normalized measure of m. The norm of $f \in L^p(E; m)$ is denoted by $||f||_p,\ p > 0$. For a finite signed measure μ which can be written as $\mu = \mu^+ - \mu^-$, by non-negative finite measures μ^+ and μ^- such that $\mu^+(B) \wedge \mu^-(B) = 0$ for any Borel set B, let $|\mu|$ be the measure defined by $|\mu|(A) = \mu^+(A) + \mu^-(A)$ and $\|\mu\| = |\mu|(E)$ be the total variation of μ.

Lemma 3.2 *There exists $\gamma < 1$ satisfying*

$$\frac{1}{2}\sup_{x,x'\in E}\|R_1(x,\cdot)-R_1(x',\cdot)\|\le\gamma. \tag{3.3}$$

Proof Put $D_x = \{y \in E : r_1(x,y) > c\}$. Then $1 \ge \int_{D_x} r_1(x,y)m(dy) \ge c\, m(D_x) \ge ac\, m(E)$. Let $D^+_{x,x'} = \{y \in D_x \cap D_{x'} : r_1(x,y) - r_1(x',y) > 0\}$ and $D^-_{x,x'} = (D_x \cap D_{x'}) \setminus D^+_{x,x'}$. Since $m(D_x \cap D_{x'}) \ge m(D_x) + m(D_{x'}) - m(E) \ge (2a-1)m(E)$ for all $x, x' \in E$, we have

$$\begin{aligned}\|R_1(x,\cdot)-R_1(x',\cdot)\| &\le \int_{D^+_{x,x'}}(r_1(x,y)-r_1(x',y))m(dy)+\int_{D^-_{x,x'}}(r_1(x',y)-r_1(x,y))m(dy)\\ &\quad +R_1(x,E\setminus(D_x\cap D_{x'}))+R_1(x',E\setminus(D_x\cap D_{x'}))\\ &\le \int_{D^+_{x,x'}}r_1(x,y)m(dy)+\int_{D^-_{x,x'}}r_1(x',y)m(dy)-c\,m(D_x\cap D_{x'})\\ &\quad +R_1(x,E\setminus(D_x\cap D_{x'}))+R_1(x',E\setminus(D_x\cap D_{x'})\\ &\le 2-c\,m(D_x\cap D_{x'})\le 2-c(2a-1)m(E).\end{aligned}$$

Hence it is enough to put $\gamma = 1 - \frac{(2a-1)c}{2}m(E) < 1$. □

Let $\{R_1^n(x,B), n \ge 1\}$ be the kernels defined inductively by

$$R_1^1(x,B)=R_1(x,B),\qquad R_1^n(x,B)=\int_E R_1^{n-1}(x,dy)R_1(y,B).$$

In view of the proof of [8, Lemma 4.8.2], Eq. 3.3 implies

$$\sup_{x,x'\in E}\|R_1^n(x,\cdot)-R_1^n(x',\cdot)\|\le 2\gamma^n. \tag{3.4}$$

and, for $f \in L^2(E;m)$,

$$\|R_1^n f-\langle m_1,f\rangle\|_2\le 2(\sqrt{\gamma})^n\,\|f-\langle m_1,f\rangle\|_2 \tag{3.5}$$

for all $n \ge 1$. We let $R_1^o(x,B) = \delta_x(B)$ by convention.

By virtue of Eq. 3.5, the sum $\sum_{n=0}^{\infty}(R_1^n f - \langle m_1, f\rangle)$ is L^2-convergent for $f \in L^2(E;m)$, and so we can define a bounded linear operator R from $L^2(E;m)$ into it by

$$Rf=R_1h,\quad h=\sum_{n=0}^{\infty}(R_1^n f-\langle m_1,f\rangle),\ \text{for } f\in L^2(E;m). \tag{3.6}$$

R satisfies the bound

$$||Rf||_2\le\frac{2\sqrt{\gamma}}{1-\sqrt{\gamma}}||f||_2. \tag{3.7}$$

Proposition 3.3 (i) *For any $f \in L^2(E;m)$, Rf defined by* Eq. 3.6 *belongs to the space $\mathcal{F}$ and satisfies*

$$\mathcal{E}(Rf,u)=(f,u-\langle m_1,u\rangle),\quad \textit{for every } u\in\mathcal{F}. \tag{3.8}$$

(ii) *It holds that*

$$\mathcal{E}(Rf,Rf)\le\frac{2\sqrt{\gamma}}{1-\sqrt{\gamma}}\|f\|_2^2,\quad f\in L^2(E;m). \tag{3.9}$$

(iii) *For any* $f \in L^\infty(E;m)(\subset L^2(E;m))$, *the sum* $h = \sum_{n=0}^\infty (R_1^n f - \langle m_1, f\rangle)$ *is convergent in* $L^\infty(E;m)$ *and* $Rf = R_1 h$ *satisfies a bound*

$$||Rf||_\infty \leq \frac{2}{1-\gamma}||f||_\infty. \tag{3.10}$$

Proof (i). By the expression (3.6), Rf belongs to $\mathcal{F}$ and satisfie

$$\begin{aligned}\mathcal{E}(Rf,u) &= \mathcal{E}_1(R_1h,u) - (R_1h,u)\\ &= (h,u) - (R_1h,u) = (f,u) - \langle m_1, f\rangle\langle m,u\rangle\end{aligned}$$

yielding the desired equation.

(ii). For $f \in L^2(E;m)$, since $\langle Rf, m\rangle = 0$, we have from Eq. 3.8

$$\mathcal{E}(Rf,Rf) = (f,Rf) - \frac{1}{m(E)}\langle f,m\rangle\langle Rf,m\rangle \leq \|f\|_2\|Rf\|_2,$$

which combined with Eq. 3.7 leads us to Eq. 3.9.
(iii), As is shown in [8, p 212], Eq. 3.4 implies that, for $f \in L^\infty(E;m)$,

$$\|R_1^n f - \langle m_1, f\rangle\|_\infty \leq 2\gamma^n\|f\|_\infty, \quad \text{for all } n \geq 1,$$

and consequently $\|h\|_\infty \leq \frac{2}{1-\gamma}\|f\|_\infty$, yielding (3.10). □

We call tbe above Rf the *recurrent potential* of $f \in L^2(E;m)$.

3.2 Construction of Recurrent Potentials of Measures in the General Case

We now work under the general setting stated in the beginning of this section. In the rest of this section, we fix an admissible compact set $F \subset E\setminus N$. For a Borel set $A \subset E \setminus N$, the measure $1_A \cdot m$ will be designated by m_A.

We consider the PCAF $\{C_t, t \geq 0\}$ of the Hunt process $\mathbb{M}$ on $E\setminus N$ defined by

$$C_t = \int_0^t 1_F(X_s)ds, \quad t > 0.$$

and let $\widetilde{F}$ be its *support*:

$$\widetilde{F} = \{x \in E\setminus N : \mathbb{P}_x(R=0)=1\}, \quad \text{for} \quad R = \inf\{t>0 : C_t > 0\}. \tag{3.11}$$

Clearly $\widetilde{F} \subset F$. As $\widetilde{F}$ is a quasi-support of m_F [8, Theorem 5.2.1], $m(F\setminus\widetilde{F}) = 0$. $\widetilde{F}$ is a Borel set because $\widetilde{F} = \{x \in E\setminus N : \varphi(x) = 1\}$ for $\varphi(x) = \mathbb{E}_x[e^{-R}]$, $x \in E\setminus N$, which is 1-excessive and hence a Borel function due to the absolute continuity condition (2.6).

Let $\check{\mathbb{M}} = (X_{\tau_t}, \{\mathbb{P}_x\}_{x\in\widetilde{F}})$ be the time changed process of $\mathbb{M}$ by the PCAF C_t, where τ_t is the right continuous inverse of C_t. According to [3, Section 5.2], $\check{\mathbb{M}}$ is $m_{\widetilde{F}}$-symmetric right process on $\widetilde{F}$ and its Dirichlet form $(\check{\mathcal{E}}, \check{\mathcal{F}})$ on $L^2(\widetilde{F} : m_{\widetilde{F}})$ is given by

$$\check{\mathcal{F}} = \check{\mathcal{F}}_e \cap L^2(\widetilde{F}; m_{\widetilde{F}}), \quad \check{\mathcal{F}}_e = \mathcal{F}_e\big|_F, \quad \check{\mathcal{E}}(u,v) = \mathcal{E}(H_{\widetilde{F}}u, H_{\widetilde{F}}v), \quad u,\ v \in \check{\mathcal{F}}_e. \tag{3.12}$$

Here all functions in the extended Dirichlet space $\mathcal{F}_e$ are assumed to be quasi-continuous and, $H_{\widetilde{F}}u(x) = \mathbb{E}[u(X_{\sigma_{\widetilde{F}}})]$, $x \in E$, for $\sigma_{\widetilde{F}} = \inf\{t > 0 : X_t \in \widetilde{F}\}$. $(\check{\mathcal{E}}, \check{\mathcal{F}})$ is recurrent in view of [3, Theorem 5.2.5]).

Denote by $\{\check{R}_p; p > 0\}$ the resolvent of the time-changed process $\check{\mathbb{M}}$. We have then, for any bounded Borel function f on $\widetilde{F}$, the identity

$$\check{R}_p f(x) = R^{p1_F}(f\cdot 1_F)(x), \quad x \in \widetilde{F}, \tag{3.13}$$

where R^{p1_F} is the 0-order resolvent of the canonical subprocess $\mathbb{M}^g$ of $\mathbb{M}$ relative to $g = p1_F$ having been studied in Section 2 [3, (5.2.2)].

Recall the function $r^g(x,y),\ x,y \in E \setminus N$ defined by Eq. 2.12. Put

$$\check{r}_p(x,y) = r^{p1_F}(x,y), \quad x,y \in \widetilde{F}, \quad p>0. \tag{3.14}$$

It follows from Eqs. 3.13 and 2.13 that $\check{r}_p(x,y)$ is a density function of $\check{R}_p(x,dy)$ with respect to $m_{\widetilde{F}}$:

$$\check{R}_p f(x) = \int_{\widetilde{F}} \check{r}_p(x,y) f(y) m_{\widetilde{F}}(dy), \quad p>0, \tag{3.15}$$

for every $x \in \widetilde{F}$ and bounded Borel function f on $\widetilde{F}$.

From those identities, one can draw two conclusions as follows.

First, the time changed process $\check{\mathbb{M}}$ is actually a Borel right process on the Lusin space $\widetilde{F}$. Indeed, Eq. 3.13 combined with Lemma 2.4 (ii) implies that $\check{R}_p$ makes the space of bounded Borel measurable functions on $\widetilde{F}$ invariant. One can then use the absolute continuity (3.15) to obtain the same property of the transition function of $\check{\mathbb{M}}$ as the proof of Lemma 2.4 (ii).

Second, combining (3.14) with inequality (2.18) and the admissibility (3.1) of F, we are led to

$$m_{\widetilde{F}}(\{y \in \widetilde{F} : \check{r}_1(x,y) > c\}) \geq m_{\widetilde{F}}(\{y \in \widetilde{F} : r_1(x,y) > c\}) > a m_{\widetilde{F}}(\widetilde{F}),$$

holding for every $x \in \widetilde{F}$. Here c and a are some constants with $c>0$, $\frac{1}{2} < a < 1$.

Thus the trace Dirichlet form $(\check{\mathcal{E}}, \check{\mathcal{F}})$ on $L^2(\widetilde{F}; m_{\widetilde{F}})$ and the associated time changed process $\check{\mathbb{M}}$ on $\widetilde{F}$ fulfill all conditions required in Section 3.1 so that Proposition 3.3 is well applicable to them.

For $\varphi \in L^2(\widetilde{F}; m_{\widetilde{F}})$, the sum $\sum_{n=0}^{\infty}(\check{R}_1^n\varphi - \frac{1}{m(\widetilde{F})}\langle m_{\widetilde{F}}, \varphi\rangle)$ is convergent in $L^2(\widetilde{F}; m_{\widetilde{F}})$. Define

$$\check{R}\varphi = \check{R}_1\eta, \quad \eta = \sum_{n=0}^{\infty}(\check{R}_1^n\varphi - \frac{1}{m(\widetilde{F})}\langle m_{\widetilde{F}}, \varphi\rangle), \text{ for } \varphi \in L^2(\widetilde{F}; m_{\widetilde{F}}). \tag{3.16}$$

Proposition 3.4 (i) *For any $\varphi \in L^2(\widetilde{F}; m_{\widetilde{F}})$, the function $\check{R}\varphi$ defined by* Eq. 3.16 *belongs to $\check{\mathcal{F}}$ and satisfies*

$$\check{\mathcal{E}}(\check{R}\varphi, \psi) = (\varphi, \psi - \frac{1}{m(\widetilde{F})}\langle m_{\widetilde{F}}, \psi\rangle)_{m_{\widetilde{F}}}. \tag{3.17}$$

for all $\psi \in \check{\mathcal{F}}$.

(ii) *It holds that*

$$\check{\mathcal{E}}(\check{R}\varphi, \check{R}\check{\varphi}) \leq C_1\|\varphi\|_2^2, \quad \varphi \in L^2(\widetilde{F}; m_{\widetilde{F}}), \tag{3.18}$$

for some constant $C_1 > 0$.

(iii) *For any $\varphi \in L^2(\widetilde{F}; m_{\widetilde{F}})$, $\check{R}\varphi$ is the restriction to $\widetilde{F}$ of a quasi-continuous function on E belonging to the space $\mathcal{F}_e^g$.*

(iv) *For any $\varphi \in L^\infty(\widetilde{F}; m_{\widetilde{F}})$, η in* Eq. 3.16 *is convergent in L^∞ and $\check{R}\varphi = \check{R}_1\eta$ satisfies a bound*

$$||\check{R}\varphi||_\infty \leq C_2||\varphi||_\infty \tag{3.19}$$

for some constant $C_2 > 0$.

Proof (i), (ii) and (iv) follow from Proposition 3.3. (iii) is a consequence of Eqs. 3.13, 3.16 and Proposition 2.5 (iii). □

$\check{R}\varphi$ in the above proposition is called the *recurrent potential* of $\varphi \in L^2(\widetilde{F}; m_{\widetilde{F}})$ for the trace Dirichlet form $(\check{\mathcal{E}}, \check{\mathcal{F}})$ on $L^2(\widetilde{F}; m_{\widetilde{F}})$.

In what follows, we let $g = 1_{\widetilde{F}}$ and consider the perturbed Dirichlet form $(\mathcal{E}^g, \mathcal{F})$ on $L^2(E; m)$ and the related objects $\mathcal{F}_e^g$, $\mathcal{S}_0^{g,(0)}$, $r^g(x, y)$ and $R^g\mu$ for a positive Radon measure μ on E having been considered in the last part of Section 2. By Eq. 2.14,

$$\mathcal{F}_e^g = \mathcal{F}_e \cap L^2(E; m_{\widetilde{F}}). \tag{3.20}$$

A positive Radon measure μ on E belongs to $\mathcal{S}_0^{g,(0)}$ if and only if $\langle \mu, R^g\mu \rangle < \infty$ and in this case $R^g\mu$ is a quasi-continuous function in $\mathcal{F}_e^g$ satisfying the equation

$$\mathcal{E}^g(R^g\mu, u) = \langle \mu, \widetilde{u} \rangle, \quad u \in \mathcal{F}_e^g. \tag{3.21}$$

Define

$$\mathcal{M}_0 = \{\mu = \mu_1 - \mu_2 : \mu_i \in \mathcal{S}_0^{g,(0)},\ \mu_i(E) < \infty,\ i = 1, 2\}, \quad \mathcal{M}_{00} = \{\mu \in \mathcal{M}_0 : \mu(E) = 0\}. \tag{3.22}$$

For $\mu = \mu_1 - \mu_2 \in \mathcal{M}_0$, we let $R^g\mu = R^g\mu_1 - R^g\mu_2$. We then define

$$R\mu = H_{\widetilde{F}}\check{R}(1_{\widetilde{F}}R^g\mu) + R^g\mu - \frac{1}{m(\widetilde{F})}\langle \mu, 1 \rangle, \quad \mu \in \mathcal{M}_0. \tag{3.23}$$

Theorem 3.5 (i) *If $\mu \in \mathcal{M}_0$, then $R\mu$ is a quasi-continuous function in $\mathcal{F}_e^g$ satisfying*

$$\mathcal{E}(R\mu, u) = \left\langle \mu, \tilde{u} - \frac{1}{m(F)}\langle m_F, u \rangle \right\rangle, \quad \textit{for any } u \in \mathcal{F}_e^g. \tag{3.24}$$

(ii) *It holds that*

$$\mathcal{E}(R\mu, R\nu) = \langle \mu, R\nu \rangle \quad \textit{for any } \mu, \nu \in \mathcal{M}_0. \tag{3.25}$$

(iii) *It holds that*

$$\mathcal{E}(R\mu, R\mu) \le C_3\langle \mu, R^g\mu \rangle, \quad \mu \in \mathcal{M}_0, \tag{3.26}$$

for some constant $C_3 > 0$.

(iv) *It holds that*

$$\|R\mu\|_\infty \le C_4\|R^g\mu\|_\infty + \frac{|\langle \mu, 1 \rangle|}{m(F)}, \quad \mu \in \mathcal{M}_0, \tag{3.27}$$

for some constant $C_4 > 0$.

Proof (i). For $\mu \in \mathcal{S}_0^{g,(0)}$, $1_{\widetilde{F}} \cdot R^g\mu \in L^2(\widetilde{F}; m_{\widetilde{F}})$ by Eq. 3.20 so that $\check{R}(1_{\widetilde{F}}R^g\mu)$ is the restriction to $\widetilde{F}$ of a quasi-continuous function on E belonging to $\mathcal{F}_e^g$ by Proposition 3.4 (iii). Hence, for $\mu \in \mathcal{M}_0$, $R\mu$ is well defined by Eq. 3.23 as a quasi-continuous function in $\mathcal{F}_e^g$.

Noting that $\mathcal{E}(1, u) = 0$ for all $u \in \mathcal{F}_e^g$, we obtain from Eqs. 3.12, 3.17 and 3.21 that

$$\begin{aligned}
\mathcal{E}(R\mu, u) &= \mathcal{E}(H_{\widetilde{F}}\check{R}(1_{\widetilde{F}}R^g\mu) + R^g\mu, u) \\
&= \mathcal{E}(H_{\widetilde{F}}\check{R}(1_{\widetilde{F}}R^g\mu), H_{\widetilde{F}}\tilde{u}) + \mathcal{E}(R^g\mu, u) \\
&= \check{\mathcal{E}}(\check{R}(1_{\widetilde{F}}R^g\mu), u|_{\widetilde{F}}) + \mathcal{E}^g(R^g\mu, u) - (R^g\mu, u)_{m_{\widetilde{F}}} \\
&= (R^g\mu, u)_{m_{\widetilde{F}}} - \frac{1}{m(\widetilde{F})}\langle m_{\widetilde{F}}, R^g\mu \rangle\langle m_{\widetilde{F}}, u \rangle + \langle \mu, \tilde{u} \rangle - (R^g\mu, u)_{m_{\widetilde{F}}} \\
&= \langle \mu, \tilde{u} \rangle - \frac{1}{m(F)}\langle m_{\widetilde{F}}, R^g\mu \rangle\langle m_F, u \rangle \quad \text{for all } u \in \mathcal{F}_e^g.
\end{aligned}$$

On the other hand, we get from Eq. 2.8

$$R^g g(x) = 1 - \mathbb{E}_x[e^{-C_\infty}] = 1, \quad x \in E \setminus N, \tag{3.28}$$

and consequently,

$$\langle m_{\widetilde{F}}, R^g\mu\rangle = \langle R^g g, \mu\rangle = \langle \mu, 1\rangle, \tag{3.29}$$

yielding Eq. 3.24.

(ii). By Eq. 3.24, $\mathcal{E}(R\mu, R\nu) = \langle \mu, R\nu\rangle - \frac{1}{m(F)}\langle \mu, 1\rangle\langle m_F, R\nu\rangle$. From Eq. 3.23 we have $\langle m_F, R\nu\rangle = (1_F, \check{R}(1_F R^g\nu))_{m_F} + \langle m_F, R^g\nu\rangle - \langle \nu, 1\rangle$. The first term of the right hand side of this identity vanishes in view of Eq. 3.16 so that we get from Eq. 3.29

$$\langle m_F, R\nu\rangle = 0, \tag{3.30}$$

yielding Eq. 3.25.

(iii). For $\mu \in \mathcal{M}_0$, we have from the above and Eq. 3.18

$$\begin{aligned}\mathcal{E}(R\mu, R\mu) &\le 2\check{\mathcal{E}}(\check{R}(1_F \cdot R^g\mu), \check{R}(1_F \cdot R^g\mu)) + 2\mathcal{E}(R^g\mu, R^g\mu) \\ &\le 2C_1\|R^g\mu\|^2_{m_F} + 2\mathcal{E}^g(R^g\mu, R^g\mu) \le 2(C_1+1)\mathcal{E}^g(R^g\mu, R^g\mu),\end{aligned}$$

yelding Eq. 3.26.

(iv). It follows from Eqs. 3.19 and 3.23 that $\|R\mu\|_\infty \le (C_2+1)\|R^g\mu\|_\infty + \frac{|\langle \mu, 1\rangle|}{m(F)}$. □

The collection $\{R\mu : \mu \in \mathcal{M}_0\}$ defined by Eqs. 3.22 and 3.23 will be called the *family of recurrent potentials relative to the admissible set F*.

3.3 Basic Properties of $\mathcal{F}_e$ and Recurrent Potentials

We continue to work under the general setting of the preceding subsection. F is an arbitrarily fixed admissible compact subset of $E \setminus N$. We are setting $g = 1_F$.

Proposition 3.6 *It holds that*

$$\mathcal{F}_e \subset L^1(E; m_F). \tag{3.31}$$

Furthermore, there exist a bounded strictly positive integrable function h such that $\|R^g h\|_\infty < \infty$, $h \ge 1_F$ and a constant C_h satisfying

$$\int_E \left| u(x) - \frac{1}{m(F)}\langle m_F, u\rangle \right| h(x)m(dx) \le C_h\, \mathcal{E}(u, u)^{1/2} \tag{3.32}$$

for all $u \in \mathcal{F}_e$.

Proof Take a bounded positive m-integrable function h such that $\|R^g h\|_\infty < \infty$. See the first paragraph in the proof of Theorem 4.8.2 (ii) of [8] for the existence of such function h. On account of Eq. 3.28, we may assume that $h \ge 1_F$ by replacing h with $h+1_F$ if necessary.

We first prove that the Poincaré type inequality (3.32) holds for h and for any $u \in \mathcal{F}_e^g$. Define

$$f = \mathrm{sgn}(u - (1/m(F))\langle m_F, u\rangle) \cdot h, \quad \mu = f \cdot m - (\langle m, f\rangle/m(F))m_F.$$

Since $\mu \in \mathcal{M}_0$ and $\mu(E) = 0$, we obtain from Eq. 3.24 that

$$\begin{aligned}&\int_E \left| u(x) - \frac{1}{m(F)}\langle m_F, u\rangle \right| h(x)m(dx) = \int_E \left(u(x) - \frac{1}{m(F)}\langle m_F, u\rangle \right) f(x)m(dx) \\ &= \int_E u(x)\mu(dx) = \mathcal{E}(R\mu, u) \le \mathcal{E}(R\mu, R\mu)^{1/2}\mathcal{E}(u, u)^{1/2}, \quad \text{for all } u \in \mathcal{F}_e^g.\end{aligned}$$

Hence, by Eqs. 3.26 and 3.28, we obtain Eq. 3.32 holding for any $u \in \mathcal{F}_e^g$ with a constant

$$C_h = (2C_3)^{1/2}\left[\|R^g h\|_\infty + (h,1)/m(F)\right]^{1/2}\ (h,1)^{1/2}.$$

Next, for any $u \in \mathcal{F}_e$, choose an approximating sequence $\{u_n\} \subset \mathcal{F}$ for u. Since $\mathcal{F} \subset \mathcal{F}_e^g$, the inequality (3.32) for $\{u_n\}$ implies that $\{u_n - c_n\}$ with $c_n = \frac{1}{m(F)}\langle m_F, u_n\rangle,\ n \geq 1$, is a Cauchy sequence in $L^1(E; h\cdot m)$. As $u_n \to u,\ n \to \infty$, m-a.e., we see by taking a subsequence if necessary that c_n converges to a constant as $n \to \infty$ and $u \in L^1(E; h\cdot m)$. Consequently $u \in L^1(E; m_F)$, $\lim_{n\to\infty} c_n = \frac{1}{m(F)}\langle m_F, u\rangle$ and the inequality (3.32) extends to u by Fatou's lemma. □

Proposition 3.6 implies that $u \in \mathcal{F}_e$ satisfies $\mathcal{E}(u,u) = 0$ if and only if u is a constant m-a.e. Let $\dot{\mathcal{F}}_e$ be the quotient space of $\mathcal{F}_e$ by the family of constant functions. For any $\dot{u} = \{u + c : c \in \mathbb{R}\} \in \dot{\mathcal{F}}$, $\mathcal{E}(\dot{u},\dot{u})$ is determined uniquely by $\mathcal{E}(u,u)$.

Theorem 3.7 (i) *$\dot{\mathcal{F}}_e$ is a Hilbert space with inner product $\mathcal{E}$.*
(ii) *$\{R\mu : \mu \in \mathcal{M}_{00}\}$ is dense in $(\dot{\mathcal{F}}_e, \mathcal{E})$.*

Proof (i). Let $\{u_n\} \subset \mathcal{F}_e$ be an $\mathcal{E}$-Cauchy sequence. By Eq. 3.31, $u_n \in L^1(E; m_F)$ and we put $c_n = (1/m(F))\langle m_F, u_n\rangle,\ n \geq 1$. Then, by Eq. 3.32 $\{u_n - c_n\}$ converges as $n \to \infty$ to a function $v \in L^1(E; h\cdot m)$ m-a.e. by taking a subsequence if necessary. For each n, there exists an approximating sequence $\{u_{n,\ell}\} \subset \mathcal{F}$ of $u_n - c_n$ such that $\mathcal{E}(u_n - u_{n,\ell}, u_n - u_{n,\ell}) < 1/n$ for all $\ell \geq n$. Then $\{u_{n,n}\}$ is an $\mathcal{E}$-Cauchy sequence converging to v m-a.e. Therefore, $v \in \mathcal{F}_e$ and $\lim_{n\to\infty} \mathcal{E}(u_n - v, u_n - v)^{1/2} \leq \lim_{n\to\infty}\left(\frac{1}{\sqrt{n}} + \mathcal{E}(u_{n,n} - v, u_{n,n} - v)^{1/2}\right) = 0$.

(ii). In view of [8, Corollary 1.6.3], it suffices to prove that any bounded function in $\mathcal{F}_e$ that is orthogonal to $\{R\mu : \mu \in \mathcal{M}_{00}\}$ is constant.

Assume that $u \in \mathcal{F}_e$ is bounded and $\mathcal{E}(u, R\mu) = 0$ for any $\mu \in \mathcal{M}_{00}$. Then $u \in \mathcal{F}_e^g$ by Eqs. 3.20 and 3.31, and accordingly $\langle \mu, \widetilde{u}\rangle = 0$ by Eq. 3.24. We shall show that this implies u is a constant m-a.e. To show it, let $a_1 = \sup\{c : m(\{x : \widetilde{u}(x) > c\}) > 0\}$ and $a_2 = \inf\{c : m(\{\widetilde{u}(x) < c\}) > 0\}$. Then $a_1 > -\infty$, $a_2 < \infty$ and $a_1 \geq a_2$. Further, $a_1 = a_2$ if and only if u is equal to a constant.

Suppose that u is not equal to a constant. Then there exists a positive constant ϵ such that $a_1 - \epsilon > a_2 + \epsilon$. By definition, the sets $A_1 = \{x : \widetilde{u}(x) \geq a_1 - \epsilon\}$ and $A_2 = \{x : \widetilde{u}(x) \leq a_2 + \epsilon\}$ are of positive m-measure. For a reference function h of the Dirichlet form $(\mathcal{E}^g, \mathcal{F})$ and a small positive constant a, we may assume that $B_1 = A_1 \cap \{x : h(x) \geq a\}$ and $B_2 = A_2 \cap \{x : h(x) \geq a\}$ are also of positive m-measure. Since $1_{B_i} \leq h/a$, $1_{B_i}\cdot m \in \mathcal{M}_0$ for $i = 1, 2$. Put $\mu = (1/m(B_1))m_{B_1} - (1/m(B_2))m_{B_2} \in \mathcal{M}_{00}$. Then $\langle \mu, \widetilde{u}\rangle \geq a_1 - a_2 - 2\epsilon > 0$ which contradicts to the assumption. □

Given a Borel set $B \subset E \setminus N$, the *hitting distribution* $H_B(x, \cdot)$ of the Hunt process $\mathbb{M} = (X_t, \{\mathbb{P}_x\}_{x\in E\setminus N})$ for B is defined by $H_B f(x) = \mathbb{E}_x[f(X_{\sigma_B})],\ x \in E \setminus N$, for any bounded Borel functions f on E where $\sigma_B = \inf\{t > 0 : X_t \in B\}$. For any finite measure μ on $E \setminus N$, $\langle \mu_B, f\rangle = \langle \mu, H_B f\rangle$ defines a finite measure μ_B and the correspondence $\mu \mapsto \mu_B$ is called the *balayage* relative to B.

For functions f_1, f_2 defined quasi-everywhere on E, we say that $f_1 = f_2$ q.e. *modulo a constant* if $f_1 - f_2$ is a constant q.e. on E.

Theorem 3.8 *Let $\mu \in \mathcal{M}_0$ and $B \subset E \setminus N$ be a Borel set containing F. Then $\mu_B \in \mathcal{M}_0$ and*

$$R\mu_B = H_B R\mu \quad \text{q.e. modulo a constant.} \tag{3.33}$$

Proof For $\mu \in \mathcal{S}_0^{g,(0)}$, μ satisfies the bound (2.15) for any $u \in \mathcal{F}_e$ with $\widetilde{u}$ in place of u. Further, for any $u \in \mathcal{F}_e \cap C_c(E)$, $(H_B|u|, H_B|u|)_{m_F} = (u.u)_{m_F}$, so that

$$\langle \mu_B, |u| \rangle = \langle \mu, H_B|u| \rangle \leq C\sqrt{\mathcal{E}^g(H_B|u|, H_B|u|)} \leq C\sqrt{\mathcal{E}^g(u,u)},$$

yielding $\mu_B \in \mathcal{S}_0^{g,(0)}$. Hence $\mu_B \in \mathcal{M}_0$ whenever $\mu \in \mathcal{M}_0$.

For $\mu \in \mathcal{M}_0$ and for any bounded m-integrable function f on E such that $R^g|f|$ is bounded, we have from Eq. 3.24

$$\mathcal{E}(H_B R\mu, Rf) = \left(f, H_B R\mu - \frac{1}{m(F)} \langle m_F, H_B R\mu \rangle \right).$$

On the other hand, the left hand side equals

$$\mathcal{E}(R\mu, H_B Rf) = \langle \mu, H_B Rf \rangle - \frac{1}{m(F)} \langle m_F, H_B Rf \rangle \langle \mu, 1 \rangle$$

(see the paragraph below (4.2)). The first term of the right hand side of this identity equals $\langle \mu_B, Rf \rangle = (f, R\mu_B)$ by virtue of Eq. 3.25. Since $F \subset B$, we have $\langle m_F, H_B Rf \rangle = \langle m_F, Rf \rangle$ that vanishes in view of Eq. 3.30. Therefore

$$R\mu_B = H_B R\mu - \frac{1}{m(F)} \langle m_F, H_B R\mu \rangle.$$

The above identity holds m-a.e. on E. It holds q.e. on E because the both hand sides are q.e. finite and q.e. finely continuous relative to $\mathbb{M}$ and consequently quasi-continuous [8, Theorem4.6.1]. □

We notice that the space $\mathcal{M}_0$ is not necessarily closed under the balayage operation $\mu \mapsto \mu_B$ unless $B \supset F$.

So far, we have constructed and studied the potentials $R\mu$ of a measure $\mu \in \mathcal{M}_0$ by using a fixed admissible set F. The space $\mathcal{M}_0$ of measures defined by Eq. 3.22 and the potential $R\mu$ of $\mu \in \mathcal{M}_0$ defined by Eq. 3.23 depend on the choice of the admissible set F. The following proposition concerns about their rlationship for different choices of F.

Proposition 3.9 *Assume that F_1 and F_2 are two admissible sets. Let $\{R^{(i)}\mu : \mu \in \mathcal{M}_0^{(i)}\}$ be the family of recurrent potentials relative to F_i defined by using $g_i = 1_{F_i}$ for $i = 1, 2$.*

(i) *If $\mu \in \mathcal{M}_0^{(1)} \cap \mathcal{M}_0^{(2)}$ satisfies $\mu(E) = 0$, $\|R^{g_1}|\mu|\|_\infty < \infty$ and $\|R^{g_2}|\mu|\|_\infty < \infty$, then $R^{(1)}\mu = R^{(2)}\mu$ q.e. modulo a constant.*

(ii) *For any $\mu \in \mathcal{M}_{00}^{(1)}$, there exists a sequence $\{\mu_n\} \subset \mathcal{M}_{00}^{(1)} \cap \mathcal{M}_{00}^{(2)}$ such that, for each n, supp$[|\mu_n|] \subset$ supp$[|\mu|]$, $R^{(1)}\mu_n = R^{(2)}\mu_n$ modulo a constant and*

$$\lim_{n\to\infty} \mathcal{E}\left(R^{(1)}\mu_n - R^{(1)}\mu, R^{(1)}\mu_n - R^{(1)}\mu\right) = 0.$$

Proof (i). If μ satisfies the stated condition, then $R^{(i)}\mu \in \mathcal{F}_e^{g_i} \subset \mathcal{F}_e$ and $\|R^{(i)}\mu\|_\infty < \infty$, $i = 1.2$. by Theorem 3.5 (i), Eqs. 3.20 and 3.27 so that $R^{(i)}\mu \in \mathcal{F}_e \cap L^2(E; m_{\tilde{F}^{(1)}}) \cap L^2(E; m_{\tilde{F}^{(2)}}) \subset \mathcal{F}_e^{g_1} \cap \mathcal{F}_e^{g_2}$, $i = 1, 2$, in view of Eq. 3.20. Hence Eq. 3.24 applies in getting that

$$\mathcal{E}\left(R^{(1)}\mu - R^{(2)}\mu, R^{(1)}\mu - R^{(2)}\mu\right) = 0,$$

and consequently, $R^{(1)}\mu - R^{(2)}\mu$ is equal to a constant by Theorem 3.7.

(ii). We first note that, for any $\mu \in \mathcal{M}_0^{(1)}$, there exists an increasing sequaence $\{A_n\}$ of closed sets such that

$$\mu_n = 1_{A_n} \cdot \mu \in \mathcal{M}_0^{(1)}, \ \|R^{g_1}|\mu_n|\|_\infty < \infty, \text{ for each } n, \text{ and } \lim_{n\to\infty} \mu(E \setminus A_n) = 0. \quad (3.34)$$

Given a quasi-continuous function v on E, an increasing family $\{F_n\}$ of closed sets is called a *nest associated with* v if $\mathrm{Cap}(E \setminus F_n) \to 0,\ n \to \infty$, and $v\big|_{F_n}$ is finite, continuous for each n. $R^{g_1}\mu$ is quasi-continuous by Proposition 2.5. Let $\{F_n^{(1)}\}$ be a nest associated with it and put

$$A_n = \{x \in F_n^{(1)} : R^{g_1}|\mu|(x) \le n\}, \quad \mu_n = 1_{A_n} \cdot \mu, \quad n \ge 1.$$

Then $R^{g_1}|\mu_n|(x) \le n$ q.e. on E and $\{A_n\}$ satisfies the property (3.34) according to the 0-order version of the maximum principle [8, Lemma 2.2.4 (ii)].

We next show that, for any $\mu \in \mathcal{M}_0^{(1)}$, there exists an increasing sequence $\{B_n\}$ of closed sets such that

$$\mu_n = 1_{B_n} \cdot \mu \in \mathcal{M}_0^{(1)} \cap \mathcal{M}_0^{(2)}, \ \|R^{g_2}|\mu_n|\|_\infty < \infty, \text{ for each } n, \text{ and } \lim_{n\to\infty} \mu(E \setminus B_n) = 0. \quad (3.35)$$

It suffices to consider the case that μ is a finite measure in $\mathcal{S}_0^{g_1,(0)}$. Since $(\mathcal{E}^{g_i}, \mathcal{F}),\ i = 1, 2$, and $(\mathcal{E}, \mathcal{F})$ share the common quasi-notion, μ can be regarded as a smooth measure relative to $(\mathcal{E}^{g_2}, \mathcal{F})$. Let $\{A_t, t \ge 0\}$ be the PCAF of $\mathbb{M}^{g_2} = (X_t, \mathbb{P}_x^{g_2})$ with Revuz measure μ. Take a bounded strictly positive integrable function f such that $\|R^{g_2} f\|_\infty < \infty$ and define

$$U_A^{g_2,\alpha} f(x) = \mathbb{E}_x^{g_2}\left[\int_0^\infty e^{-\alpha t} f(X_t) dA_t\right], \quad R_\alpha^{g_2,A} f(x) = \mathbb{E}_x^{g_2}\left[\int_0^\infty e^{-\alpha t - A_t} f(X_t) dt\right], \quad \alpha \ge 0.$$

$U_A^{g_2,0} f$ and $R_0^{g_2,A} f$ are denoted by $U_A^{g_2} f$ and $R^{g_2,A} f$, respectively.

By virtue of [8, (5.1.8)], we have then

$$R_\alpha^{g_2,A} f = R_\alpha^{g_2} f - U_A^{g_2,\alpha} R_\alpha^{g_2,A} f, \quad \alpha > 0.$$

By letting $\alpha \downarrow 0$, we get the identity

$$R^{g_2,A} f(x) = R^{g_2} f(x) - U_A^{g_2} R^{g_2,A} f(x), \quad x \in E \setminus N, \quad (3.36)$$

where each function is bounded in $x \in E \setminus N$ due to its $\mathbb{M}^{g_2}$-excessiveness and the absolute continuity of the transition function of $\mathbb{M}^{g_2}$.

On the other hand, it holds for any bounded non-negative Borel function u on E,

$$U_A^{g_2} u(x) = R^{g_2}(u \cdot \mu)(x), \quad x \in E \setminus N, \quad (3.37)$$

because the identity $(h, U_A^{g_2} u) = \langle u \cdot \mu, R^{g_2} h\rangle = (R^{g_2}(u \cdot \mu), h)$ is valid for any non-negative Borel function h on E on account of [8, Theorem 5.1.3], and the right hand side of Eq. 3.37 is also $\mathbb{M}^{g_2}$-excessive in view of Eq. 2.17.

$R^{g_2,A} f$ is finite and $\mathbb{M}^{g_2}$-excessive and consequently quasi-continuous by [8, Theorem 4.6.1]. Let $\{F_n^{(2)}\}$ be a nest associated with it. Let $B_n = \{x \in F_n^{(2)} : R^{g_2,A} f(x) \ge 1/n\}$, Then, by Eqs. 3.36 and 3.37

$$R^{g_2}(1_{B_n} \cdot \mu)(x) = U_A^{g_2} 1_{B_n}(x) \le n U_A^{g_2} R^{g_2,A} f(x) \le n\|R^{g_2} f\|_\infty, \quad \text{for all } x \in E \setminus N.$$

Hence the measure $\mu_n = 1_{B_n} \cdot \mu$ satisfies $\langle \mu_n, R^{g_2}\mu_n\rangle < \infty$ so that $\mu_n \in \mathcal{M}_0^{(2)}$ by Proposition 2.5 (ii). Clearly $\mu_n \in \mathcal{M}_0^{(1)}$ and $\mu(E \setminus B_n) \to 0$ as $n \to \infty$.

Let $\{A_n\}$, $\{B_n\}$ be the sequences satisfying (3.34) and (3.35), respectively, for the measure $\mu \in \mathcal{M}_0^{(1)}$. Without loss of generality, we assume that μ is a non-zero measure. Define $C_n = A_n \cap B_n$ and $\nu_n = 1_{C_n} \cdot \mu$. Then $\nu_n \in \mathcal{M}_0^{(1)} \cap \mathcal{M}_0^{(2)}$. We may assume that $\langle \nu_1, 1\rangle \neq 0$. As ν_k is non-zero for some $k \geq 1$, we can otherwise replace ν_1 by $1_{C_k'} \cdot \mu$ for some closed subset C_k' of C_k with $\mu(C_k') \neq 0$ and ν_n, $n \geq 2$ by ν_{k+n-1}, respectively.

We let

$$\mu_n = \nu_n - \frac{\langle \nu_n, 1\rangle}{\langle \nu_1, 1\rangle} \cdot \nu_1.$$

Then $\mu_n(E) = 0$ and $\|R^{g_i}|\mu_n|\|_\infty < \infty$, $i = 1, 2$. Hence $\langle |\mu_n|, R^{g_i}|\mu_n|\rangle < \infty$, $i = 1, 2$, namely, $\mu_n \in \mathcal{M}_{00}^{(1)} \cap \mathcal{M}_{00}^{(2)}$ so that $R^{(1)}\mu_n = R^{(2)}\mu_n$ q.e. modulo a constant by (i). Since $\lim_{n\to\infty} 1_{E\setminus C_n} = 0$ $|\mu|$-a.e., $\lim_{n\to\infty}\langle \nu_n, 1\rangle = \langle \mu, 1\rangle = 0$ and furthermore

$$\lim_{n\to\infty} \langle \mu - \mu_n, R^{g_1}(\mu - \mu_n)\rangle = 0.$$

The $\mathcal{E}$-convergence of $R^{(1)}\mu_n$ to $R^{(1)}\mu$ follows from this combined with the bound (3.26). □

4 Markov property of Gaussian field indexed by $\mathcal{F}_e$

In this section, we continue to assume that $(\mathcal{E}, \mathcal{F})$ is an irreducible recurrent regular Dirichlet form on $L^2(E; m)$ satisfying the absolute continuity condition (2.6). For any function $u \in \mathcal{F}_e$, an open set G is called a *regular set* of u if $\mathcal{E}(u, v) = 0$ for any $v \in \mathcal{F} \cap \mathcal{C}_c(E)$ with supp$[v] \subset G$. The complement of the largest regular set of u is called the *spectrum* of u and denoted by $s(u)$.

For a given admisible set F, let $\{R\mu : \mu \in \mathcal{M}_{00}\}$ be the family of recurrent potentials relative to F. For each $\mu \in \mathcal{M}_{00}$, it holds that $R\mu \in \mathcal{F}_e$ and $\mathcal{E}(R\mu, v) = \langle \mu, v\rangle$ for any $v \in \mathcal{F} \cap C_c(E)$ by virtue of Eq. 3.24. Hence we see that

$$s(R\mu) = \text{supp}[|\mu|]. \tag{4.1}$$

For any open set $G \subset E$, put $B = E \setminus G$ and let $\mathcal{F}_{e,G}$ be a linear subspace of $\mathcal{F}_e$ defined by

$$\mathcal{F}_{e,G} = \{u \in \mathcal{F}_e : \tilde{u} = 0 \text{ q.e. on } B\}.$$

By [8, Theorem 2.3.3], $s(u) \subset B$ if and only if

$$\mathcal{E}(u, v) = 0, \qquad \forall v \in \mathcal{F}_{e,G}. \tag{4.2}$$

In view of [8, Theorem 4.6.5] or [3, Theorem 3.4.8], it holds for any $u \in \mathcal{F}_e$ that $H_B|\widetilde{u}|(x) < \infty$ for q.e. $x \in E$ and $H_B\widetilde{u}$ is a quasi-continuous element of $\mathcal{F}_e$ satisfying (4.2), where $\widetilde{u}$ is a quasi-continuous version of u. Consequently

$$s(H_B\widetilde{u}) \subset B, \quad \text{for any } u \in \mathcal{F}_e. \tag{4.3}$$

The following theorem is a weak counterpart of the *spectral synthesis theorem* having been formulated for transient Dirichlet forms in [4, p168] and [8, Theorem 2.3.2].

Theorem 4.1 *Suppose that the spectrum $s(u)$ of $u \in \mathcal{F}_e$ is contained in a closed set B with $m(B) > 0$. Choose an admissible set F to be a subset of B in accordance with Lemma 3.1 and let $\mathcal{M}_{00}$ and $R\mu$, $\mu \in \mathcal{M}_{00}$, be the associated family of measures and potentials. Then there exists a sequence $\mu_n \in \mathcal{M}_{00}$ such that supp$[|\mu_n|] \subset B$ and*

$$\lim_{n\to\infty} \mathcal{E}(R\mu_n - u, R\mu_n - u) = 0.$$

In particular, if $m(s(u)) > 0$, then we can take $s(u)$ as the set B.

Proof Put $G = E \setminus s(u) \supset E \setminus B$. The spectrum of $u - H_B\widetilde{u}$ is contained in B by Eq. 4.3, while $u - H_B\widetilde{u} \in \mathcal{F}_{e,G}$. Hence $u = H_B\tilde{u} + k$ for some constant k by Eq. 4.2 and Theorem 3.7 (i). Since $u = H_B\tilde{u}$ a.e. on B, $k = 0$.

As $\{R\mu : \mu \in \mathcal{M}_{00}\}$ is dense in $(\dot{\mathcal{F}}_e, \mathcal{E})$ by Theorem 3.7 (ii), there exists a sequence $\{\nu_n\} \subset \mathcal{M}_{00}$ such that $\lim_{n\to\infty} \mathcal{E}(u - R\nu_n, u - R\nu_n) = 0$. Put $\mu_n = (\nu_n)_B$. Then, by virtue of Theorem 3.8, $H_B R\nu_n = R\mu_n$ q.e modulo a constant and consequently

$$\mathcal{E}(R\mu_n - u, R\mu_n - u) = \mathcal{E}(H_B(R\nu_n - u), H_B(R\nu_n - u)) \le \mathcal{E}(R\nu_n - u, R\nu_n - u) \to 0$$

as $n \to 0$. □

The Dirichlet form $(\mathcal{E}, \mathcal{F})$ is called *local* if $\mathcal{E}(u, v) = 0$ for any $u, v \in \mathcal{F} \cap \mathcal{C}_c(X)$ with disjoint support. $(\mathcal{E}, \mathcal{F})$ is local if and only if the corresponding Hunt process $\mathbb{M}$ is a diffusion process.

Proposition 4.2 *Assume that $(\mathcal{E}, \mathcal{F})$ is local. Let G be an open subset of E such that $m(E \setminus G) > 0$. If $u \in \mathcal{F}_e$ satisfies $s(u) \subset E \setminus G$, then $s(H_{\overline{G}}u) \subset \partial G$.*

Proof Put $B = E \setminus G$, choose an admissible set F contained in B in accordance with Lemma 3.1 and let $\{R\mu : \mu \in \mathcal{M}_{00}\}$ be the family of recurrent potentials relative to F. By the preceding theorem, there then exists a sequence of measures $\mu_n \in \mathcal{M}_{00}$ with $\mathrm{supp}[|\mu_n|] \subset B$ and $\lim_{n\to\infty} \mathcal{E}(R\mu_n - u, R\mu_n - u) = 0$.

Next, we choose another admissible set $\widehat{F}$ contained in G in accordance with Lemma 3.1. We then let $\{\widehat{R}\nu : \nu \in \widehat{\mathcal{M}}_{00}\}$ be the family of recurrent potentials relative to $\widehat{F}$. By making use of Proposition 3.9 for each μ_n, we can select a sequence $\{\nu_n\} \subset \mathcal{M}_{00} \cap \widehat{\mathcal{M}}_{00}$ such that

$$\mathrm{supp}[|\nu_n|] \subset \mathrm{supp}[|\mu_n|], \qquad \lim_{n\to\infty} \mathcal{E}(\widehat{R}\nu_n - u, \widehat{R}\nu_n - u) = 0.$$

We can then use Theorem 3.8 to get $H_{\overline{G}}\widehat{R}\nu_n = \widehat{R}\nu_{n,\overline{G}}$ q.e modulo a constant where $\nu_{n,\overline{G}} = (\nu_n)_{\overline{G}}$. Therefore $\widehat{R}\nu_{n,\overline{G}}$ is $\mathcal{E}$-convergent to $H_{\overline{G}}u$ as $n \to \infty$. As $\mathrm{supp}[|\nu_n|] \subset B$, the continuity of the paths of $\mathbb{M}$ implies that $\nu_{n,\overline{G}} = \nu_{n,\partial G}$. Therefore $\widehat{R}\nu_{n,\partial G}$ is $\mathcal{E}$-convergent to $H_{\overline{G}}u$. This implies the assertion of the proposition because $s(\widehat{R}\nu_{n,\partial G}) \subset \partial G$ by Eq. 4.1. □

Let us consider a system of centered Gaussian random variables $\mathbb{G}(\mathcal{E}) = \{X_u : u \in \dot{\mathcal{F}}_e\}$ on a probability space $(\Omega, \mathcal{B}, \mathbb{P})$ with covariance

$$\mathbb{E}[X_u X_v] = \mathcal{E}(u, v), \qquad u, v \in \mathcal{F}_e. \tag{4.4}$$

For $A \subset E$, define the σ-field $\sigma(A) \subset \mathcal{B}$ by

$$\sigma(A) = \sigma\{X_u : u \in \mathcal{F}_e, s(u) \subset A\}.$$

For a Borel set A of E, $\mathbb{G}(\mathcal{E})$ is said to have the *Markov property* with respect to A if the identity

$$\mathbb{E}\left[YZ|\sigma(\partial A)\right] = \mathbb{E}\left[Y|\sigma(\partial A)\right]\mathbb{E}\left[Z|\sigma(\partial A)\right] \tag{4.5}$$

holds for any bounded $\sigma(\overline{A})$-measurable function Y and any bounded $\sigma(\overline{E \setminus A})$-measurable function Z on Ω. It is known (cf. [14, Proposition 6.3]) that $\mathbb{G}(\mathcal{E})$ has the Markov property with respect to A if and only if

$$\sigma\left\{\mathbb{E}\left[Y|\sigma(\overline{A})\right] : Y \text{ is bounded and } \sigma(\overline{E \setminus A})\text{-measurable}\right\} \subset \sigma(\partial A). \tag{4.6}$$

In particular, $\mathbb{G}(\mathcal{E})$ is said to have the global (resp. local) Markov property if $\mathbb{G}(\mathcal{E})$ has the Markov property for any open (resp. relatively compact open) set A.

Proposition 4.3 *For any closed set $B \subset E$ and any $u \in \mathcal{F}_e$,*

$$\mathbb{E}[X_u \,\big|\, \sigma(B)] \;=\; X_{H_B\tilde{u}}. \tag{4.7}$$

Proof Take any $v \in \mathcal{F}_e$ with $s(v) \subset B$. Since $u - H_B\tilde{u} \in \mathcal{F}_{e,E\setminus B}$, $\mathcal{E}(u - H_B\tilde{u}, v) = 0$ by Eq. 4.2. Hence $\mathbb{E}\left[(X_u - X_{H_B\tilde{u}})X_v\right] = 0$ so that $X_u - X_{H_B\tilde{u}}$ is independent of $\sigma(B)$ as all random variables involved are centered Gaussian. Consequently

$$\mathbb{E}\left[X_u - X_{H_B\tilde{u}} \middle| \sigma(B)\right] = \mathbb{E}\left[X_u - X_{H_B\tilde{u}}\right] = 0,$$

so that Eq. 4.3 implies

$$\mathbb{E}\left[X_u|\sigma(B)\right] = \mathbb{E}\left[X_{H_B\tilde{u}} \middle| \sigma(B)\right] = X_{H_B\tilde{u}}.$$

□

Theorem 4.4 *For the Gaussian field $\mathbb{G}(\mathcal{E})$ indexed by $\dot{\mathcal{F}}_e$, the following conditions are equivalent to each others.*

(i) *$\mathbb{G}(\mathcal{E})$ has the global Markov property.*
(ii) *$\mathbb{G}(\mathcal{E})$ has the local Markov property.*
(iii) *The Dirichlet form $(\mathcal{E}, \mathcal{F})$ is local.*

Proof Assume that $(\mathcal{E}, \mathcal{F})$ is local. Let G be an open set of E. We may assume that $m(E \setminus G) > 0$ because, if otherwise, $E \setminus G = \partial G$ and hence Eq. 4.6 holds for $A = G$. Take any $u \in \mathcal{F}_e$ with $s(u) \subset E \setminus G$. We have from Eq. 4.7

$$\mathbb{E}\left[X_u|\sigma(\overline{G})\right] = X_{H_{\overline{G}}\tilde{u}}.$$

The last random variable is $\sigma(\partial G)$-measurable by Proposition 4.2. As the lefthand side of Eq. 4.6 for $A = G$ coincides with $\sigma\left\{\mathbb{E}[X_u|\sigma(\overline{G})] \;:\; u \in \mathcal{F}_e,\; s(u) \subset E \setminus G\right\}$ in view of [14, Proposition 6.3], we obtain the global Markov property of $\mathbb{G}(\mathcal{E})$ by Eq. 4.6. Hence (iii) implies (i).

(i) ⇒ (ii) is clear. Assume that $\mathbb{G}(\mathcal{E})$ satisfies the local Markov property. Let G be a relatively compact open set of E and u be any function in $\mathcal{F}_e$ with $s(u) \subset E \setminus G$. Consider the random variable $Y = \mathbb{E}\left[X_u \middle| \sigma(\overline{G})\right]$. Then $Y = X_{H_{\overline{G}}\tilde{u}}$ by Eq. 4.7. Now take any open subset A of G with $\overline{A} \subset G$ and let $B = G \setminus \overline{A}$. Then $\sigma(\overline{G}) \supset \sigma(\overline{B}) \supset \sigma(\partial G)$. By the assumption of the Markov property of $\mathbb{G}(\mathcal{E})$ and Eq. 4.6, Y is $\sigma(\partial G)$-measurable and hence $\sigma(\overline{B})$-measurable so that $Y = \mathbb{E}[X_u \big| \sigma(\overline{B})]$ which equals $X_{H_{\overline{B}}\tilde{u}}$ by Eq. 4.7 again. Therefore

$$\mathbb{E}\left[(X_{H_{\overline{G}}\tilde{u}} - X_{H_{\overline{B}}\tilde{u}})^2\right] = 0, \quad \text{that is,} \quad \mathcal{E}\left(H_{\overline{G}}\tilde{u} - H_{\overline{B}}\tilde{u}, H_{\overline{G}}\tilde{u} - H_{\overline{B}}\tilde{u}\right) = 0.$$

We now make a special choice of an admissible set F to be a compact subset of B in accordance with Lemma 3.1 and consider the family $\{R\mu \;:\; \mu \in \mathcal{M}_0\}$ of recurrent potentials relative to F. Substitute in the above identity $u = R\mu$ for any $\mu \in \mathcal{M}_0$ with $\text{supp}[\mu] \subset E \setminus G$. By virtue of Theorem 3.8, we have $H_{\overline{G}}R\mu = R\mu_{\overline{G}}$ and $H_{\overline{B}}R\mu = R\mu_{\overline{B}}$ q.e. modulo constants. Hence

$$\mathcal{E}\left(R(\mu_{\overline{G}} - \mu_{\overline{B}}), R(\mu_{\overline{G}} - \mu_{\overline{B}})\right) = 0.$$

By Schwarz inequality

$$\mathcal{E}\left(R(\mu_{\overline{G}} - \mu_{\overline{B}}), f\right) = 0,$$

for any $f \in \mathcal{F} \cap \mathcal{C}_c(E)$. By noting that $\mu_{\overline{G}}(E) - \mu_{\overline{B}}(E) = 0$, we are then led from Eq. 3.24 to

$$\left\langle \mu_{\overline{G}} - \mu_{\overline{B}}, f \right\rangle = 0,$$

for any $f \in \mathcal{F} \cap \mathcal{C}_c(E)$. Hence $\langle \mu, H_{\overline{G}} f - H_{\overline{B}} f \rangle = 0$.

In particular, if the support of f is contained in A, then $H_{\overline{B}} f = 0$ so that

$$\langle \mu, H_{\overline{G}} f \rangle = 0, \quad \text{for any } \mu \in \mathcal{M}_0 \quad \text{with } \mathrm{supp}[|\mu|] \subset E \setminus \overline{G}.$$

This implies that $H_{\overline{G}} f = 0$ q.e. on $E \setminus \overline{G}$, and consequently $H_{\overline{G}}(x, \cdot)$ is concentrated on $\overline{B}$ for q.e. $x \in E \setminus \overline{G}$. Since this holds for any open subset A of G such that $\overline{A} \subset G$, $H_{\overline{G}}(x, \cdot)$ is concentrated on ∂G for q.e. $x \in E \setminus G$. Then the local property of $\mathbb{M}$ follows from [8, Lemma 4.5.1], that is (ii) implies (iii). □

Remark 4.5 (a). Similarly to the above proof of the implication (iii) ⇒ (i), it should have been noted in the proof of [7, Theorem 2.9] that the lefthand side of [7, (2.23)] coincides with $\sigma\left\{\mathbb{E}[X_\mu|\sigma(\overline{G})] : \mu \in \mathcal{M}_{00}(\mathbb{C}),\ \mathrm{supp}[|\mu|] \subset \mathbb{C} \setminus G\right\}$.

(b). The above proof of Propoition 4.2 and Theorem 4.4 indicates that, for a non-local Dirichlet form $\mathcal{E}$, one may still formulate and prove the equivalence between a short range property of the Lévy system $(N(x, dy), H)$ of the associated Hunt process $\mathbb{M}$ and an appropriately weak Markov property of the associated Gaussian field $\mathbb{G}(\mathcal{E})$.

5 Examples

5.1 Relation to Logarithmic Potentials Over $\mathbb{C}$ and $\overline{\mathbb{H}}$

In this subsection, we first consider the special case where E is the complex plane $\mathbb{C}$, m is the Lebesgue measure on $\mathbb{C}$ and the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathbb{C}) = L^2(\mathbb{C}; m)$ is given by $(\frac{1}{2}\mathbf{D}, H^1(\mathbb{C}))$ which is regular, irreducible and recurrent. Here

$$\mathbf{D}(u, v) = \int_{\mathbb{C}} \nabla u(\mathbf{x}) \cdot \nabla v(\mathbf{x}) dx, \quad H^1(\mathbb{C}) = \{u \in L^2(\mathbb{C}), |\nabla u| \in L^2(\mathbb{C})\}.$$

Its extended Dirichlet space $\mathcal{F}_e$ coincides with the Beppo Levi space BL($\mathbb{C}$) defined by

$$\mathrm{BL}(\mathbb{C}) = \{u \in L^2_{\mathrm{loc}}(\mathbb{C}) : |\nabla u| \in L^2(\mathbb{C})\}.$$

We let

$$p_t(\mathbf{x}) = \frac{1}{2\pi t} \exp\left(-\frac{|\mathbf{x}|^2}{2t}\right), \quad t > 0,\ \mathbf{x} \in \mathbb{C}, \qquad k(\mathbf{x}) = \frac{1}{\pi} \log \frac{1}{|\mathbf{x}|},\ \mathbf{x} \in \mathbb{C}. \tag{5.1}$$

The Hunt process $\mathbb{M}$ on $\mathbb{C}$ associated with the above Dirichlet form is the Brownian motion with the transition probability density $p_t(\mathbf{x} - \mathbf{y})$, $t > 0$, $\mathbf{x}, \mathbf{y} \in \mathbb{C}$, so that the condition (2.6) is fulfilled with no exceptional set N. By Theorem 4.4, the Gaussian field $\mathbb{G}(\mathcal{E})$ (for $\mathcal{E} = \frac{1}{2}\mathbf{D}$) indexed by BL($\mathbb{C}$) enjoys the Markov property.

Notice that the 1-order resolvent density $r_1(\mathbf{x}, \mathbf{y}) = \int_0^\infty e^{-t} p_t(\mathbf{x} - \mathbf{y}) d\mathbf{y}$ of $\mathbb{M}$ satisfies $\inf_{\mathbf{x},\mathbf{y} \in F} r_1(\mathbf{x}, \mathbf{y}) > 0$ for any compact set $F \subset \mathbb{C}$. Therefore any compact set $F \subset \mathbb{C}$ with positive Lebesgue measure is trivially admissible in the sense that it satisfies condition (3.1). Accordingly we can consider the family $\{R\mu : \mu \in \mathcal{M}_{00}\}$ of recurrent potentials relative to any compact set F with positive Lebesgue measure.

For a finite signed measure μ on $\mathbb{C}$ of compact support, its *logarithmic potential* $U\mu$ is defined by

$$U\mu(\mathbf{x}) = \int_{\mathbb{C}} k(\mathbf{x} - \mathbf{y}) \mu(d\mathbf{y}) \quad \mathbf{x} \in \mathbb{C},$$

which is locally integrable and locally bounded below. Define

$$\overset{\circ}{\mathcal{M}}_{00}(\mathbb{C}) = \{\mu : \text{ finite signed measure of compact support on } \mathbb{C},\ \langle|\mu|, U|\mu|\rangle < \infty,\ \mu(\mathbb{C}) = 0\}. \quad (5.2)$$

This space of measures on $\mathbb{C}$ is denoted by $\mathcal{M}_{00}(\mathbb{C})$ in [7]. But, in this section, we shall use the notion $\overset{\circ}{\mathcal{M}}_{00}(\mathbb{C})$ for it in order to distinguish it from the space $\mathcal{M}_{00}$ of measures being used in the present paper. It is shown in [7] that, for any $\mu \in \overset{\circ}{\mathcal{M}}_{00}(\mathbb{C})$, its logarithmic potential $U\mu$ is a quasi-continuous function belonging to BL($\mathbb{C}$) satisfying the Poisson equation

$$\frac{1}{2}\mathbf{D}(U\mu, v) = \langle\mu, \widetilde{v}\rangle, \quad \text{for any } v \in \mathrm{BL}(\mathbb{C}). \quad (5.3)$$

It is further shown in [7] that the space $\overset{\circ}{\mathcal{M}}_{00}(\mathbb{C})$ is closed under the balayage operation to any compact subset of $\mathbb{C}$.

Proposition 5.1 (i) *Let μ be a finite signed measure on $\mathbb{C}$ of compact support with $\mu(\mathbb{C}) = 0$. Let G be any relatively compact open set containing the support of $|\mu|$ and $\{R\nu : \nu \in \mathcal{M}_{00}\}$ be the family of recurrent potentials relative to $\overline{G}$. Then $\mu \in \mathcal{M}_{00}$ if and only if $\mu \in \overset{\circ}{\mathcal{M}}_{00}(\mathbb{C})$, and in this case, $R\mu = U\mu$ q.e. modulo a constant.*

(ii) *Let $F \subset \mathbb{C}$ be any compact set with positive Lebesgue measure and $\{R\mu : \mu \in \mathcal{M}_{00}\}$ be the family of recurrent potentials relative to F. Then, for any $\mu \in \mathcal{M}_{00}$, there exists a sequence $\{\nu_n\} \subset \overset{\circ}{\mathcal{M}}_{00}(\mathbb{C})$ such that supp$[\nu_n] \subset$ supp$[\mu]$ for each n and* $\lim_{n\to\infty} \boldsymbol{D}(U\nu_n - R\mu, U\nu_n - R\mu) = 0$.

For a proof of this proposition, we prepare a lemma. Let G be a relatively compact open subset of $\mathbb{C}$ and $\{r_{G,\alpha}(\mathbf{x}, \mathbf{y}),\ \alpha \geq 0, \mathbf{x}, \mathbf{y} \in G\}$ be the resolvent density of the part $\mathbb{M}_G$ of $\mathbb{M}$ on G obtained by killing upon leaving the set G. The Dirichlet form of $\mathbb{M}_G$ on $L^2(G) = L^2(G; m)$ equals $(\frac{1}{2}\mathbf{D}, H_0^1(G))$ [8, Example 4.4.1]. For a finite measure μ on G, define $R_{G,\alpha}\mu$ by $R_{G,\alpha}\mu(\mathbf{x}) = \int_G r_{G,\alpha}(\mathbf{x}, \mathbf{y})\mu(d\mathbf{y}),\ \mathbf{x} \in G$. $R_{G,0}\mu$ is denoted by $R_G\mu$.

Lemma 5.2 *For a positive finite measure μ on $\mathbb{C}$ with supp$[\mu] \subset G$, the following conditions are equivalent:*

$$\langle\mu, R_G\mu\rangle < \infty. \quad (5.4)$$

$$\langle\mu, |v|\rangle \leq C\,\boldsymbol{D}(v, v)^{1/2} \quad \textit{for any } v \in C_c^1(G), \quad (5.5)$$

for some constant $C > 0$.

$$\langle\mu, |v|\rangle \leq \widetilde{C}\left[\frac{1}{2}\boldsymbol{D}(v, v) + \int_G v(x)^2 dx\right]^{1/2} \quad \textit{for any } v \in C_c^1(\mathbb{C}), \quad (5.6)$$

for some constant $\widetilde{C} > 0$.

Proof By [8, Exercise 4.2.2], a positive finite measure μ on $\mathbb{C}$ with supp$[\mu] \subset G$ satisfies $\langle\mu, R_{G,\alpha}\mu\rangle < \infty$ for $\alpha > 0$ if and only if μ is of finite energy relative to the Dirichlet form $(\frac{1}{2}\mathbf{D}, H_0^1(G))$ on $L^2(G)$ and, in this case, $R_{G,\alpha}\mu$ is a quasi-continuous version of the α-potential of μ. The equivalence of Eqs. 5.4 and 5.5 is an easy consequence of this.

Suppose a positive finite measure μ on $\mathbb{C}$ with supp$[\mu] \subset G$ satisfies (5.5). Take $f \in C_c^1(G)$ with $f = 1$ on supp$[\mu]$, $0 \leq f \leq 1$. Then, for any $v \in C_c^1(\mathbb{C})$, $fv \in C_c^1(G)$ and $\langle\mu, |v|\rangle \leq C\mathbf{D}(fv, fv)^{1/2}$, whose right hand side is dominated by the right hand side of Eq. 5.6 with $\widetilde{C} = 2C(1 \vee C_1)^{1/2}$ for $C_1 = \sup_{\mathbf{x}\in G, i=1,2} |f_{\mathbf{x}_i}(\mathbf{x})|$.

Conversely (5.6) implies (5.5) due to a Poincaré inequality [8, Example 1.5.1]. □

Proof of Proposition 5.1 (i). Under the stated condition, we can draw from the fundamental identity for the logarithmic potential [12, Theorem 3.4.2] the following identity holding for the planar Brownian motion $\mathbb{M} = (X_t, \{\mathbb{P}_{\mathbf{x}}\}_{\mathbf{x}\in\mathbb{C}})$:

$$U|\mu|(\mathbf{x}) = R_G|\mu|(\mathbf{x}) + \mathbb{E}_{\mathbf{x}}\left[U|\mu|(X_{\sigma_{\partial G}}); \sigma_{\partial G} < \infty\right] - W_{\partial G}(\mathbf{x})|\mu|(\mathbb{C}), \quad \mathbf{x} \in G, \quad (5.7)$$

where $W_{\partial G}$ is some bounded function on G.

The identity (5.7) implies that $\langle|\mu|, U|\mu|\rangle$ is finite if and only if $\langle|\mu|, R_G|\mu|\rangle$ is finite. By Lemma 5.2, the latter condition is equivalent to $|\mu| \in \mathcal{S}_0^{(g,0)}$ relative to the perturbed Dirichlet form of $(\frac{1}{2}\mathbf{D}, H^1(\mathbb{C}))$ by the function $g = 1_{\overline{G}}$. In other words, $\mu \in \overset{\circ}{\mathcal{M}}_{00}$ if and only if $\mu \in \mathcal{M}_{00}$, where $\mathcal{M}_{00}$ is the space of measures defined by Eq. 3.22 relative to the admissible set $\overline{G}$. In this case, $U\mu$ satisfies the Poisson equation (5.3), while $R\mu$ does the same in view of Eq. 3.24. Therefore $R\mu = U\mu$ q.e. modulo a constant.

(ii). It suffices to prove this assertion by assuming that $\mu \in \mathcal{M}_{00}$ is of compact support because generally we can make an approximation by such measures using the bound (3.26). We then take a relatively compact open set G containing the support of $|\mu|$. Let $\{\widehat{R}\nu : \nu \in \widehat{\mathcal{M}}_{00}\}$ be the family of recurrent potentials relative to the admissible set $\overline{G}$. By virtue of Propoition 3.9, there exists a sequence $\{\nu_n\} \subset \mathcal{M}_{00} \cap \widehat{\mathcal{M}}_{00}$ such that $\text{supp}[\nu_n] \subset \text{supp}[\mu] \subset G$, $R\nu_n = \widehat{R}\nu_n$ modulo a contant and $R\nu_n$ is $\mathbf{D}$-convergent to $R\mu$ as $n \to \infty$. In view of (i), $\nu_n \in \overset{\circ}{\mathcal{M}}_{00}$ and $\widehat{R}\nu_n = U\nu_n$ modulo a constant for each n, and consequently $U\nu_n$ is $\mathbf{D}$-convergent to $R\mu$. □

Precisely analogous conideration can be made for the upper half plane $\mathbb{H}$ and the accompanied form

$$\mathbf{D}_{\mathbb{H}}(u, v) = \int_{\mathbb{H}} \nabla u(\mathbf{x}) \cdot \nabla v(\mathbf{x}) dx, \quad H^1(\mathbb{H}) = \{u \in L^2(\mathbb{H}), |\nabla u| \in L^2(\mathbb{H})\}.$$

$(\frac{1}{2}\mathbf{D}_{\mathbb{H}}, H^1(\mathbb{H}))$ is a regular, irreducible and recurrent Dirichlet form on $L^2(\overline{\mathbb{H}})$, and the associated Hunt process on $\overline{\mathbb{H}}$ is the reflecting Brownian motion (RBM in abbreviation). For $\mathbf{x} = (x, y) \in \mathbb{C}$, $\mathbf{x}^* = (x, -y)$ denotes its reflection relative to $\partial\mathbb{H}$. Using the logarithmic kernel $k(\mathbf{x})$ of Eq. 5.1, the *logarithmic kernel for the RBM* on $\overline{\mathbb{H}}$ is defined by

$$\widehat{k}(\mathbf{x}, \mathbf{y}) = k(\mathbf{x} - \mathbf{y}) + k(\mathbf{x} - \mathbf{y}^*), \quad \mathbf{x}, \mathbf{y} \in \overline{\mathbb{H}}.$$

For a finite signed meaure μ on $\overline{\mathbb{H}}$ with compact support, its logarithmic potential $\widehat{U}\mu$ for the RBM is defined by

$$\widehat{U}\mu(\mathbf{x}) = \int_{\overline{\mathbb{H}}} \widehat{k}(\mathbf{x}, \mathbf{y})\mu(d\mathbf{y}), \quad \mathbf{x} \in \overline{\mathbb{H}}.$$

We can then formulate a counterpart of Proposition 5.1 for $\overline{\mathbb{H}}$, $\widehat{U}$, $\mathbf{D}_{\mathbb{H}}$ in place of $\mathbb{C}$, U, $\mathbf{D}$, repectively, and prove it by using [7, Proposition 3.2] a counterpart of the identity (5.7).

5.2 Dirichlet Forms of Reflecting Diffusions

We consider the case where E is a domain D of the Euclidean d-space $\mathbb{R}^d$ for $d \geq 2$, and m is the Lebesgue measure on D. Put

$$\text{BL}(D) = \{u \in L^2_{\text{loc}}(D) : |\nabla u| \in L^2(D)\}, \quad H^1(D) = \text{BL}(D) \cap L^2(D).$$

The Dirichlet integral $\int_D \nabla u(x) \cdot \nabla v(x) dx$ is denoted by $\mathbf{D}_D(u, v)$. We write $\mathbf{D}_{D,1}(u, v) = \mathbf{D}_D(u, v) + (u, v)_{L^2(D)}$. A linear map $u \in H^1(D) \mapsto \widehat{u} \in H^1(\mathbb{R}^d)$ is called an *extension operator* if $\widehat{u}\big|_D = u$ a.e., and $\mathbf{D}_{\mathbb{R}^d,1}(\widehat{u},\widehat{u}) \le C\ \mathbf{D}_{D,1}(u,u),\ u \in H^1(D)$, for some constant $C > 0$. A domain admitting an extension operator is called an *extension domain*. Denote by $\mathcal{D}$ the collection of all extension domains. Any domain D with Lipschitz boundary ∂D is in $\mathcal{D}$ (cf. [15, p 181]). It holds for any $D \in \mathcal{D}$ that $H^1(\mathbb{R}^d)\big|_D = H^1(D)$.

We fix a domain $D \in \mathcal{D}$. Given measurable functions $a_{ij}(x),\ 1 \le i, j \le d$, on D such that

$$a_{ij}(x) = a_{ji}(x), \quad \Lambda^{-1}|\xi|^2 \le \sum_{1\le i,j\le d} a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2, \quad x \in D,\ \xi \in \mathbb{R}^d,$$

for some constant $\Lambda \ge 1$, let us consider the form

$$(\mathcal{E},\mathcal{F}) = (\mathbf{a}, H^1(D)) \tag{5.8}$$

on $L^2(D)$ where

$$\mathbf{a}(u,v) = \int_D \sum_{i,j=1}^d a_{ij}(x)\frac{\partial u}{\partial x_i}(x)\frac{\partial v}{\partial x_j}(x)dx, \quad u, v \in H^1(D).$$

The form (5.8) is a regular, irreducible and strongly local Dirichlet form on $L^2(\overline{D})$. The associated diffusion process $\mathbb{M}$ on $\overline{D}$ is called the *reflecting diffusion* determined by the uniformly elliptic diffusion coefficients $\{a_{ij}(x)\}$. The transition function of the reflecting diffusion $\mathbb{M}$ on $\overline{D}$ satisfies the absolute continuity condition (2.6). To see this, recall that the Dirichlet form $(\frac{1}{2}\mathbf{D}_{\mathbb{R}^d}, H^1(\mathbb{R}^d))$ on $L^2(\mathbb{R}^d)$ satisfies the Sobolev inequality (2.7) for some $q > 2$ (cf. [8, Theorem 2.4.3]). Since D is an extension domain, this inequality is honestly inherited by the form $(\frac{1}{2}\mathbf{D}_D, H^1(D))$ on $L^2(D)$, and accordingly by the form (5.8) on $L^2(D)$ as well. Therefore $\mathbb{M}$ has the desired property (2.6) by virtue of [8, Theorem 4.2.7].

Equation 5.8 is either recurrent or transient. It is always recurrent when $d = 2$ [3, Theorem 2.2.13]. When $d \ge 3$, it is recurrent if D has a finite Lebesgue measure. In the recurrent case, its extended Dirichlet space coincides with its reflected Dirichlet space $(\mathrm{BL}(D), \mathbf{a})$ [3, Theorem 6.3.2] and the Gaussian field indexed by $\mathrm{BL}(D)$ with covariance $\mathbf{a}$ has the Markov property by Theorem 4.4. In the transient case, the extended Dirichlet space $H^1_e(D)$ of Eq. 5.8 is a proper subspace of $\mathrm{BL}(D)$ and yet the Gaussian field indexed by $H^1_e(D)$ with covariance $\mathbf{a}$ has also the Markov property by [14].

5.3 Energy Forms

We consider a measurable function $\rho(x)$ on $\mathbb{R}^d$ for $d \ge 1$ such that

$$0 < \lambda_\ell \le \rho(x) \le \Lambda_\ell < \infty, \quad \text{for every } x \in B_\ell := \{|x| < \ell\}, \quad \ell > 0. \tag{5.9}$$

for constants $\lambda_\ell,\ \Lambda_\ell$ depending on $\ell > 0$, and the associated spaces $\mathcal{F}^\rho,\ \mathcal{G}^\rho$ and form $\mathbf{D}^\rho$ defined respectively by

$$\mathcal{F}^\rho = \{u \in L^2(\mathbb{R}^d; \rho dx)\ :\ |\nabla u| \in L^2(\mathbb{R}^d; \rho dx)\},$$
$$\mathcal{G}^\rho = \{u \in L^2_{\mathrm{loc}}(\mathbb{R}^d)\ :\ |\nabla u| \in L^2(\mathbb{R}^d; \rho dx)\},$$
$$\mathbf{D}^\rho(u,v) = \int_{\mathbb{R}^d} \nabla u(x)\cdot\nabla v(x)\ \rho(x)\ dx.$$

When ρ is a positive constant, $\mathcal{F}^\rho,\ \mathcal{G}^\rho$ are reduced to $H^1(\mathbb{R}^d),\ \mathrm{BL}(\mathbb{R}^d)$, respectively.

$(\mathbf{D}^\rho, \mathcal{F}^\rho)$ is a regular, irreducible and strongly local Dirichlet form on $L^2(\mathbb{R}^d; \rho dx)$ ([6]). This form is called an *energy form* and the associated diffusion process $\mathbb{M}^\rho$ on $\mathbb{R}^d$ is called the *distorted Brownian motion*. Let $\mathbb{M}_{\ell,0}$ be the part of $\mathbb{M}$ on the ball B_ℓ obtained by killing upon leaving B_ℓ. On account of Eq. 5.9, the Dirichlet form of $\mathbb{M}_{\ell,0}$ on $L^2(B_\ell, \rho dx) = L^2(B_\ell)$ is equivalent to the form $(\mathbf{D}, H_0^1(B_\ell))$ that satisfies the Sobolev inequality (2.7) for some $q > 2$. Hence the transition function of $\mathbb{M}_{\ell,0}$ fulfills the absolute continuity condition (2.6) by [8, Theorem 4.2.7] and so does the transition function of the distorted Brownian motion $\mathbb{M}$ as one can see by letting $\ell \to \infty$.

The energy form $(\mathbf{D}^\rho, \mathcal{F}^\rho)$ on $L^2(\mathbb{R}^d : \rho dx)$ is either recurrent or transient. It is recurrent if $\int_{\mathbb{R}^d} \rho dx < \infty$. In the recurrent case, its extended Dirichlet space coincides with its reflected Dirichlet space $(\mathcal{G}^\rho, \mathbf{D}^\rho)$ [6] and the Gaussian field indexed by $\mathcal{G}^\rho$ with covariance $\mathbf{D}^\rho$ has the Markov property by Theorem 4.4. In the transient case, the extended Dirichlet space $\mathcal{F}_e^\rho$ of the energy form is a proper subspace of $\mathcal{G}^\rho$ and yet the Gaussian field indexed by $\mathcal{F}_e^\rho$ with covariance $\mathbf{D}^\rho$ enjoys also the Markov property by [14].

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