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One-point extensions of Markov processes by darning

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Abstract This paper is a continuation of the works by Fukushima–Tanaka (Ann Inst Henri Poincaré Probab Stat 41: 419–459, 2005) and Chen–Fukushima–Ying (Stochastic Analysis and Application, p.153–196. The Abel Symposium, Springer, Heidelberg) on the study of one-point extendability of a pair of standard Markov processes in weak duality. In this paper, general conditions to ensure such an extension are given. In the symmetric case, characterizations of the one-point extensions are given in terms of their Dirichlet forms and in terms of their L^2-infinitesimal generators. In particular, a generalized notion of flux is introduced and is used to characterize functions in the domain of the L^2-infinitesimal generator of the extended process. An important role in our investigation is played by the α-order approaching probability u_α.

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Contents

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1 Introduction

The purpose of this paper is twofold. The first is to present in Sect. 3 general conditions to ensure the one-point extendability of a pair of standard Markov processes in weak duality by developing those methods employed in Fukushima–Tanaka [14] and Chen–Fukushima–Ying [8]. The second is to characterize in Sect. 4 the Dirichlet form and the L^2-infinitesimal generator of the extended process in the symmetric case by making use the results in Chen–Fukushima–Ying [6]. In both subjects, we shall focus our attention on the significant roles played by the α-order approaching probability u_α.

Let E be a locally compact separable metric space and m a σ-finite measure on E. Consider a closed subset K of E and put $E_0 = E \setminus K$. We assume that either K is compact or E_0 is relatively compact in E. We then consider the topological extension $E^* = E_0 \cup \{a\}$ of E_0 obtained by regarding K as a single point a. When the open set E_0 is relatively compact in E, E^* is nothing but the one point compactification of E_0. The restriction of the measure m to E_0 will be denoted by m_0, which is then extended to E^* by setting $m_0(\{a\}) = 0$.

Let $X = (X_t, \mathbf{P}_x)$, $\widehat{X} = (\widehat{X}_t, \widehat{\mathbf{P}}_x)$ be a pair of standard processes on E which are in weak duality with respect to m. They are assumed to be of no jumps from E_0 to K but approachable to K from E_0 in the sense that

$$\mathbf{P}_x(\sigma_K < \infty) > 0, \quad \widehat{\mathbf{P}}_x(\sigma_K < \infty) > 0, \quad \text{for q.e. } x \in E_0,$$

where $\sigma_K := \inf\{t > 0 : X_t \in K\}$ is the first hitting time of K by X. When there is no possibility for confusion, the first hitting time of K by $\widehat{X}$ will also be denoted as σ_K.

Let $X^0 = (X_t^0, \mathbf{P}_x^0, \zeta^0)$ and $\widehat{X}^0 = (\widehat{X}_t^0, \widehat{\mathbf{P}}_x^0, \widehat{\zeta}^0)$ be the subprocesses of X and $\widehat{X}$ on E_0, respectively, killed upon leaving E_0. They are in weak duality with respect to m_0. We shall call them part processes or absorbed processes occasionally. We are concerned with extending X^0 and $\widehat{X}^0$ on E_0 to a pair of standard processes X^* and $\widehat{X}^*$ on E^* that are in weak duality with respect to measure m_0. When X is m-symmetric, X^0 is m_0-symmetric on E_0 and we will look for m_0-symmetric extensions X^* of X^0 to E^*.

To get X^* and $\widehat{X}^*$ from X^0 and $\widehat{X}^0$, there are two different but closely related point of views. The first one is just what is stated above: we start with a pair of processes X and $\widehat{X}$ on state space E that are in weak duality with respect to measure m and X^0 and $\widehat{X}^0$ are the subprocesses of X and $\widehat{X}$ killed upon leaving $E_0 := E \setminus K$, where K is a closed subset of E. The second approach is that the metric space $E^* := E_0 \cup \{a\}$ is prescribed a priori and what we are given is a pair of standard processes X^0 and $\widehat{X}^0$

on E_0 that are in weak duality with respect to a measure m_0 on E_0. In this paper, we will adopt the first point of view.

For the second approach, in order to construct X^* and $\widehat{X}^*$ from X^0 and $\widehat{X}^0$, it is necessary to assume that X^0 and $\widehat{X}^0$ approach at finite lifetimes to the point a in the topology of E^* with positive probability. In fact, this approach of constructions of X^* and $\widehat{X}^*$ from X^0 and $\widehat{X}^0$ together with their characterizations has been accomplished by [14] in the symmetric diffusion case and by [8] in the weak dual standard processes case under certain additional conditions on X^0 and $\widehat{X}^0$. We have made use of Itô's Poisson point processes of excursions of X^0 and $\widehat{X}^0$ around the point a whose characteristic measures are governed by the uniquely determined entrance laws $\{\mu_t, t > 0\}$ of X^0 and $\{\widehat{\mu}_t, t > 0\}$ of $\widehat{X}^0$ from the one point a. The processes X^* and $\widehat{X}^*$ on E^* are then constructed by stitching up those excursions of X^0 and $\widehat{X}^0$ according to the rules $\{\mu_t, t > 0\}$ and $\{\widehat{\mu}_t, t > 0\}$.

We shall call such a procedure of obtaining X^* from X (or from X^0) *darning a hole* K. Kiyosi Itô [20] introduced the notion of the Poisson point process of excursions around one point a in the state space of a standard Markov process X. He was motivated by giving systematic constructions of Markovian extensions of the absorbed diffusion process X^0 on the half line $(0, \infty)$ subjected to Feller's general boundary conditions [22]. Itô had constructed the most general jump-in process from the exit boundary 0 by using the Poisson point process in his unpublished lecture notes [19] that preceded [20]. In this case, a is just the point 0. However recent papers [14] and [8] show that Itô's program works equally well in the construction of X^* by conceiving a certain set K as a single point a.

We point out that it is assumed in both [8] and [14] that X^0 and $\widehat{X}^0$ admit no killings inside E_0. Furthermore, to ensure the finiteness of $\mu_t(E_0)$ for each $t > 0$, a crucial condition imposed in Sect. 4 of [14] and in the first part of Sect. 5 of [8] is the m_0-integrability of the α-order approaching probability u_α of X^0 defined by

$$u_\alpha(x) := \mathbf{E}_x^0 \left[e^{-\alpha\zeta^0}; X_{\zeta^0-} \in K \right] \left(= \mathbf{E}_x \left[e^{-\alpha\sigma_K} \right] \right), \quad x \in E_0, \ \alpha > 0.$$

Note that in [8] and [14], the set K is identified with a single point $\{a\}$. At the end of Sect. 5 of [8], the integrability condition on u_α is eventually removed by reducing the situation to the case that $m_0(E_0) < \infty$ using a certain time-change of X^0. But for the time-change argument to go through, we need to assume, in addition to the finiteness of m_0 on some neighborhood of K, a property of X^0 that is invariant under time-change

$$\mathbf{P}_x^0 \left(X_{\zeta^0-}^0 \in K \cup \{\Delta\} \right) = 1 \quad \text{for } x \in E_0, \tag{1.1}$$

which is required to hold regardless the length of the life time ζ^0 finite or infinite. Here Δ is the point at infinity of E. Then the integrability of u_α becomes a consequence of the possibility of the darning rather than an assumption for it. So it is very important to know under what condition on X its part process X^0 has the desired property (1.1). Note that under the assumption that X^0 admit no killings inside E_0, condition (1.1) is equivalent to a more general one

$$\mathbf{P}_x^0\left(X^0_{\zeta^0-}\in E_\Delta\right)=1\quad\text{for } x\in E_0. \tag{1.2}$$

We shall start the present paper with an answer to the above question in a general setting. In Sect. 2, we study a general strong Markov process X on E in relation to a fixed excessive measure m and establish the dichotomy of recurrence and transience under the m-irreducibility assumption together with the sample path behaviors near the end of its life time, following the corresponding arguments in Fukushima–Oshima–Takeda [13] and Getoor [15]. Under the m-irreducibility of X together with an additional lower semicontinuity assumption for excessive functions of X when X is not m-symmetric, we then derive that the property (1.2) for the part process X^0 on E_0 holds for quasi-every $x\in E_0$ (see Theorem 2.5 below).

In Sect. 3, by combining Theorem 2.5 with the results of Sect. 5 of [8], we formulate those conditions on X and X^0 that will enable us to construct X^* by darning the hole K (see Theorem 3.1 below). The m_0-integrability of u_α is just a consequence of those conditions. In Theorem 3.1, we shall also see how some basic properties of X and X^0 are honestly inherited by X^*, which will be important when darning more than one hole (see [5]).

As mentioned earlier, in both [14] and [8] an additional strong condition is assumed that X^0 and $\widehat{X}^0$ admit no killings inside E_0. This condition in fact can be much relaxed to only require that X^0 and $\widehat{X}^0$ have no killings inside some neighborhood of K, as is done in this paper. We will explain in the Appendix (Sect. 6) of this paper in some details how this can be done and that the main results of [8] (especially those in Sect. 5) are still valid under this weaker condition. Theorem 3.1 of this paper will be formulated by incorporating this relaxation of the condition. In non-symmetric case, generally speaking, the darned processes X^* and $\widehat{X}^*$ on E^* will preserve the m_0-weak duality only if we allow some killings at the point a for them. Therefore the relaxation to allow X and $\widehat{X}$ have killings inside E is important when we apply the darning procedures repeatedly to non-symmetric Markov processes (see [5]).

In Sect. 4, we shall specialize Theorem 3.1 to the case where X is m-symmetric and formulate in Theorem 4.1 the unique one-point symmetric extension X^* of X^0 from E_0 to $E^*=E_0\cup\{a\}$ such that X^* admits neither sojourn nor killing at the point a. But, as is noted previously, we may instead start with an m_0-symmetric standard process X^0 on E_0 without referring to X and consider its unique m_0-symmetric extension X^* as above. Denote by $(\mathcal{E}^*,\mathcal{F}^*)$ and $(\mathcal{E}^0,\mathcal{F}^0)$ the Dirichlet spaces of X^* and X^0 on $L^2(E^*;m_0)=L^2(E_0;m_0)$, respectively. The second purpose of the present paper is to characterize $(\mathcal{E}^*,\mathcal{F}^*)$ in terms of the reflected Dirichlet space of $(\mathcal{E}^0,\mathcal{F}^0)$ and to give a lateral condition that characterizes the domain of the L^2-infinitesimal generator of X^*.

Let $(\mathcal{E}^{\rm ref},(\mathcal{F}^0)^{\rm ref})$ be the reflected Dirichlet space of $(\mathcal{E}^0,\mathcal{F}^0)$ and let $(\mathcal{F}^0)^{\rm ref}_a:=(\mathcal{F}^0)^{\rm ref}\cap L^2(E_0;m_0)$ be its active part. The notion of reflected Dirichlet space was introduced by M. L. Silverstein in [25] and [26], and was further studied by Chen in [3] (see also [6, Sect. 3]). By making use of Theorem 3.4 of [6], we shall show in Theorem 4.4 that $u_\alpha\in(\mathcal{F}^0)^{\rm ref}$ and furthermore the space $(\mathcal{F}^*,\mathcal{E}^*_\alpha)$ is just the subspace of $((\mathcal{F}^0)^{\rm ref}_a, \mathcal{E}^{\rm ref}_\alpha)$ spanned by $\mathcal{F}^0$ and u_α. We introduce a linear operator $\mathcal{L}$ on $L^2(E_0; m_0)$ by

$$\begin{aligned} &f \in \mathcal{D}(\mathcal{L}) \text{ with } \mathcal{L}f = g \in L^2(E_0; m_0) \quad \text{if and only if} \\ &f \in (\mathcal{F}^0)^{\rm ref} \quad \text{such that} \quad \mathcal{E}^{\rm ref}(f, v) = -(g, v) \quad \text{for every} \quad v \in \mathcal{F}^0. \end{aligned} \tag{1.3}$$

We also define the *flux* $\mathcal{N}(f)$ of $f \in \mathcal{D}(\mathcal{L})$ at a by

$$\mathcal{N}(f) = \mathcal{E}^{\rm ref}(f, u_\alpha) + (\mathcal{L}f, u_\alpha), \tag{1.4}$$

which will be shown to be independent of $\alpha > 0$. Theorem 4.4 implies that $\mathcal{L}$ is an extension of the L^2-generator $\mathcal{A}^*$ of X^*. We shall show in Theorem 4.8 that a function $f \in \mathcal{D}(\mathcal{L})$ is in $\mathcal{D}(\mathcal{A}^*)$ if and only if f admits a fine limit at a along the path of X^0 and satisfies the *lateral condition* $\mathcal{N}(f) = 0$. In Sect. 5, explicit expressions for $\mathcal{N}(f)$ will be derived in concrete examples.

Note that the subprocess X^0 of X is symmetric with respect to the measure $m_0 := m|_{E_0}$. However since X^0 may not be irreducible, the symmetrizing measure for X^0 can be non-unique in general. Suppose that E_0 is a disjoint union of open subsets $E_{01}, \ldots, E_{0k}$, each of which is X^0-invariant. Then for any choice of k-vector $\mathbf{p} = (p_1, \ldots, p_k)$ with positive entries,

$$\widetilde{m}_0 := \sum_{i=1}^k p_i \cdot m_0^i \quad \text{with } m_0^i := m_0|_{E_{0i}} \qquad \text{for } 1 \le i \le k,$$

is again a symmetrizing measure of X^0. So we can consider the $\widetilde{m}_0$-symmetric extension $\widetilde{X}^*$ of X^0 to E^* and, as will be formulated in Theorem 4.11 at the end of Sect. 4, our characterization in terms of the reflected Dirichlet space of $\mathcal{F}^0$ will be useful to identify the Dirichlet form $(\widetilde{\mathcal{E}}^*, \widetilde{\mathcal{F}}^*)$ of $\widetilde{X}^*$ on $L^2(E^*; \widetilde{m}_0)$ and the L^2-generator $\widetilde{\mathcal{A}}^*$ of $\widetilde{X}^*$.

The first four subsections of Sect. 5 will treat some basic examples in symmetric cases. From Sect. 5.1 to Sect. 5.3, we shall examine how the above mentioned *skew* darned processes $\widetilde{X}^*$ of symmetric diffusions X^0 look like by exhibiting their Dirichlet forms and lateral conditions.

In part **(ii)** of Sect. 5.1, we consider the case that $E = \mathbb{R}$, $K = \{a\} = \{0\}$, $E_0 = (-\infty, 0) \cup (0, \infty)$, m is the Lebesgue measure and X is the one-dimensional standard Brownian motion. In this case, $E^* = \mathbb{R}$. We shall prove in Theorem 5.1 that the darned diffusion $\widetilde{X}^*$ on $\mathbb{R}$ of the absorbed Brownian motion X^0 on E_0 can be identified with the well-known *skew Brownian motion* (cf. [18,24]). Its construction by darning and its characterization by the reflected Dirichlet space seem to be new. We shall also derive a Skorohod stochastic equation for $\widetilde{X}^*$

$$\widetilde{X}^*_t = \widetilde{X}^*_0 + B_t + (p^+ - p^-)\ell_t,$$

where B_t is a Brownian motion and ℓ_t is a positive continuous additive functional of $\widetilde{X}^*$ with Revuz measure δ_0, by using a decomposition theorem of strict additive functionals for a symmetric diffusion [12].

In Sect. 5.1, we shall also construct the reflecting Brownian motion on $[0, \infty)$ and the Brownian motion on a circle by darning and further present the lateral condition for a darned Brownian motion obtained by identifying multi-points.

In Sect. 5.2, we shall consider the case where X^0 is a collection of absorbed diffusions on half lines merging at one point. Its possible extensions $\widetilde{X}^*$ have been mentioned already in Example 6.3 of [14], but we shall identify their Dirichlet forms and lateral conditions in the present framework. The skew Brownian motion on the real line may be regarded as a special case of this example.

Section 5.3 will treat the case where X is the absorbed Brownian motion on an open set of $\mathbb{R}^n$, which has appeared in Examples 6.4 and 6.2 of [14]. Applying Theorem 4.4 and Theorem 4.8, we shall characterize the Dirichlet space and the lateral condition of X^* in terms of the Sobolev space $\mathbb{H}^1$.

In Sect. 5.4, we shall consider the darning of a censored stable process X^0 on an Euclidian open set, which is a jump-type Markov process and has been discussed in Sect. 6.1 of [8]. We shall examine how the conditions for Theorem 4.1 are fulfilled for different type of open sets and identify the Dirichlet forms of the darned processes X^* using Theorem 4.4.

Deviating from above subsections, in Sect. 5.5 we will examine the darning of non-symmetric diffusions in Euclidean domains, giving an example to illustrate the scope of Theorem 3.1.

In the rest of this section, for illustrating the stated properties of the α-order hitting probability and the notion of the reflected Dirichlet space and the lateral condition, we shall consider the absorbed diffusion process $X^0 = (X_t^0, \zeta^0, \mathbf{P}_x^0)$ on the open half line $(0, \infty)$ with Feller's local generator

$$\frac{d}{dm}\frac{d}{ds}. \tag{1.5}$$

Define for $\alpha > 0$ the α-order approaching probability to $\{0\}$ by

$$u_\alpha(x) = \mathbf{E}_x^0\left[e^{-\alpha\zeta^0}; X_{\zeta^0-}^0 = 0\right], \quad x \in (0, \infty).$$

Then

$$u_\alpha(x) > 0 \quad \text{and} \quad \int_0^\infty u_\alpha(x) m(dx) < \infty \tag{1.6}$$

if and only if the boundary 0 is regular (exit and entrance in the terminology of [23, Sect. 4.6]), namely,

$$s(0+) > -\infty \quad \text{and} \quad m((0, 1)) < \infty.$$

In fact, the first condition of (1.6) is known to be fulfilled if and only if 0 is exit. On the other hand, u_α is a non-negative decreasing solution of

$$\left(\alpha - \frac{d}{dm}\frac{d}{ds}\right)u = 0$$

and hence

$$\alpha \int_0^x u(x)m(dx) = \frac{du}{ds}(x) - \frac{du}{ds}(0+) \leq -\frac{du}{ds}(0+),$$

which is known to be finite if and only if 0 is entrance (see [21, Sect. 5.18] or [23, Sect. 4.6]). In particular, the integrability of u_α in (1.6) never depends on the behavior of X^0 near ∞ at all.

The process X^0 is known to be m-symmetric. Its Dirichlet space $(\mathcal{F}^0, \mathcal{E}^0)$ on $L^2(0, \infty); m)$ and the reflected Dirichlet space $((\mathcal{F}^0)^{\text{ref}}, \mathcal{E}^{\text{ref}})$ of $\mathcal{F}^0$ can be described as follows:

$$(\mathcal{F}^0)^{\text{ref}} = \left\{ f : f \text{ is absolutely continuous with respect to } s \text{ and } \mathcal{E}^{\text{ref}}(f, f) < \infty \right\}, \quad (1.7)$$

where

$$\mathcal{E}^{\text{ref}}(f, f) = \int_0^\infty \left(\frac{df(x)}{ds(x)}\right)^2 ds(x), \quad (1.8)$$

while

$$\mathcal{F}^0 = \left\{ f \in (\mathcal{F}^0)_a^{\text{ref}} : f(0+) = 0 \right\}$$
$$\text{and} \quad \mathcal{E}^0(f_1, f_2) = \mathcal{E}^{\text{ref}}(f_1, f_2) \quad \text{for } f_1, f_2 \in \mathcal{F}^0, \quad (1.9)$$

under the condition that 0 is regular but ∞ is non-regular. This condition is assumed from now on.

In this case, the operator $\mathcal{L}$ defined by (1.3) is equal to the expression (1.5) with

$$\mathcal{D}(\mathcal{L}) = \left\{ f \in (\mathcal{F}^0)_a^{\text{ref}} : d\frac{df}{ds} \text{ is absolutely continuous with respect to } m, \right.$$
$$\left. \mathcal{L}f \in L^2((0, \infty); m) \right\}, \quad (1.10)$$

while we shall see in Lemma 5.3 of Sect. 5.2 that

$$\mathcal{N}(f) = -\frac{df}{ds}(0+), \quad f \in \mathcal{D}(\mathcal{L}). \quad (1.11)$$

Let us denote by $\mathcal{F}^*$ the active reflected Dirichlet space $(\mathcal{F}^0)^{\text{ref}}_a$. Then $(\mathcal{F}^*, \mathcal{E}^{\text{ref}})$ is a local regular Dirichlet space on $L^2([0,\infty); m)$ and, by a general existence theorem [13], it has an associated m-symmetric diffusion $X^* = (X^*_t, \mathbf{P}^*_x)$ on $[0,\infty)$, which is a unique m-symmetric one-point extension of X^0 to $[0,\infty)$ with no sojourn nor killing at $\{0\}$. Hence the function u_α can be described by the hitting time $\sigma_{\{0\}}$ of $\{0\}$ as

$$u_\alpha(x) = \mathbf{E}^*_x\left[e^{-\alpha\sigma_{\{0\}}}; \sigma_{\{0\}} < \infty\right] \quad \text{for } x \in (0,\infty),$$

which in turn implies that u_α is a member of $\mathcal{F}^*$ and that the space $(\mathcal{F}^*, \mathcal{E}^{\text{ref}}_\alpha)$ is spanned by $\mathcal{F}^0$ and u_α (cf. [13, Sect. 4]). By Theorem 4.8 and (1.11), we see that f is in the domain of the L^2-generator of X^* if and only if

$$f \in \mathcal{D}(\mathcal{L}), \quad |f(0+)| < \infty \quad \text{and} \quad \frac{df}{ds}(0+) = 0. \tag{1.12}$$

The above example corresponds to the simplest case that

$$E_0 = (0,\infty), \quad E = E^* = [0,\infty), \quad K = \{a\} = \{0\},$$

X^0 is the absorbed diffusion on $(0,\infty)$ and X as well as X^* is the reflected diffusion on $[0,\infty)$.

In Example 5.2, we shall exhibit a family of symmetric skew extensions of a collections of such absorbed diffusions on half lines merging at one point.

2 Recurrence, transience and path behaviors at $\zeta-$

The main goal of this section is to extend the notion of transience and recurrence in Fukushima–Oshima–Takeda [13] and Getoor[15] to a right process X having excessive measure m under quasi-every (q.e. in abbreviation) setting and derive the corresponding global behaviors of the sample paths. We start with formulating transience, recurrence and m-irreducibility analytically, similar to that in [13] and then relate them to the notions in [15].

Let $X = (X_t, \zeta, \mathbf{P}_x)$ be a Borel right processes on a locally compact separable metric space E and m be a σ-finite excessive measure on E with full support. Let $E_\Delta = E \cup \{\Delta\}$ be the one-point compactification of E. Any function f on E will be extended to E_Δ by setting $f(\Delta) = 0$ unless otherwise specified.

We call a nearly Borel set $A \subset E$ *m-polar* if $\mathbf{P}_m(\sigma_A < \infty) = 0$, where $\sigma_A := \inf\{t > 0 : X_t \in A\}$. A statement is said to be true q.e. on E if there is a nearly Borel m-polar set A such that the statement holds for every $x \in E \setminus A$. The 0-*order resolvent* operator G of X is defined by

$$Gf(x) = \mathbf{E}_x\left[\int_0^\infty f(X_t)dt\right], \quad x \in E,$$

for every non-negative nearly Borel function f on E.

The process X is said to be *transient, recurrent* and *m-irreducible* if

$$
\begin{aligned}
&Gf(x) < \infty \quad \text{q.e. on } E \quad \text{for some strictly positive } f \in L^1(E;m),\\
&Gf(x) = \infty \quad \text{q.e. on } E \quad \text{for every } f \in L^1_+(E;m) \text{ with } \int_E f(x)m(dx) > 0, \qquad (2.1)\\
&\mathbf{P}_x(\sigma_B < \infty) > 0 \quad \text{for q.e. } x \in E \quad \text{whenever} \quad m(B) > 0,
\end{aligned}
$$

respectively. Here $L^1_+(E,m) = \{f \in L^1(E,m) : f \geq 0 \ m\text{-a.e. on } E\}$.

In view of [17, Sect. 6], any m-negligible nearly Borel finely open set is m-polar and furthermore $A \subset E$ is m-polar if and only if it is contained in a Borel m-inessential set B in the sense that $m(B) = 0$ and $E \setminus B$ is absorbing. So if X is transient, then for some strictly positive $f \in L^1(E,m)$, there is a Borel m-inessential set $A \subset E$ such that

$$Gf(x) < \infty \quad \text{for every } x \in E \setminus A.$$

Thus $X|_{E\setminus A}$ is transient in the sense of Getoor [15, Proposition 2.2].

By [16, Theorem 2.4], X is transient if and only if

$$Gf(x) < \infty \quad \text{q.e. for every } f \in L^1_+(E;m). \qquad (2.2)$$

Note that the above requires no irreducibility assumption on X. The next lemma can also be deduced from [16, Theorem 2.4]. However we present an alternative proof here.

Lemma 2.1 *Assume the process X is m-irreducible. Then X is either transient or recurrent.*

Proof Let $f \geq 0$ and define $\mathcal{N}_f := \{x : Gf(x) = \infty\}$, which is a finely closed subset of E. Note that

$$Gf(x) \geq \mathbf{E}_x\left[Gf(X_{\sigma_{\mathcal{N}_f}}); \sigma_{\mathcal{N}_f} < \infty\right], \quad x \in E. \qquad (2.3)$$

It follows that under the m-irreducibility condition (2.1), if $m(\mathcal{N}_f) > 0$, then $\mathbf{P}_x(\sigma_{\mathcal{N}_f} < \infty) > 0$ for q.e. $x \in E$ and consequently, by (2.3), $Gf = \infty$ q.e. on E. This proves that for every $f \geq 0$ on E,

$$\text{either } Gf < \infty \ \ m\text{-a.e. on } E, \quad \text{or } Gf \equiv \infty \text{ q.e. on } E. \qquad (2.4)$$

If X is not transient, then there is some $g \in L^1_+(E;m)$ with $m\{x : Gg(x) = \infty\} > 0$. So by (2.4), $Gg = \infty$ q.e. on E. Now by [13, Lemma 1.6.3], for every $f \in L^1_+(E;m)$, we have

$$m\{x : Gf(x) = \infty\} \geq m\{x : f(x) > 0\}.$$

Therefore, for every $f \in L^1_+(E; m)$ with $m\{f > 0\} > 0$, we have by (2.4) again that $Gf = \infty$ q.e. on E, proving the recurrence of X. □

We have defined the transience and recurrence of X analytically in terms of the convergence and divergence of the 0-order resolvent of X, respectively. In order to relate them to the sample path behaviors of X, we prepare two theorems. We denote by $\{P_t, t \geq 0\}$ the transition semigroup of X; that is, for any measurable $f \geq 0$,

$$P_t f(x) := \mathbf{E}_x \left[f(X_t) \right] \quad \text{for } x \in E \text{ and } t \geq 0.$$

Lemma 2.2 (i) *If X is m-irreducible, then any non-m-polar Borel set B satisfies*

$$\mathbf{P}_x(\sigma_B < \infty) > 0 \quad \text{for q.e. } x \in E.$$

(ii) *If X is recurrent, then X is conservative in the sense that $\mathbf{P}_x(\zeta = \infty) = 1$ for* q.e. $x \in E$. *Moreover, any bounded excessive function u of X is constant* q.e.

Proof (i). Let $D = \{x \in E : \mathbf{P}_x(\sigma_B < \infty) > 0\}$. Then $m(D) > 0$ and $\mathbf{P}_x(\sigma_D < \infty) > 0$ for q.e. $x \in E$. On the other hand, we see as in the proof of [13, Lemma 4.6.4] that $\mathbf{P}_x(\sigma_D < \infty) = 0$ for $x \in E \setminus D$. Hence $E \setminus D$ must be m-polar.
(ii). We first show that any bounded excessive function u satisfies

$$P_t u(x) = u(x) \quad \text{for every } \; t > 0, \; \text{q.e. } x \in E. \tag{2.5}$$

Its proof is similar to that in the first part of the proof of [15, Lemma 3.2]. However for reader's convenience, we spell out the details. Define

$$\psi(x) :=\downarrow \lim_{t\to\infty} P_t u \leq u.$$

Note that $P_s\psi = \lim_{t\to\infty} P_s(P_t u) = \psi$ for every $s > 0$. Let $f = u - \psi \geq 0$. Then

$$P_t f = P_t u - P_t \psi = P_t u - \psi$$

increases to $f = u - \psi$ as $t \downarrow 0$ and so f is excessive. Clearly $\downarrow \lim_{t\to\infty} P_t f = 0$ and $f \leq \|u\|_\infty$. For $n \geq 1$, define $f_n = n(f - P_{1/n} f)$, which is finely continuous with $Gf_n = n \int_0^{1/n} P_t f \, dt$ increasing to f as $n \to \infty$ (see [15, Lemma (3.1)]). If the finely open set $A_n := \{x : f_n(x) > 0\}$ has positive m-measure, then by the recurrence of X we have $f \geq Gf_n = \infty$ q.e. on E, which contradicts to the fact that f is bounded. Thus $m(A_n) = 0$ and thus $f_n = 0$ m-a.e. for every $n \geq 1$. This implies that $f = 0$ m-a.e.. As u is excessive, it follows that $u = \psi$ q.e. and thus (2.5) holds. As constant function 1 is excessive, it follows immediately from (2.5) that for q.e. $x \in E$, $\mathbf{P}_t 1(x) = 1$ for every $t > 0$; that is, for q.e. $x \in E$, $\mathbf{P}_x(\zeta = \infty) = 1$.

We now show that every bounded excessive function u has to be constant m-a.e.. If not, there are constants $0 < a < b$ such that the finely open sets $A := \{x \in E : u(x) > b\}$ and $B := \{x \in E : u(x) < a\}$ both have positive m-measure. As $u \wedge b$ is excessive, we have by (2.5) that q.e., $u \wedge b = P_t(u \wedge b)$ for every $t > 0$. So for q.e. $x \in A$,

$$b = u(x) \wedge b = \mathbf{E}_x \left[u(X_t) \wedge b \right].$$

It follows that for q.e. $x \in A$, $\mathbf{P}_x(X_t \in B) = 0$ for every $t > 0$ and so $G1_B(x) = 0$. This is impossible since X is recurrent, $m(B) > 0$ and we must have $G1_B = \infty$ q.e. This proves that u is constant m-a.e. Since u is finely continuous, u has to be constant q.e. □

Theorem 2.3 *Assume that X is recurrent. If $B \subset E$ is a nearly Borel set such that B is not m-polar, then*

$$\mathbf{P}_x \left(\sigma_B \circ \theta_n < \infty \, \textit{for every} \, n \geq 1 \right) = 1 \quad \text{for q.e. } x \in E.$$

Proof Let $\varphi_B(x) = \mathbf{P}_x(\sigma_B < \infty)$, which is a bounded excessive function of X. Since B is not m-polar, it follows from Lemma 2.2 that $\varphi_B = c$ q.e. for some constant $c > 0$. For each $t > 0$ and q.e. $x \in E$,

$$\begin{aligned} c &= \mathbf{P}_x(\sigma_B \leq t) + \mathbf{P}_x(t < \sigma_B, \sigma_B < \infty) \\ &= \mathbf{P}_x(\sigma_B \leq t) + \mathbf{E}_x[\mathbf{P}_{X_t}(\sigma_B < \infty); t < \sigma_B] \\ &= \mathbf{P}_x(\sigma_B \leq t) + c\mathbf{P}_x(t < \sigma_B). \end{aligned}$$

By letting $t \to \infty$ along a sequence, we get $c(1-c) = 0$ and hence $c = 1$. So we have $\varphi_B = 1$ q.e. on E. Since X is conservative, we have $P_n \varphi_B = 1$ for every integer $n \geq 1$ q.e. on E, from which the conclusion of the theorem follows. □

We consider the condition of the *existence of the left limit of X_t at finite life time*:
(LLL): $\mathbf{P}_x \left(\zeta < \infty \text{ and } X_{\zeta-} \text{ exists with value in } E_\Delta \right) = \mathbf{P}_x(\zeta < \infty) \quad \text{for } x \in E.$

We also consider the condition that
(LSC): Gf is lower semi-continuous on E for every Borel $f \geq 0$.

Theorem 2.4 *Assume that X is a Borel right process on E satisfying condition* **(LLL)** *and that X is transient. Suppose that one of the following conditions* **(a)** *and* **(b)** *holds*:

(a) *X satisfies the condition* **(LSC)**,
(b) *X is m-symmetric and its associated Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(E; m)$ is regular.*

Then

$$\mathbf{P}_x(X_{\zeta-} \in E_\Delta) = 1 \quad \text{for q.e. } x \in E, \tag{2.6}$$

and more specifically, the path wanders out to infinity whenever its life time is infinite:

$$\mathbf{P}_x(\zeta = \infty \textit{ and } X_{\zeta-} = \Delta) = \mathbf{P}_x(\zeta = \infty) \quad \text{for q.e. } x \in E. \tag{2.7}$$

Proof Assume first that the condition **(a)** holds. Take a strictly positive continuous function $f \in L^1(E; m)$ and define

$$N = \{x : \ Gf(x) = \infty\}.$$

Since X is transient, N is m-polar. Taking a Borel m-inessential set $N_1 \supset N$ and applying [15, Corollary 2.3] (where the condition **(LSC)** is used) to the restriction $X\big|_{E\setminus N_1}$ of X to $E \setminus N_1$, we have for every compact subset $K \subset E$,

$$L_K := \sup\{t > 0 : X_t \in K\} < \infty \quad \mathbf{P}_x\text{-a.s. for every } x \in E \setminus N_1, \tag{2.8}$$

As Δ is a one-point compactification for the locally compact separable metric space E, we conclude from above that (2.7) holds.

Now assume that the condition **(b)** holds. Let $f > 0$ with $f \in L^1(E; m)$. By the transience assumption on X, there is a Borel m-inessential set N_0 such that

$$Gf(x) > 0 \quad \text{for } x \in E \quad \text{and} \quad Gf(x) < \infty \quad \text{for } x \in E \setminus N_0,$$

For a Borel set B, define $p_B(x) := \mathbf{P}_x(\sigma_B < \infty)$. Since $G_{1/k}(f \wedge k)$ is quasi continuous and increases to Gf as $k \to \infty$, Gf is quasi lower semicontinuous: there exists a decreasing sequence of open sets $\{A_n, n \geq 1\}$ such that $A_n \supset N_0$, $\text{Cap}_{(0)}(A_n) \to 0$ as $n \to \infty$, and the restriction of Gf to each $E \setminus A_n$ is lower semicontinuous.

p_{A_n} is a quasi continuous version of the 0-order equilibrium potential of A_n (cf. [13, Theorem 4.3.3]) and $\text{Cap}_{(0)}(A_n) = \mathcal{E}(p_{A_n}, p_{A_n}) \to 0$. It follows that

$$\lim_{n\to\infty} p_{A_n}(x) = 0, \quad \text{q.e.}$$

Thus there is a Borel m-inessential set N_1 containing N_0 such that

$$\mathbf{P}_x\left(\bigcup_{n=1}^{\infty}\{\sigma_{A_n} = \infty\}\right) = 1 \quad \text{for every } x \in E \setminus N_1. \tag{2.9}$$

Take an increasing sequence of compact sets $\{K_n, n \geq 1\}$ that increases to E, and let $F_n = K_n \setminus A_n = K_n \cap (E \setminus A_n)$. Since F_n is compact and Gf is lower semicontinuous on F_n, $c_n := \inf_{x\in F_n} Gf(x) > 0$ and so

$$c_n\, p_{F_n}(x) \leq \mathbf{E}_x\left[(Gf)(X\sigma_{F_n})\right] \leq Gf(x) < \infty \quad \text{for } x \in E \setminus N_1.$$

Therefore for $x \in E \setminus N_1$,

$$P_t p_{F_n}(x) \leq c_n^{-1}\, P_t Gf(x) \downarrow 0 \quad \text{as } t \uparrow \infty$$

and consequently,

$$\mathbf{P}_x\left(\cap_{j=1}^{\infty} \Lambda_\ell\right) = \lim_{j\to\infty} \mathbf{P}_x\left(\Lambda_j\right) = 0,$$

where $\Lambda_j := \{\sigma_{F_n} \circ \theta_j < \infty\} \cap \{j < \zeta\}$. Since $\bigcap_{j=1}^{\infty} \Lambda_j = \bigcap_{j=1}^{\infty} \{\sigma_{F_n} \circ \theta_j < \infty\} \cap \{\zeta = \infty\}$, we get

$$\mathbf{P}_x \left(\cup_{j=1}^{\infty} \{\sigma_{F_n} \circ \theta_j = \infty\} \cap \{\zeta = \infty\} \right) = \mathbf{P}_x(\zeta = \infty) \quad \text{for } x \in E \setminus N_1.$$

This holds for every $n \geq 1$ and so for $x \in E \setminus N_1$,

$$\mathbf{P}_x \big(\zeta = \infty \text{ and for every } n \geq 1 \text{ there exists } j \geq 1 \\ \text{so that } X(j, \infty) \subset E \setminus F_n \big) = \mathbf{P}_x(\zeta = \infty).$$

By (2.9), we can replace F_n with K_n in the above to get the desired identity (2.7).

□

The next theorem will be utilized in Sect. 3.

Theorem 2.5 *Let X be an m-irreducible Borel right process on E satisfying condition* **(LLL)** *of this section. Assume that one of the conditions* **(a)** *and* **(b)** *in Theorem 2.4 holds when X is transient.*

Let K be a closed subset of E such that K is not m-polar and X admits no jump from $E_0 = E \setminus K$ to K in the sense that

$$\mathbf{P}_x\, (X_{t-} \in E_0 \text{ and } X_t \in K \text{ for some } t > 0) = 0 \quad \text{for q.e. } x \in E. \tag{2.10}$$

Assume further that there exists a neighborhood U_0 of K such that X admits no killing inside $U_0 \setminus K$ in the sense that

$$\mathbf{P}_x \left(X_{\zeta-} \in U_0 \setminus K,\ \zeta < \infty \right) = 0 \quad \text{for q.e. } x \in E. \tag{2.11}$$

Consider the part process $X^0 = \{X_t^0, \zeta^0, P_x^0\}$ of X killed upon leaving E_0. Then

$$\mathbf{P}_x^0 \left(X_{\zeta^0-}^0 \in K \cup (E_\Delta \setminus U_0) \right) = 1, \quad \text{q.e. } x \in E_0. \tag{2.12}$$

Proof By the assumption (2.10),

$$\mathbf{P}_x \left(X_{\sigma_K -} \in K \text{ and } \sigma_K < \infty \right) = \mathbf{P}_x(\sigma_K < \infty) \quad \text{for q.e } x \in E_0. \tag{2.13}$$

By Lemma 2.1, X is either recurrent or transient. When X is transient, (2.12) follows from Theorem 2.4, (2.11) and (2.13). When X is recurrent, by Theorem 2.3 and (2.13) above, we have

$$\mathbf{P}_x^0 \left(X_{\zeta^0-}^0 \in K \text{ and } \zeta^0 < \infty \right) = 1 \quad \text{for q.e. } x \in E_0,$$

which is a stronger version than (2.12). □

Any Hunt process is known to satisfy the condition **(LLL)** of this section. Next lemma gives a sufficient condition for a more general process to satisfy **(LLL)**.

Lemma 2.6 *Suppose X is a special standard process on E satisfying condition* **(LSC)** *of this section. Then X satisfies* **(LLL)** *and accordingly,* (2.6) *and* (2.7) *as well.*

Proof [7, Lemma 2.11] and its proof tell us that the special standard process X is a Hunt process under the Ray topology and

$$\{\zeta < \infty \text{ and } X_{\zeta-} \text{ does not exist in } E\} \subset \left\{\zeta < \infty \text{ and } X^r_{\zeta-} = \Delta\right\}$$

almost surely, where $X^r_{\zeta-}$ denotes the left hand side limit of X under the Ray topology. Furthermore there exist a sequence of stopping times $\{T_n\}$ such that $T_n < \zeta_p,\ T_n \uparrow \zeta_p$, where

$$\zeta_p := \begin{cases} \zeta & \text{if } \zeta < \infty \text{ and } X^r_{\zeta-} = \Delta, \\ \infty & \text{otherwise.} \end{cases}$$

For every compact set $K \subset E$, $c_K := \inf_{x\in K} Gf(x)$ is positive by the condition **(LSC)** and so

$$c_K\, \mathbf{P}_x(\sigma_K < \infty) \le Gf(x) < \infty, \quad x \in E \setminus N_1,$$

which in turn implies for $x \in E \setminus N_1$ that

$$\begin{aligned} c_K\, \mathbf{P}_x\left(\sigma_K \circ \theta_{T_n} < \infty\right) &= c_K\, \mathbf{E}_x\left[\mathbf{P}_{X_{T_n}}(\sigma_K < \infty)\right] \\ &\le \mathbf{E}_x\left[\int_{T_n}^{\infty} f(X_t)dt\right] \downarrow 0 \quad \text{as } n \to \infty. \end{aligned}$$

In other words, we have for $x \in E \setminus N_1$,

$$\mathbf{P}_x\left(\sigma_K \circ \theta_{T_n} = \infty \text{ for some } n\right) = 1.$$

We consider an increasing sequence of relative compact open sets $\{D_k, k \ge 1\}$ such that $\overline{D}_k \subset D_{k+1}$ and $\bigcup_{k\ge1} D_k = E$. Taking $K = \overline{D}_k$, we see that for every $x \in E\setminus N_1$,

$$\mathbf{P}_x\left(\bigcap_{k=1}^{\infty} \left\{\sigma_{D_k} \circ \theta_{T_n} = \infty \text{ for some } n\right\}\right) = 1.$$

It follows then for $x \in E \setminus N_1$, $\mathbf{P}_x$-a.s.,

$$\left\{\zeta < \infty,\ X^r_{\zeta-} = \Delta\right\} \subset \left\{\zeta < \infty,\ X_{\zeta-} = \Delta\right\}$$

and consequently X enjoys the property **(LLL)**. □

3 One-point extensions of X^0 and $\widehat{X}^0$ by darning a hole K

Let E, E_Δ be as in the previous section and m be a σ-finite Borel measure on E. Throughout this section, we shall fix a pair of Borel standard processes $X = (X_t, \zeta, \mathbf{P}_x)$ and $\widehat{X} = (\widehat{X}_t, \widehat{\zeta}, \widehat{\mathbf{P}}_x)$ on E which are in weak duality with respect to m:

$$\int_E \widehat{G}_\alpha f(x) g(x) m(dx) = \int_E f(x) G_\alpha g(x) m(dx), \quad \alpha > 0, \quad f, g \in \mathcal{B}^+(E),$$

where $\{G_\alpha,\ \alpha > 0\}$ and $\{\widehat{G}_\alpha,\ \alpha > 0\}$ denote the resolvent of X and $\widehat{X}$, respectively. The notion of m-polarity for process X is the same as that for $\widehat{X}$ and there will be no ambiguity in using the term q.e.

Let K be a closed subset of E and define $E_0 = E \setminus K$. We shall assume that either K is compact or $\overline{E_0}$ is compact. Denote by m_0 the restriction of m to E_0.

We shall assume once and for all that X satisfies the following conditions **(B.1), (B.2), (B.3)** and that $\widehat{X}$ satisfies the corresponding conditions **$(\widehat{\mathbf{B}}.1)$, $(\widehat{\mathbf{B}}.2)$, $(\widehat{\mathbf{B}}.3)$**.

(B.1) X is m-irreducible, X satisfies condition **(LLL)** of Sect. 2 and X admits no killings inside $U_0 \setminus K$ for some open neighborhood U_0 of K in E.

(B.2) $m_0(U \cap E_0)$ is finite for some neighborhood U of K, and the set K is non-m-polar with respect to X.

(B.3) X admits no jumps from E_0 to K.

We shall be also concerned with the next two conditions **(B.4)** and **(B.5)** on X and the corresponding ones **$(\widehat{\mathbf{B}}.4)$, $(\widehat{\mathbf{B}}.5)$** on $\widehat{X}$.

(B.4) Every semipolar set is m-polar for X.

(B.5) X satisfies one of the conditions **(a), (b)** in Theorem 2.4 whenever X is transient.

We consider the subprocesses $X^0 = (X_t^0, \zeta^0, \mathbf{P}_x)$ and $\widehat{X}^0 = (\widehat{X}_t^0, \widehat{\zeta}^0, \widehat{\mathbf{P}}_x)$ of X and $\widehat{X}$ killed upon leaving E_0, respectively. The subprocesses X^0 and $\widehat{X}^0$ are in weak duality with respect to m_0 (cf. [7]). The resolvents of X^0 and $\widehat{X}^0$ are denoted by $\{G_\alpha^0, \alpha > 0\}$ and $\{\widehat{G}_\alpha^0, \alpha > 0\}$, respectively.

For $\alpha > 0$, define the functions $\varphi,\ u_\alpha$ on E by

$$\varphi(x) = \mathbf{P}_x(\sigma_K < \infty) \quad \text{and} \quad u_\alpha(x) = \mathbf{E}_x\left[e^{-\alpha\sigma_K}\right] \quad \text{for } x \in E.$$

The corresponding functions for $\widehat{X}$ will be denoted by $\widehat{\varphi},\ \widehat{u}_\alpha$. By **(B.1)**, **(B.2)**, **$(\widehat{\mathbf{B}}.1)$**, **$(\widehat{\mathbf{B}}.2)$** and Lemma 2.2(i), we get

$$\varphi(x) > 0 \quad \text{and} \quad \widehat{\varphi}(x) > 0 \quad \text{for q.e. } x \in E_0. \tag{3.1}$$

Moreover, by **(B.3), $(\widehat{\mathbf{B}}.3)$** and [1, p. 59], we have for $x \in E_0$,

$$\varphi(x) = \mathbf{P}_x^0\left(\zeta^0 < \infty \text{ and } X_{\zeta^0-}^0 \in K\right) \quad \text{and}$$
$$\widehat{\varphi}(x) = \widehat{\mathbf{P}}_x^0\left(\widehat{\zeta}^0 < \infty \text{ and } \widehat{X}_{\widehat{\zeta}^0-}^0 \in K\right), \tag{3.2}$$
$$u_\alpha(x) = \mathbf{E}_x^0\left[e^{-\alpha\zeta^0};\ X_{\zeta^0-}^0 \in K\right] \quad \text{and} \quad \widehat{u}_\alpha(x) = \widehat{\mathbf{E}}_x^0\left[e^{-\alpha\widehat{\zeta}^0};\ \widehat{X}_{\widehat{\zeta}^0-}^0 \in K\right]. \tag{3.3}$$

and also, by **(B.1), ($\widehat{\mathbf{B}}$.1)**.

$$\begin{aligned} &\mathbf{P}_x^0\left(\zeta^0 < \infty,\ X^0_{\zeta^0-} \in K \cup (E_\Delta \setminus U_0)\right) = \mathbf{P}_x^0(\zeta^0 < \infty) \\ &\widehat{\mathbf{P}}_x^0\left(\widehat{\zeta}^0 < \infty,\ \widehat{X}^0_{\widehat{\zeta}^0-} \in K \cup (E_\Delta \setminus U_0)\right) = \mathbf{P}_x^0(\widehat{\zeta}^0 < \infty). \end{aligned} \tag{3.4}$$

Here for a Borel set $B \subset E_\Delta$, the notation "$X^0_{\zeta^0-} \in B$" means that the left limit of $t \mapsto X^0_t$ at $t = \zeta^0$ exists under the topology of E_Δ and takes values in B.

We consider several conditions on X^0 corresponding to **(A.3), (A.4), (A.6)** and **(A.5)** in [8, Sect. 5]:

(C.1) u_α is m_0-integrable on E_0 for every $\alpha > 0$.
(C.2) $G^0_{0+}f$ is lower semicontinuous on E_0 for any non-negative Borel function f on E_0.
$\varphi(x) > 0$ for any $x \in U \cap E_0$ for some neighborhood U of K.
(C.3) Either $E \setminus U$ is compact for any open neighborhood U of K in E, or for any open neighborhood U_1 of K in E, there exists an open neighborhood U_2 of K in E such that

$$\overline{U}_2 \subset U_1 \quad \text{and} \quad J_0(U_2 \setminus K, E_0 \setminus U_1) < \infty.$$

Here J_0 denotes the jumping measures of X^0.
(C.4) $\lim_{x \to K} u_\alpha(x) = 1$ for every $\alpha > 0$.

Corresponding conditions on $\widehat{X}^0$ are designated by **($\widehat{\mathbf{C}}$.1), ($\widehat{\mathbf{C}}$.2), ($\widehat{\mathbf{C}}$.3), ($\widehat{\mathbf{C}}$.4).**

Let us extend the topological space E_0 to $E^* = E_0 \cup \{a\}$ by adding an extra point a to E_0 whose topology is prescribed as follows: a subset U of E^* containing the point a is an open neighborhood of a if there is an open set $U_1 \subset E$ containing K such that $U_1 \cap E_0 = U \setminus \{a\}$. In other words, E^* is obtained from E_0 by identifying K into the one point $\{a\}$. Notice that, in the special case that $\overline{E}_0$ is compact in E, $E^* = E \cup \{a\}$ is nothing but the one-point compactification of E_0.

The measure m_0 is extended from E_0 to E^* by setting $m_0(\{a\}) = 0$. For functions f, g on E_0, we denote by (f, g) the integral of $f \cdot g$ over E_0 against m_0. For an X^0-excessive measure μ and an X^0-excessive function f on E_0, $L^{(0)}(\mu, f)$ will denote the X^0-energy functional defined by using the transition function p_t^0 of X^0 as

$$L^{(0)}(\mu, f) = \lim_{t \downarrow 0} \frac{1}{t} \langle \mu - \mu P_t^0, f \rangle.$$

An analogous quantity $\widehat{L}^{(0)}$ can be defined for $\widehat{X}^0$.

A strong Markov process X^* on E^* is said to be a q.e. extension of X^0 if the subprocess of X^* killed upon leaving E_0 coincides with X^0 for q.e. starting points $x \in E_0$.

We now state a theorem consisting of three items (i), (ii), (iii) which can be regarded as generalized versions (in the darning context) of Theorems 5.15, 5.16 and 5.17 of [8], respectively.

Theorem 3.1 *Let us assume the conditions* **(B.1)**, **(B.2)**, **(B.3)** *on X and the corresponding conditions on $\widehat{X}$. We further assume for X^0 condition* **(C.2)** *as well as condition* **(C.3)** *when X^0 is not a diffusion and condition* **(C.4)** *when X^0 is not symmetric. Corresponding conditions on $\widehat{X}^0$ are also imposed.*

Then the following are true:

(i) *Suppose that the integrability conditions* **(C.1)** *and* **($\widehat{\mathbf{C}}$.1)** *are fulfilled. Then there exist right processes X^* and $\widehat{X}^*$ on E^* possessing the next properties*:

(i.1) *X^* and $\widehat{X}^*$ are* q.e. *extensions of X^0 and $\widehat{X}^0$ respectively.*

(i.2) *X^* and $\widehat{X}^*$ are in weak duality with respect to m_0. The resolvents G^*_α and $\widehat{G}^*_\alpha$ of X^* and $\widehat{X}^*$ admit the expressions*

$$\begin{aligned} G^*_\alpha f(a) &= \frac{(\widehat{u}_\alpha, f)}{\alpha(\widehat{u}_\alpha, \varphi) + L^{(0)}(\widehat{\varphi}\cdot m_0, 1-\varphi) + \delta_0}, \\ G^*_\alpha f(x) &= G^0_\alpha f(x) + u_\alpha(x) G^*_\alpha f(a), \quad \text{q.e. } x \in E_0, \end{aligned} \tag{3.5}$$

$$\begin{aligned} \widehat{G}^*_\alpha f(a) &= \frac{(u_\alpha, f)}{\alpha(u_\alpha, \widehat{\varphi}) + \widehat{L}^{(0)}(\varphi\cdot m_0, 1-\widehat{\varphi}) + \widehat{\delta}_0}, \\ \widehat{G}^*_\alpha f(x) &= \widehat{G}^0_\alpha f(x) + \widehat{u}_\alpha(x) \widehat{G}^*_\alpha f(a), \quad \text{q.e. } x \in E_0, \end{aligned} \tag{3.6}$$

respectively, where δ_0, $\widehat{\delta}_0$ are any preassigned non-negative numbers satisfying

$$L^{(0)}(\widehat{\varphi}\cdot m_0, 1-\varphi) + \delta_0 = \widehat{L}^{(0)}(\varphi\cdot m_0, 1-\widehat{\varphi}) + \widehat{\delta}_0.$$

(i.3) *The sample path of X^* is cadlag and enjoys the property* **(LLL)** *of Sect. 2. X^* is m_0-irreducible.*

(i.4) *$X^* = (X^*_t, \zeta^*, \mathbf{P}^*_x)$ admits no sojourn at a, namely,*

$$\mathbf{P}^*_x\left(\int_0^\infty 1_{\{a\}}(X^*_s)ds = 0\right) = 1 \quad \text{q.e. } x \in E^*.$$

(i.5) *The point a is regular for itself with respect to X^* and $\{a\}$ is not m_0-polar for X^*.*

(i.6) *X^* admits no jumps from E_0 to a nor from a to E_0, namely,*

$$\mathbf{P}^*_x\left(X^*_{t-} \in E_0 \textit{ and } X^*_t = a \ \text{ for some } t > 0\right) = 0 \quad \text{q.e. } x \in E^*$$

and

$$\mathbf{P}^*_x\left(X^*_{t-} = a \textit{ and } X^*_t \in E_0 \ \text{ for some } t > 0\right) = 0 \quad \text{q.e. } x \in E^*.$$

(i.7) *A subset of E_0 is m_0-polar with respect to X^* if and only if it is so with respect to X^0.*

(i.8) $\delta_0 = 0$ *if and only if* X^* *admits no killings at* a *in the sense that*

$$\mathbf{P}^*_x \left(\zeta^* < \infty, \quad X^*_{\zeta^*-} = a \right) = 0 \quad \text{q.e. } x \in E^*.$$

(i.9) *If* X^0 *is a diffusion, then so is* X^* *for* q.e. *starting points* $x \in E^*$.
The process $\widehat{X}^*$ *also enjoy the properties corresponding to* (i.3)–(i.9).

(ii) *In addition to the conditions* **(C.1)** *and* **($\widehat{\mathbf{C}}$.1)**, *we suppose the condition* **(B.4)** *on* X *holds. Then the processes* X^* *and* $\widehat{X}^*$ *in* (i) *are quasi-left continuous for* q.e. *starting points. Every semipolar set is* m_0*-polar for* X^*. *Furthermore, for a subset of* E_0, *its* m_0*-polarity with respect to* X^* *and* m*-polarity with respect to* X *are equivalent.*

(iii) *Suppose* X *and* $\widehat{X}$ *satisfy the conditions* **(B.5)** *and* **($\widehat{\mathbf{B}}$.5)**. *Then the integrability conditions* **(C.1)** *and* **($\widehat{\mathbf{C}}$.1)** *are fulfilled.*

Proof (i). Note that, in view of the definition of $E^* = E_0 \cup \{a\}$, the set K can be replaced by the one-point set $\{a\}$ in the identities (3.2), (3.3) and (3.4) and also in the conditions **(C.1) (C.2), (C.3)**. The set E_Δ appearing in (3.4) can also be replaced by E^*_Δ the one-point compactification of E^*.

Therefore, under the stated assumptions, X^0 and $\widehat{X}^0$ satisfy conditions **(A.1), (A.3), (A.4)** as well as **(A.5)** (in non diffusion case) and **(A.6)** (in non-symmetric case) of [8, Theorem 5.15]. They also satisfy condition **(A.2a)** of Sect. 6 the Appendix of the present paper—a weakened version of **(A.2)** of [8, Sect. 5] allowing X^0 and $\widehat{X}^0$ to have killings inside E_0, except that the property (3.1) corresponding to the first half of condition **(A.2a)** in Sect. 6 is valid only for q.e. starting point $x \in E_0$ rather than for every $x \in E_0$.

Suppose that (3.1) hold for every starting point $x \in E_0$. Then we can produce from the generalized version of [8, Theorem 5.15] (see the Appendix Sect. 6 below) those extensions X^* and $\widehat{X}^*$ of X^0 and $\widehat{X}^0$, respectively, on E^*, by constructing absorbed Poisson point processes taking values of excursions around the point a and by piecing together those excursions and the paths of X^0 and $\widehat{X}^0$. The resolvents of X^* and $\widehat{X}^*$ satisfy (3.5) and (3.7), respectively, with 'q.e. $x \in E_0$' being replaced by 'for every $x \in E_0$'.

Particularly, the entrance law $\{\mu_t, t > 0\}$ from a governing the characteristic measure $\mathbf{n}$ of the absorbed Poisson point process taking part in the construction of X^* is uniquely specified by the equation

$$\widehat{\varphi} \cdot m_0 = \int_0^\infty \mu_t \, dt. \tag{3.7}$$

In view of [8, Theorem 5.15], the process X^* thus constructed satisfies the first properties in (i.3) and (i.5) as well as the first one in (i.6), the equivalence statements (i.8) and statement (i.9) with 'q.e. $x \in E^*$' being replaced by 'every $x \in E^*$'. The second property in (i.5) and the second one in (i.6) for X^* follow from (iv) and (iii) of Proposition 4.1 in [8], respectively. Property (i.4) for X^* can be shown in a similar manner to the proof of Proposition 4.1(i) of [8].

The second property of (i.3), the m_0-irreducibility of X^*, follows from the assumption (3.1) holding for every $x \in E_0$ and (3.5) because, for any Borel set $B \subset E^*$ with a positive m_0-measure, we see that $G^*_\alpha 1_B(a) > 0$ and

$$G^*_\alpha 1_B(x) \geq u_\alpha(x) G^*_\alpha 1_B(a) > 0, \quad x \in E_0.$$

To prove (i.7) for X^*, it suffices to show its 'if' part. Let B be an m_0-polar nearly Borel subset of E_0 with respect to X^0. By virtue of (i.6), we have the inclusion

$$\{0 < \sigma_B < \infty\} \subset \bigcup_{r \in \mathbb{Q}^+} \Lambda_r, \quad \text{where } \Lambda_r = \{r < \sigma_B,\ \sigma_B \circ \theta_r < \sigma_a \circ \theta_r\}$$

holding $\mathbf{P}^*_x$-a.s. for q.e. $x \in E_0$. By the Markov property of X^*, we get

$$\mathbf{P}^*_{m_0}(\Lambda_r) = \mathbf{E}^*_{m_0}\left[\mathbf{P}^0_{X^*_r}(\sigma_B < \infty);\ X^*_r \in E_0\right] \leq \mathbf{P}^0_{m_0}(\sigma_B < \infty) = 0,$$

and consequently, $\mathbf{P}^*_{m_0}(\sigma_B < \infty) = 0$. So B is m_0-polar with respect to X^*.

In general, (3.1) holds only for every $x \in E_0 \backslash N$ for some m_0-polar set $N \subset E_0$ under the present assumptions. But we can then find a Borel set $B \subset E_0$ containing N that is properly exceptional for both X^0 and $\widehat{X}^0$ as in the proof of [13, Theorem 4.1.1]. Let us remark that B can be contained in $E_0 \backslash U$ for a neighborhood U of K appearing in the conditions **(C.2), ($\widehat{\mathbf{C}}$.2)**. Let $E_0' = E_0 \backslash B$. Then the restrictions of X^0 and $\widehat{X}^0$ to E_0' are standard again and $m(B) = 0$. (3.1) holds true for every $x \in E_0'$. Accordingly, we can apply the same arguments as above to E_0' and $E_0' \cup \{a\}$ instead of E_0 and E^* in obtaining all conclusions desired. Especially, due to **(C.2), ($\widehat{\mathbf{C}}$.2)** and the above remark, the validity of the key lemma [8, Lemma 5.4] is not violated by these restrictions of the state space.

(ii). The first conclusion follows from [8, Theorem 5.16 (ii)]. The second one is shown as follows. The one point set $\{a\}$ is not semipolar with respect to X^* by (i.5). Suppose a set $N \subset E_0$ is semipolar with respect to X^*. Then so it is semipolar with respect to X^0. Hence N is m_0-polar with respect to X^0 by assumption (**B.4**). Thus N is m_0-polar with respect to X^* by (i.7).

Finally let us show for a set $N \subset E_0$

$$N \text{ is } m\text{-polar for } X \iff N \text{ is } m_0\text{-polar for } X^0, \tag{3.8}$$

which combined with (i.7) implies the third assertion of (ii). Notice that, under the assumption of (**B.4**), $K \backslash K^r$ is $\mathbf{P}_m$-polar for X. On the other hand, both X and $\widehat{X}$ admit no jumps from E_0 to K $\mathbf{P}_m$-a.s. and $\widehat{\mathbf{P}}_m$-a.s. on account of **(B.3)** and **($\widehat{\mathbf{B}}$.3)**, respectively. Hence the same argument as in the proof of [8, Proposition 4.1] works in proving that X admits no jumps from K to E_0 $\mathbf{P}_m$-a.s. We can therefore repeat a similar argument as in the proof of (i.7) to obtain (3.8).

(iii) Under the stated condition, we can invoke Theorem 2.5 to conclude that X^0 has the property (2.12) and $\widehat{X}^0$ also has the corresponding one. In other words, X^0 and $\widehat{X}^0$ satisfy the condition **(A.2a)**$'$ of Sect. 6 below in the topology of E^* but holding only for q.e. $x \in E_0$ rather than every $x \in E_0$. **(A.2a)**$'$ is a weakened version of **(A.2)**$'$ of [8, Theorem 5.17] allowing for X^0 and $\widehat{X}^0$ to have killings inside E_0.

Moreover, condition **(B.2)** implies that $m_0(U \cap E_0)$ is finite for some neighborhood U of $\{a\}$ in E^*. If **(A.2a)**$'$ were true for every $x \in E_0$, then all conditions for the generalized version of [8, Theorem 5.17] (see Sect. 6) are fulfilled and we can deduce from it the integrability **(C.1)**. In general, we use the same reasoning as in the last part of the proof of (i) to get **(C.1)**. □

Remark 3.2 (i) The non-negative numbers δ_0 and $\widehat{\delta}_0$ in Theorem 3.1 are killing rates of X^* and $\widehat{X}^*$ at $\{a\}$, respectively, in the sense of [8, (3.6)].

(ii) Condition **(C.2)** can be replaced by

(C.2)$'$ $\varphi(x) > 0$ for any $x \in U \cap E_0$ for some neighborhood U of K and $\inf_{x \in C} G_1^0 \varphi(x) > 0$ for any compact set $C \subset U \cap E_0$.

In fact, **(C.2)** implies **(C.2)**$'$ and the latter is what we really need in the proof of [8, Lemma 5.4].

4 Symmetric one-point extensions of X^0 and their characterizations

In this section, we continue to work under the same assumption as in the preceding section for

$$E, \ K, \ E_0 = E \setminus K, \ m, \ m_0 = m|_{E_0},$$

but with an additional assumption that $X = \widehat{X}$. Let us first restate Theorem 3.1 under this assumption.

Assume that we are given an m-symmetric Borel standard process X on E satisfying the conditions **(B.1), (B.2), (B.3)**. We note that every semipolar set is m-polar by the symmetry of X. Let $X^0 = (X_t^0, \zeta^0, \mathbf{P}_x^0)$ be the subprocess of X killed upon leaving E_0. The process X^0 is m_0-symmetric.

We shall also assume once and for all that X^0 satisfies **(C.2)** as well as **(C.3)** of Sect. 3 in non-diffusion case.

For $x \in E_0$ and $\alpha > 0$, define

$$\varphi(x) = \mathbf{P}_x^0 \left(\zeta^0 < \infty \text{ and } X_{\zeta^0-}^0 \in K \right) \quad \text{and} \quad u_\alpha(x) = \mathbf{E}_x^0 \left[e^{-\alpha \zeta^0}; X_{\zeta^0-}^0 \in K \right].$$

It follows from **(B.1)–(B.3)**, [1, p59] and Lemma 2.2(i) that

$$\varphi(x) > 0 \quad \text{for q.e. } x \in E_0. \tag{4.1}$$

Let $E^* = E_0 \cup \{a\}$ be the one-point extension of E_0, by regarding the set K as a one point $\{a\}$ as was done in the preceding section. We extend measure m_0 to E^* by defining $m_0(\{a\}) = 0$. We now restate Theorem 3.1 in the case that $X = \widehat{X}$ and $\delta_0 = \widehat{\delta}_0 = 0$. The $\delta_0 = \widehat{\delta}_0 > 0$ case can be dealt with similarly and we leave the details to the interested readers.

Theorem 4.1 *Under the above stated conditions, the following are true.*

(i) *Suppose that the condition* **(C.1)** *is fulfilled. Then there exists a unique m_0-symmetric right process X^* on E^* possessing the next properties*:

(i.1) X^* *is a* q.e. *extension of* X^0.

(i.2) *The resolvent* G^*_α *of* X^* *admits the expression*

$$G^*_\alpha f(a) = \frac{(u_\alpha, f)}{\alpha(u_\alpha, \varphi) + L^{(0)}(\varphi \cdot m_0, 1 - \varphi)},$$
$$G^*_\alpha f(x) = G^0_\alpha f(x) + u_\alpha(x) G^*_\alpha f(a) \quad \text{for q.e. } x \in E_0, \tag{4.2}$$

where G^0_α *is the resolvent of* X^0 *and* (f, g) *denotes the integral* $\int_{E_0} f g dm_0$.

(i.3) *The sample path of* X^* *is cadlag and enjoys the property* **(LLL)** *of Sect. 2.* X^* *is* m_0*-irreducible.*

(i.4) $X^* = (X^*_t, \zeta^*, \mathbf{P}^*_x)$ *admits no sojourn at a, namely,*

$$\mathbf{P}^*_x\left(\int_0^\infty 1_{\{a\}}(X^*_s) ds = 0\right) = 1 \quad \textit{for q.e. } x \in E^*.$$

(i.5) *The point a is regular for itself with respect to* X^* *and* $\{a\}$ *is not* m_0*-polar for* X^*.

(i.6) X^* *admits jumps neither from* E_0 *to a nor from a to* E_0*, namely,*

$$\mathbf{P}^*_x\left(X^*_{t-} \in E_0,\ X^*_t = a,\ \text{for some } t > 0\right) = 0 \quad \textit{for q.e. } x \in E^*.$$

and

$$\mathbf{P}^*_x\left(X^*_{t-} = a,\ X^*_t \in E_0,\ \text{for some } t > 0\right) = 0 \quad \textit{for q.e. } x \in E^*.$$

(i.7) X^* *admits no killings at a in the sense that*

$$\mathbf{P}^*_x\left(\zeta^* < \infty, \quad X^*_{\zeta^*-} = a\right) = 0 \quad \textit{for q.e. } x \in E^*.$$

(i.8) X^* *is quasi-left continuous for q.e. starting point* $x \in E^*$.

(i.9) *If* X^0 *is a diffusion, then so is* X^* *for* q.e. *starting points* $x \in E^*$.

(i.10) *The entrance law* $\{\mu_t, t > 0\}$ *from a governing the characteristic measure of the absorbed Poisson point process taking part in the construction of* X^* *is uniquely specified by the equation*

$$\varphi \cdot m_0 = \int_0^\infty \mu_t \, dt. \tag{4.3}$$

(ii) *Suppose X satisfies the condition* **(B.5)**. *Then the integrability condition* **(C.1)** *is fulfilled.*

In what follows, we shall assume **(B.1), (B.2), (B.3), (B.5)** of Sect. 2 for X and **(C.2)** of Sect. 3 for X^0 as well as **(C.3)** in non-diffusion case so that u_α is m_0-integrable and X^0 admits a unique extension X^* to E^* as is described in (i).

For later convenience, we call $\{\mu_t, t > 0\}$ specified by the equation (4.3) the *entrance law from a for* X^*. We also note the following: by the quasi-homeomorphism method due to [4] and by [13, Sect. 4.4], we may and do assume that the Dirichlet form of X (resp. X^0) on $L^2(E; m)$ (resp. $L^2(E_0; m_0)$) is regular with X (resp. X^0) being an associated Hunt process on E (resp. E_0).

Let $(\mathcal{E}^0, \mathcal{F}^0)$ and $(\mathcal{E}^*, \mathcal{F}^*)$ be the Dirichlet forms on $L^2(E_0; m_0)$ of the m_0-symmetric standard processes X^0 and X^*, respectively. The Dirichlet form $(\mathcal{E}^0, \mathcal{F}^0)$ is transient in view of (4.1) and [13, Lemma 1.6.4]. We aim at characterizing $(\mathcal{E}^*, \mathcal{F}^*)$ in terms of the active reflected Dirichlet space $((\mathcal{F}^0)^{\rm ref}_a, \mathcal{E}^{\rm ref})$ of $(\mathcal{F}^0, \mathcal{E}^0)$. The notion of the reflected Dirichlet space of a transient regular Dirichlet space $\mathcal{F}^0$ was first introduced by M.L. Silverstein in [25,26] in two different ways, which were later on made precise and shown to be equivalent in [3] by the first author of the present paper. The first way is to add to $\mathcal{F}^0$ *the space of all harmonic functions on* E_0 *with finite Dirichlet integrals* by using the equilibrium measures ([26] and [3]) or the energy functional ([6]), while the second way is to consider *the space of all functions on* E_0 *with finite Dirichlet integrals* by using the energy measures of $u \in \mathcal{F}_{\rm loc}$ ([3,25]).

Here we adopt the definition in [6, Sect. 3] where X^0 was assumed to be of no killing inside E_0. We shall remove this condition and show that the current definition coincides with the one given in [3].

For convenience, we introduce the following notions related to the standard process X^0 on E_0. A nearly Borel set $A \subset E_0$ is called X^0-invariant if $\mathbf{P}_x(\Omega_A) = 1$ for every $x \in A$, where

$$\Omega_A = \left\{\omega \in \Omega : X^0_t(\omega) \in A \text{ and } X^0_{t-}(\omega) \in A \ \text{ for every } t \in [0, \zeta^0)\right\}. \tag{4.4}$$

Then the restriction $X^0|_A$ defined in a natural way is a standard process on A. We say a random variable Φ on Ω is $X^0|_A$-measurable if the restriction $\Phi|_{\Omega_A}$ is measurable with respect to the σ-field $\mathcal{M}^0 \cap A$, where $\mathcal{M}^0$ denotes the natural σ-field generated by $\{X^0_t, t \geq 0\}$. The random variable Φ needs not to be defined on $\Omega \setminus \Omega_A$ in this case. Recall that a nearly Borel set $N \subset E_0$ is called X^0-properly exceptional if $E_0 \setminus N$ is X^0-invariant and $m(N) = 0$.

We call a random variable $\Phi = \Phi(\omega)$ on Ω a *terminal variable* if there exists an X^0-properly exceptional set $N \subset E_0$ such that

(i) Φ is $X^0|_{E_0 \setminus N}$-measurable,
(ii) $\Phi(\theta_t(\omega)) = \Phi(\omega)$ for every $\omega \in \Omega_{E_0 \setminus N}$ and for every $t < \zeta^0(\omega)$, and
(iii) $\{\Phi \neq 0\} \subset \{\zeta^0 < \infty \text{ and } X^0_{\zeta^0-} = \partial\}$, where θ_t is the shift operator on Ω and ∂ is the point at infinity of E_0.

By convention, we let $X^0_\infty = \partial$ and a function f on E_0 is extended to $E_0 \cup \{\partial\}$ by setting $f(\partial) = 0$. A function f on E_0 is called X^0-*harmonic* if, for any relatively compact open subset D of E_0,

$$\mathbf{E}_x^0\left[\left|f\left(X^0_{\sigma_{E_0\setminus D}}\right)\right|\right] < \infty \quad \text{and} \quad f(x) = \mathbf{E}_x^0\left[f\left(X^0_{\sigma_{E_0\setminus D}}\right)\right] \quad \text{for q.e. } x \in E_0.$$

Denote by $(\mathcal{F}_e^0, \mathcal{E}^0)$ the extended Dirichlet space of $(\mathcal{E}^0, \mathcal{F}^0)$.

Lemma 4.2 (i) *Let Φ be a terminal variable with* $\mathbf{E}_x^0[|\Phi|] < \infty$ for q.e. $x \in E_0$. *Put*

$$h(x) = \mathbf{E}_x^0[\Phi], \quad x \in E_0. \tag{4.5}$$

Then h is X^0-harmonic. Furthermore, for any relatively compact open subsets D_k of E_0 increasing to E_0,

$$\lim_{k\to\infty} h(X^0_{\sigma_k}) = \Phi \quad \mathbf{P}_x^0\text{-a.s. and in } L^1(\mathbf{P}_x^0) \quad \text{for q.e. } x \in E_0,$$

where σ_k denotes the hitting time of $E_0 \setminus D_k$ for X^0.

(ii) *For any $f \in \mathcal{F}_e^0$,*

$$\lim_{k\to\infty} f(X^0_{\sigma_k}) = 0 \quad \text{in } L^1(\mathbf{P}_x^0) \text{ and in probability } (\mathbf{P}_x^0) \text{ for q.e. } x \in E_0,$$

where $\{\sigma_k, k \geq 1\}$ is as in (i).

Proof (i) is contained in [3, Lemma 1.5]. To prove (ii), we may assume that $f \in \mathcal{F}_e^0$ is non-negative. Put

$$H^k f(x) = \mathbf{E}_x^0\left[f(X_{\sigma_k})\right] = \mathbf{E}_x^0\left[f(X_{\sigma_k}); \sigma_k < \infty\right], \quad x \in E_0,$$

which is an $\mathcal{E}^0$-quasi-continuous function in $\mathcal{F}_e^0$ and $\mathcal{E}^0$-orthogonal to the space

$$\mathcal{F}_{e,k}^0 = \{g \in \mathcal{F}_e^0 : g = 0 \quad \text{q.e. on } E_0 \setminus D_k\},$$

and accordingly, $\mathcal{E}^0(H^j f, H^k f) = \mathcal{E}^0(H^j f, H^j f)$, $k < j$ (cf. [13, Sect. 4.3]). Hence

$$\mathcal{E}^0(H^k f - H^j f, H^k f - H^j f) = \mathcal{E}^0(H^k f, H^k f) - \mathcal{E}^0(H^j f, H^j f), \quad k < j,$$

which implies that $\{H^k f\}$ is $\mathcal{E}^0$-convergent to a function $u \in \mathcal{F}_e^0$. Since u is orthogonal to $\mathcal{F}_{e,k}^0$ for all k, we conclude that $u = 0$ from the regularity of $(\mathcal{F}^0, \mathcal{E}^0)$. In particular, we get $\lim_{k\to\infty} H^k f(x) = 0$, q.e. on E_0 the desired conclusion. □

We denote by p_t^0 the transition function of X^0.

Lemma 4.3 *Let Φ be a terminal variable with $\mathbf{E}_x^0\left[\Phi^2\right] < \infty$ for* q.e. $x \in E_0$. *Let $h(x)$ be the function defined by* (4.5) *and define*

$$g(x) = \mathbf{E}_x^0\left[\Phi^2\right] - h(x)^2, \quad x \in E_0, \tag{4.6}$$

$$Mh(t) = h(X_t^0)\mathbf{1}_{\{t<\zeta^0\}} + \Phi\mathbf{1}_{\{t\geq\zeta^0\}} - h(X_0^0) \quad t \geq 0. \tag{4.7}$$

Then g is X^0-excessive, $\{Mh(t)\}_{t\geq 0}$ is a $\mathbf{P}_x^0$-square integrable, uniformly integrable martingale additive functional of X^0 and

$$g(x) = p_t^0 g(x) + \mathbf{E}_x^0\left[(Mh(t))^2\right], \quad t \geq 0, \quad \text{q.e. } x \in E_0. \tag{4.8}$$

In particular

$$\frac{1}{2}L^{(0)}(m_0, g) = e(Mh)(\leq \infty), \tag{4.9}$$

where $L^{(0)}$ denotes the energy functional of an X^0-excessive measure and an X^0-excessive function defined before the statement of Theorem 3.1 *and e denotes the energy of an additive functional defined in* [13, Sect. 5.2].

Proof Since $\mathbf{E}_x^0\left[\Phi^2\right]$ is X^0-excessive and finite q.e., we see from $h(x)^2 \leq \mathbf{E}_x^0[\Phi^2]$ that for q.e. $x \in E_0$, $|p_t^0 h(x)|^2 \leq p_t^0 h^2(x) < \infty$, $\mathbf{E}_x^0(Mh(t)) = 0$, and

$$\begin{aligned}\mathbf{E}_x^0\left[(Mh(t))^2\right] &= p_t^0 h^2(x) + \mathbf{E}_x^0\left[\Phi^2 I_{\{t\geq\zeta^0\}}\right] + h(x)^2 \\ &\quad -2h(x)p_t^0 h(x) - 2h(x)\mathbf{E}_x^0\left[\Phi I_{\{t\geq\zeta^0\}}\right] \\ &= g(x) - p_t^0 g(x).\end{aligned}$$

□

Let

$$\mathbf{N} = \{\Phi : \ \Phi \text{ is a terminal variable with } \mathbf{E}_x^0(\Phi^2) < \infty \text{ q.e. } x \in E_0,\ L^{(0)}(m_0, g) < \infty\}, \tag{4.10}$$

where g is defined by (4.6) for Φ. The reflected Dirichlet space $((\mathcal{F}^0)^{\text{ref}}, \mathcal{E}^{\text{ref}})$ and the active reflected Dirichlet space $(\mathcal{F}^0)_a^{\text{ref}}$ of $(\mathcal{F}^0, \mathcal{E}^0)$ are then defined as follows:

$$(\mathcal{F}^0)^{\text{ref}} = \mathcal{F}_e^0 + \mathbf{HN}, \quad (\mathcal{F}^0)_a^{\text{ref}} = (\mathcal{F}^0)^{\text{ref}} \cap L^2(E_0; m_0), \tag{4.11}$$

where

$$\mathbf{HN} = \{h : \ h(x) = \mathbf{E}_x[\Phi] \ \text{ for q.e. } x \in E_0 \text{ with } \Phi \in \mathbf{N}\}. \tag{4.12}$$

For $f = f_0 + h \in (\mathcal{F}^0)^{\rm ref}$, where $f_0 \in \mathcal{F}^0_e$ and $h = \mathbf{E}_\cdot[\Phi]$ with $\Phi \in \mathbf{N}$, we let

$$\mathcal{E}^{\rm ref}(f, f) = \mathcal{E}(f_0, f_0) + \tfrac{1}{2}L^{(0)}(m_0, g), \tag{4.13}$$

for g defined by (4.6) for Φ.

On account of Lemma 4.3 and [3, Theorem 1.8], it is clear that the above definition of the reflected Dirichlet space coincides with the one given in [3]. We also notice that the space $\mathcal{F}^0_e \cap \mathbf{HN}$ consists only of zero function because of Lemma 4.2.

The following is one of the main theorems of this section.

Theorem 4.4 *Under the conditions of this section, we have*

(i) $\mathcal{F}^*_e\big|_{E_0} \subset (\mathcal{F}^0)^{\rm ref}$, $\mathcal{F}^*\big|_{E_0} \subset (\mathcal{F}^0)^{\rm ref}_a$ *and* $\mathcal{E}^*(u, v) = \mathcal{E}^{\rm ref}(u|_{E_0}, v|_{E_0})$ *for* $u, v \in \mathcal{F}^*_e$.

(ii) $\varphi \in (\mathcal{F}^0)^{\rm ref}$ *and* $u_\alpha \in (\mathcal{F}^0)^{\rm ref}_a$ *for every* $\alpha > 0$.

(iii) *For* $\alpha > 0$, *consider the one-dimensional subspace*

$$\mathcal{H}_\alpha = \{c\, u_\alpha \, : \, c \in \mathbb{R}\}$$

of $(\mathcal{F}^0)^{\rm ref}_a$. *The space* $\left(\mathcal{F}^*, \mathcal{E}^*_\alpha\right)$ *can be decomposed in the Hilbert space* $\left((\mathcal{F}^0)^{\rm ref}_a, \mathcal{E}^{\rm ref}_\alpha\right)$ *as the direction sum:*

$$\mathcal{F}^*\big|_{E_0} = \mathcal{F}^0 \oplus \mathcal{H}_\alpha.$$

(iv) *It holds that*

$$\mathcal{F}^*_e\big|_{E_0} = \left\{ f = f_0 + c\varphi \, : \, f_0 \in \mathcal{F}^0_e, \quad c \in \mathbb{R} \right\},$$
$$\mathcal{E}^*(f, f) = \mathcal{E}^{\rm ref}(f, f) = \mathcal{E}^0(f_0, f_0) + c^2 V \quad \text{for } f = f_0 + c\varphi \text{ with } f_0 \in \mathcal{F}^0_e, \tag{4.14}$$

where

$$V = L^{(0)}(\varphi \cdot m_0, 1 - \varphi). \tag{4.15}$$

Proof Let $\{D_k, k \geq 1\}$ be an increasing sequence of relatively compact open subset of E_0. On account of (5) of Theorem 4.1, we then have

$$\lim_{k\to\infty} X_{\sigma_{E\setminus D_k}} = X_{\sigma_K} \quad \text{on } \{\sigma_K < \infty\}.$$

Now the same proof for [6, Theorem 3.4 (i)] establishes the first half of the assertion (i).

To prove the second half of (i), let $(N(x, dy), H)$ be a Lévy system of X^* and μ_H be the Revuz measure of the positive continuous additive functional H with respect to m_0. We can then deduce from (i.5) to (i.7) of in Theorem 4.1 for X^* that

$$\mu_H(\{a\})N(a, E_0 \cup \{a\}) = 0.$$

On the other hand, if we denote by $\mu_{\langle u\rangle} = \mu^c_{\langle u\rangle} + \mu^j_{\langle u\rangle} + \mu^k_{\langle u\rangle}$ the energy measure of $u \in \mathcal{F}^*_e$, then

$$\mu^j_{\langle u\rangle}(\{a\}) = \mu_H(\{a\}) \int_{E_0} ((u(a) - u(y))^2 N(a, dy) \quad \text{and}$$
$$\mu^k_{\langle u\rangle}(\{a\}) = \mu_H(\{a\}) u(a)^2 N(a, \{\Delta\}).$$

Since $\{a\}$ is just a one point set, it follows from [6, Theorem 2.10] by taking $F = \{a\}$ as the trace space that $\mu^c_{\langle u\rangle}(\{a\}) = 0$. Therefore we have $\mu_{\langle u\rangle}(\{a\}) = 0$ for every $u \in \mathcal{F}^*_e$. The second half of (i) now follows from [6, Theorem 3.4 (iii)].

Let σ_a be the hitting time of the point a by the process X^*. Since X^* is an extension of X^0 from E_0 to $E^* = E_0 \cup \{a\}$, we can conclude from (i.6) of Theorem 4.1 and [1, p. 59] that

$$\varphi(x) = \mathbf{P}^*_x(\sigma_a < \infty) \quad \text{and} \quad u_\alpha(x) = \mathbf{E}^*_x\left[e^{-\alpha\sigma_a}\right] \quad \text{for q.e } x \in E_0. \tag{4.16}$$

Moreover, in view of (i.5) of Theorem 4.1, the above identities hold for $x = a$ if we extend the functions φ and u_α to E^* by setting

$$\varphi(a) = 1 = u_\alpha(a). \tag{4.17}$$

Note that the functions φ and u_α are members of $\mathcal{F}^*_e$ and $\mathcal{F}^*$, respectively. In fact, by the quasi-homeomorphism method (see [4]), we may and do assume without loss of generality that $(\mathcal{E}^*, \mathcal{F}^*)$ is a regular Dirichlet form and X^* is an associated Hunt process, and so [13, Theorem 4.6.5, Theorem 4.3.1] applies. Moreover, for $f \in \mathcal{F}^*$, letting $f_0 := f - f(a)\, u_\alpha$,

$$f = f_0 + f(a)\, u_\alpha$$

gives the decomposition of f into $\mathcal{F}^0$ and its $\mathcal{E}^*_\alpha$-orthogonal complement. These combined with (i) yield (ii) and (iii).

Similarly,

$$f = f_0 + f(a)\, \varphi \quad \text{with } f_0 = f - f(a)\, \varphi,$$

represents a decomposition of $f \in \mathcal{F}^*_e$ into an element of $\mathcal{F}^0$ and a constant multiple of φ which are mutually $\mathcal{E}^*$-orthogonal. This combined with (i) implies (iv) except for the identity

$$\mathcal{E}^*(c\varphi,\, c\varphi) = c^2\, V, \tag{4.18}$$

where V is given by (4.15). But (4.18) is just a special case of the much more general formula (3.19) in [6] which is established for a general non-m-polar quasi-closed set rather than a one point set. □

We remark that Theorem 4.4(iv) generalizes Theorem 5.1(ii) of [14] where X^0 and X^* were assumed to be symmetric diffusions.

The next lemma is a variant of [25, Theorem 14.5].

Lemma 4.5 *We assume for $\Phi \in \mathbf{N}$ that the function h defined by (4.5) belongs to the active reflected Dirichlet space $(\mathcal{F}^0)_a^{\mathrm{ref}}$. Then*

$$\{\Phi \neq 0\} \subset \{\zeta^0 < \infty\} \quad \mathbf{P}_x\text{-a.e. for q.e. } x \in E_0.$$

Proof We may assume that $\Phi \geq 0$. We let $h_\alpha(x) = \mathbf{E}_x^0 \left[e^{-\alpha\zeta^0} \Phi \right]$. Since $h \in L^2(E_0; m_0)$, the function $h - h_\alpha = \alpha G_\alpha^0 h$ is in $\mathcal{F}^0$ and

$$\begin{aligned}\mathcal{E}^0(h - h_\alpha, h - h_\alpha) &= \alpha^2 \mathcal{E}_\alpha^0(G_\alpha^0 h, G_\alpha^0 h) - \alpha^3 (G_\alpha^0 h, G_\alpha^0 h) \\ &= \alpha^2 (G_\alpha^0 h,\ h - \alpha G_\alpha^0 h) \\ &\leq \alpha^2 \|G_\alpha^0 h\|_2 \, \|h\|_2 \leq \alpha \|h\|_2^2.\end{aligned}$$

Since $(\mathcal{E}^0, \mathcal{F}^0)$ is transient, it follows that there is a subsequence $\{\alpha_k, k \geq 1\}$ decreasing to zero as $k \to \infty$ so that $\lim_{k\to\infty}(h(x) - h_{\alpha_k}(x)) = 0$ for $\mathcal{E}^0$-q.e. $x \in E_0$. Note that $\mathcal{E}^0$-q.e is the same as $\mathcal{E}$-q.e. on E_0 (see [13]). Hence for q.e. $x \in E_0$,

$$\mathbf{E}_x^0 \left[\Phi \, \mathbf{1}_{\{\zeta^0 = \infty\}} \right] = \lim_{k\to\infty} (h(x) - h_{\alpha_k}(x)) = 0.$$

□

A Borel function f on E_0 will be said to have an X^0-fine limit at a if there exists a constant c such that

$$\mathbf{P}_x^0 \left(\lim_{t \uparrow \zeta^0} f(X_t^0) = c \;\middle|\; \zeta^0 < \infty \text{ and } X_{\zeta^0-}^0 \in K \right) = 1 \quad \text{for q.e. } x \in E_0.$$

In this case, we write as $\gamma f(a) = c$.

We shall consider the condition that

(D.1) If a function $f \in (\mathcal{F}^0)_a^{\mathrm{ref}}$ admits the X^0-fine limit 0 at a, then $f \in \mathcal{F}^0$.

This is a condition imposed on the process X^0 and on the set K. In view of Lemma 4.2 and Lemma 4.5, a sufficient condition for **(D.1)** to be fulfilled is

$$\mathbf{P}_x^0 \left(\lim_{k\to\infty} X_{\sigma_{E_0 \setminus D_k}}^0 \in K \;\middle|\; \lim_{k\to\infty} \sigma_{E_0 \setminus D_k} < \infty \right) = 1 \quad \text{for q.e. } x \in E_0,$$

for some relatively compact open sets D_k increasing to E_0, where the left limit of the path is taken in the topology of E. For instance, when X^0 is the absorbed Brownian motion on the interval $(0, 1)$, then **(D.1)** is fulfilled if $K = \{0\} \cup \{1\}$ but not if $K = \{0\}$. On the other hand, when X^0 is a diffusion on $(0, \infty)$ with generator (1.5) for which 0 is regular and ∞ is non-regular, then **(D.1)** is fulfilled if $K = \{0\}$ (see Sect. 5.2).

Lemma 4.6 *If f is the restriction to E_0 of an X^*-q.e. finely continuous function f^* on E^*, then f admits an X^0-fine limit at a and $\gamma f(a) = f^*(a)$. In particular, u_α and φ both admit the X^0-fine limit 1 at a.*

Proof We may assume without loss of generality that $(\mathcal{E}^*, \mathcal{F}^*)$ is regular and X^* is an associated Hunt process. By [13, Theorem 4.2.2], for every an X^*-q.e. finely continuous function f^* on E^*,

$$\mathbf{P}_x^*\left(\lim_{t'\uparrow t} f(X_{t'}^*) = f(X_{t-}^*) \ \text{ for every } t \in [0, \zeta)\right) = 1 \quad \text{for q.e. } x \in X^*.$$

Since X^* admits no jump from E^0 to a by Theorem 4.1(i.6), it follows then $\gamma f(a) = f^*(a)$. □

Let us consider a linear operator $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ on $L^2(E_0; m_0)$ specified by the following condition:

$$f \in \mathcal{D}(\mathcal{L}) \quad \text{with} \quad \mathcal{L} f = g \ (\in L^2(E_0; m_0)),$$

if and only if

$$f \in (\mathcal{F}^0)_a^{\text{ref}} \quad \text{such that} \quad \mathcal{E}^{\text{ref}}(f, v) = -(g, v) \quad \text{for every } \ v \in \mathcal{F}^0. \tag{4.19}$$

Lemma 4.7 *Assume condition* **(D.1)** *holds. Suppose that* $f \in \mathcal{D}(\mathcal{L})$ *such that* f *admits an* X^0*-fine limit* c *at* a *and*

$$\mathcal{L} f = \alpha f \quad \textit{for some } \alpha > 0.$$

Then $f = c\, u_\alpha$.

Proof By (4.19), $f \in (\mathcal{F}^0)_a^{\text{ref}}$ and $\mathcal{E}_\alpha^{\text{ref}}(f, v) = 0$ for any $v \in \mathcal{F}^0$. By Theorem 4.4. $u_\alpha \in (\mathcal{F}^0)_a^{\text{ref}}$ and u_α satisfies the same equation. Put $f_0 = f - c\, u_\alpha$. Then $f_0 \in (\mathcal{F}^0)^{\text{ref}}$ and $\mathcal{E}_\alpha^{\text{ref}}(f_0, v) = 0$ for any $v \in (\mathcal{F}^0)^{\text{ref}}$. Since f_0 admits an X^0-fine limit 0 at a, we have by **(D.1)** that $f_0 \in \mathcal{F}^0$ and so $\mathcal{E}_\alpha^{\text{ref}}(f_0, f_0) = 0$. This implies $f_0 = 0$. □

For $f \in \mathcal{D}(\mathcal{L}) (\subset \mathcal{F}_a^{\text{ref}})$, we let

$$\mathcal{N}(f) = \mathcal{E}^{\text{ref}}(f, u_\alpha) + (\mathcal{L} f, u_\alpha). \tag{4.20}$$

For $\alpha, \beta > 0$, we can easily verify the identity $u_\alpha - u_\beta = (\alpha - \beta) G_\alpha^0 u_\beta$, which is a member of $\mathcal{F}^0$ because $u_\beta \in L^2(E_0; m_0)$. Hence $\mathcal{N}(f)$ defined by (4.20) is independent of the choice of $\alpha > 0$ in view of (4.19). We call $\mathcal{N}(f)$ the *flux* of f at a.

Denote by $\mathcal{A}^*$ the L^2 infinitesimal generator of X^*: $\mathcal{A}^*$ is a self adjoint operator on $L^2(E^*; m_0)$ $(= L^2(E_0; m_0))$ such that

$$\begin{aligned} f \in \mathcal{D}(\mathcal{A}^*) \text{ with } \mathcal{A}^* f = g \quad &\text{if and only if} \quad f \in \mathcal{F}^* \\ &\text{with } \mathcal{E}^*(f, v) = -(g, v) \ \text{ for every } v \in \mathcal{F}^*. \end{aligned} \tag{4.21}$$

We see from Theorem 4.4 that $\mathcal{L}$ is an extension of $\mathcal{A}^*$:

$$\mathcal{D}(\mathcal{A}^*) \subset \mathcal{D}(\mathcal{L}) \quad \text{and} \quad \mathcal{A}^* f = \mathcal{L} f \quad \text{for } f \in \mathcal{D}(\mathcal{A}^*). \tag{4.22}$$

We are now in a position to present a *lateral condition* in terms of $\mathcal{N}(f)$ to characterize for a function $f \in \mathcal{D}(\mathcal{L})$ to be a member of $\mathcal{D}(\mathcal{A}^*)$.

Theorem 4.8 (i) *If* $f \in \mathcal{D}(\mathcal{A}^*)$, *then*

$$f \in \mathcal{D}(\mathcal{L}), \quad f \text{ admits an } X^0\text{-fine limit at } a, \tag{4.23}$$

and

$$\mathcal{N}(f) = 0. \tag{4.24}$$

(ii) *Assume condition* **(D.1)** *holds. If a function* f *satisfies the conditions* (4.23), (4.24), *then* $f \in \mathcal{D}(\mathcal{A}^*)$.

Proof If $f \in \mathcal{D}(\mathcal{A}^*)$, then (4.21), (4.22) and Theorem 4.4 imply that $f \in \mathcal{D}(\mathcal{L})$ and

$$\mathcal{E}^{\text{ref}}(f, u_\alpha) = -(\mathcal{L}f, u_\alpha)$$

which reads as $\mathcal{N}(f) = 0$. By Lemma 4.6, f also admits an X^0-fine limit at a. Conversely, suppose a function f satisfies conditions (4.23) and (4.24). Put, for $\alpha > 0$,

$$g := (\alpha - \mathcal{L})f, \quad f_1 := G^*_\alpha g \quad \text{and} \quad v = f - f_1.$$

Then $v \in \mathcal{D}(\mathcal{L})$ with $(\alpha - \mathcal{L})v = 0$, and v admits an X^0-fine limit, say c, at a. Then by Lemma 4.7, $v = c\, u_\alpha$. Since $\mathcal{N}(v) = 0$,

$$c\left(\mathcal{E}^{\text{ref}}(u_\alpha, u_\alpha) + \alpha(u_\alpha, u_\alpha)\right) = 0,$$

which together with (4.1) implies $c = 0$ and thus $v = 0$. Consequently, $f = G^*_\alpha g \in \mathcal{D}(\mathcal{A}^*)$. □

In Sect. 5, explicit formulae for the flux $\mathcal{N}(f)$ will be given in some concrete examples. The next Lemma gives other ways to evaluate $\mathcal{N}(f)$.

Lemma 4.9 *Assume condition* **(D.1)**. *If* $f \in \mathcal{D}(\mathcal{L})$ *admits an* X^0*-fine limit at a, then*

$$\mathcal{N}(f) = \mathcal{E}^{\text{ref}}(f, \varphi) + \lim_{\alpha \downarrow 0}(\mathcal{L}f, u_\alpha). \tag{4.25}$$

If the Dirichlet form $(\mathcal{E}^0, \mathcal{F}^0)$ *is further assumed to satisfy the Poincaré inequality in the sense that there exists a constant* $C > 0$ *with*

$$(u, u) \le C\, \mathcal{E}^0(u, u) \quad \text{for every } u \in \mathcal{F}^0, \tag{4.26}$$

then

$$\mathcal{N}(f) = \mathcal{E}^{\text{ref}}(f, \varphi) + (\mathcal{L}f, \varphi) \quad \text{for } f \in \mathcal{D}(\mathcal{L}). \tag{4.27}$$

Proof (i) From the equation $\varphi = u_\alpha + \alpha G^0_{0+} u_\alpha$, we get

$$(u_\alpha, u_\alpha) + \alpha(u_\alpha, G^0_{0+} u_\alpha) = (u_\alpha, \varphi) < \infty,$$

which particularly means

$$G^0_{0+} u_\alpha \in \mathcal{F}^0 \quad \text{and so} \quad \mathcal{E}(G^0_{0+} u_\alpha, v) = (u_\alpha, v) \quad \text{for any } v \in \mathcal{F}^0. \tag{4.28}$$

We further get from the equation (4.3) the representation

$$\alpha(u_\alpha, \varphi) = \int_0^\infty (1 - e^{-\alpha u}) \Theta(du),$$

as in [14, Lemma 2.3] where $\Theta(du)$ is a positive measure on $(0, \infty)$ defined by $\Theta((s, t]) = \mu_t(\varphi) - \mu_s(\varphi)$ for $0 < s < t$. In particular,

$$\lim_{\alpha \downarrow 0} \alpha(u_\alpha, \varphi) = 0. \tag{4.29}$$

Suppose that $f \in \mathcal{D}(\mathcal{L})$ has an X^0-fine limit c at a. Then $f - c\varphi \in \mathcal{F}^0$ by Lemma 4.6 and condition **(D.1)**, and we get by (4.28)

$$\begin{aligned}
\mathcal{E}^{\text{ref}}(f, u_\alpha) &= \mathcal{E}^{\text{ref}}(f, \varphi) - \alpha \mathcal{E}^{\text{ref}}(f, G^0_{0+} u_\alpha) \\
&= \mathcal{E}^{\text{ref}}(f, \varphi) - \alpha \mathcal{E}^{\text{ref}}(f - c\varphi, G^0_{0+} u_\alpha) \\
&= \mathcal{E}^{\text{ref}}(f, \varphi) - \alpha(f, u_\alpha) + c\alpha(u_\alpha, \varphi).
\end{aligned}$$

Therefore we have

$$\mathcal{N}(f) = \mathcal{E}^{\text{ref}}(f, \varphi) + (\mathcal{L} f, u_\alpha) + d_\alpha$$

with

$$d_\alpha = -\alpha(f, u_\alpha) + c\alpha(u_\alpha, \varphi).$$

Since $\alpha^2(f, u_\alpha)^2 \le \alpha(f, f) \cdot \alpha(u_\alpha, \varphi)$, we obtain $\lim_{\alpha \downarrow 0} d_\alpha = 0$ by (4.29). This establishes (4.25).

(ii). (4.26) implies that the 0-order resolvent G^0_{0+} of X^0 is a bounded operator on $L^2(E_0, m_0)$. Hence

$$\varphi = u_\alpha + \alpha G^0_{0+} u_\alpha \in L^2(E_0, m_0).$$

This together with dominated convergence theorem yields (4.27). □

So far we have considered the unique m_0-symmetric one-point extension X^* to E^* of X^0 on E_0, where m_0 is defined to be the restriction to $E_0 = E \setminus K$ of the symmetrizing measure m of X on E and extended to E^* by setting $m_0(\{a\}) = 0$. The subprocess X^0 is symmetric with respect to the measure m_0. But X^0 may admit other choices of symmetrizing measures on E_0 rather than m_0; and thus X^0 may admit other possible extensions.

Let us consider the assumption that

(D.2) $E_0 = E_{01} \cup \cdots \cup E_{0k}$ for some disjoint open sets $E_{0i},\ 1 \le i \le k$, and each E_{0i} is X^0-invariant.

Assumption **(D.2)** means that

$$\mathbf{P}_x(\Omega_{E_{0i}}) = 1 \quad \text{for every } x \in E_{0i} \text{ and every } i = 1, \dots, k,$$

where $\Omega_{E_{0i}}$ is defined by (4.4). Condition **(D.2)** is equivalent to say that X^0 does not travel between two different sets E_{0i} and $E_{0j}, i \neq j,\ 1 \le i, j \le k$. If X^0 is a diffusion, then **(D.2)** is automatically satisfied. The restrictions of functions and measures on E_0 to E_{0i} will be designated by the superscript $^i,\ 1 \le i \le k$.

Choosing any k-vector $\mathbf{p}$ with positive entries:

$$\mathbf{p} = (p_1, \dots, p_k) \quad \text{with } p_1, \dots, p_k > 0,$$

we define a new measure $\widetilde{m}_0$ on E_0 by

$$\widetilde{m}_0^i = p_i \cdot m_0^i, \quad 1 \le i \le k. \tag{4.30}$$

The measure $\widetilde{m}_0$ will be also designated by $m_0^{\mathbf{p}}$ to indicate its dependence on $\mathbf{p}$.

Clearly X^0 is $\widetilde{m}_0$-symmetric and we extend $\widetilde{m}_0$ to E^* by setting $\widetilde{m}_0(\{a\}) = 0$.

Theorem 4.10 *Assume the condition* **(D.2)**.

(i) *There exists a unique $\widetilde{m}_0$-symmetric standard process $\widetilde{X}^*$ on E^* whose resolvent is given by* (4.2) *with $\widetilde{m}_0$ in place of m_0.*

(ii) *The process $\widetilde{X}^*$ enjoys the properties* (i.1) *and* (i.3)–(i.9) *in* Theorem 4.1.

(iii) *The entrance laws $\{\widetilde{\mu}_t, t > 0\}$ for $\widetilde{X}^*$ and $\{\mu_t, t > 0\}$ for X^* are related by*

$$\widetilde{\mu}_t^i = p_i \cdot \mu_t^i \quad \text{for } 1 \le i \le k.$$

(iv) *The extensions $\widetilde{X}^*$ and $\widetilde{X}'^*$ corresponding to two different k-vectors* $\mathbf{p}$ *and* $\mathbf{q}$ *are equivalent in law if and only if*

$$\mathbf{p} = \lambda\, \mathbf{q} \quad \text{for some } \lambda > 0.$$

Proof Parts (i) and (ii) follow from Theorem 4.1 with $\widetilde{m}_0$ in place of m_0 there. Part (iii) is immediate from a comparison of (4.3) with

$$\varphi \cdot \widetilde{m}_0 = \int_0^\infty \widetilde{\mu}_t \, dt.$$

By substituting $m_0^{\mathbf{p}}$ and $m_0^{\mathbf{q}}$ in (4.2), the corresponding extensions can be seen to have the same resolvents if and only if (4.25) holds, proving (iv). □

Under the assumption **(D.2)**, the Dirichlet form $(\mathcal{E}^0, \mathcal{F}^0)$ of X^0 on $L^2(E_0, m_0)$ and its reflected Dirichlet space $((\mathcal{F}^0)^{\mathrm{ref}}, (\mathcal{E}^0)^{\mathrm{ref}})$ can be described as follows. For each $1 \le i \le k$, define the restriction $(\mathcal{E}^{0i}, \mathcal{F}^{0i})$ of $(\mathcal{E}^0, \mathcal{F}^0)$ to E_{0i} by

$$\mathcal{F}^{0i} = \mathcal{F}^0\Big|_{E_{0i}} \quad \text{and} \quad \mathcal{E}^{0i}(u|_{E_{0i}}, v|_{E_{0i}}) = \mathcal{E}^0(u\mathbf{1}_{E_{0i}}, v\mathbf{1}_{E_{0i}}) \quad \text{for } u, v \in \mathcal{F}^0.$$

This is a transient Dirichlet form on $L^2(E_{0i}; m_0^i)$, whose reflected Dirichlet space will be denoted by $(\mathcal{F}^{0i})^{\mathrm{ref}}, \mathcal{E}^{\mathrm{ref},i})$. It holds then that

$$\begin{aligned}
\mathcal{F}^0 &= \{u : u|_{E_{0i}} \in \mathcal{F}^{0i} \text{ for } 1 \le i \le k\} \\
\mathcal{E}^0(u, v) &= \sum_{i=1}^k \mathcal{E}^{0i}(u|_{E_{0i}}, v|_{E_{0i}}) \quad \text{for } u, v \in \mathcal{F}^0, \\
(\mathcal{F}^0)^{\mathrm{ref}} &= \{u : u|_{E_{0i}} \in (\mathcal{F}^{0i})^{\mathrm{ref}} \text{ for } 1 \le i \le k\}, \quad \text{and} \\
\mathcal{E}^{\mathrm{ref}}(u, v) &= \sum_{i=1}^k \mathcal{E}^{\mathrm{ref},i}(u|_{E_{0i}}, u|_{E_{0i}}) \quad \text{for } u, v \in (\mathcal{F}^0)^{\mathrm{ref}}.
\end{aligned} \tag{4.31}$$

By virtue of Theorem 4.4(iii), the Dirichlet form $(\mathcal{E}^*, \mathcal{F}^*)$ of X^* on $L^2(E^*; m_0)$ can be described as

$$\begin{aligned}
\mathcal{F}^* &= \left\{ f = f_0 + c\, u_\alpha : f_0 \in \mathcal{F}^0,\ c \in \mathbb{R} \right\}, \\
\mathcal{E}^*(u, v) &= \mathcal{E}^{\mathrm{ref}}(u, v) \quad \text{for } u, v \in \mathcal{F}^*,
\end{aligned} \tag{4.32}$$

where $\mathcal{E}^{\mathrm{ref}}$ is given by (4.31). Note that $\mathcal{F}^* \subset (\mathcal{F}^0)^{\mathrm{ref}}$.

Now, for measure $\widetilde{m}_0$ defined by (4.30), we denote by $(\widetilde{\mathcal{E}}^0, \widetilde{\mathcal{F}}^0)$ the Dirichlet form of X^0 on $L^2(E_0, \widetilde{m}_0)$ and by $((\widetilde{\mathcal{F}}^0)^{\mathrm{ref}}, (\widetilde{\mathcal{E}}^0)^{\mathrm{ref}})$ its reflected Dirichlet space. We then readily see that

$$\widetilde{\mathcal{F}}^0 = \mathcal{F}^0 \quad \text{and} \quad \widetilde{\mathcal{E}}^0(u, v) = \sum_{i=1}^k p_i\, \mathcal{E}^{0i}(u^i, v^i) \quad \text{for } u, v \in \mathcal{F}^0, \tag{4.33}$$

$$(\widetilde{\mathcal{F}}^0)^{\mathrm{ref}} = (\mathcal{F}^0)^{\mathrm{ref}} \quad \text{and} \quad \widetilde{\mathcal{E}}^{\mathrm{ref}}(u, v) = \sum_{i=1}^k p_i\, \mathcal{E}^{\mathrm{ref},i}(u^i, v^i) \quad \text{for } u, v \in (\mathcal{F}^0)^{\mathrm{ref}}. \tag{4.34}$$

Let $\widetilde{X}^*$ be the $\widetilde{m}_0$-symmetric extension of X^0 described in Theorem 4.10 and $(\widetilde{\mathcal{E}}^*, \widetilde{\mathcal{F}}^*)$ be its Dirichlet form on $L^2(E^*; \widetilde{m}_0)$. By virtue of Theorem 4.4(iii) again, we get analogously to (4.32)

$$\widetilde{\mathcal{F}}^* = \left\{ f = f_0 + c\, u_\alpha : \ f_0 \in \widetilde{\mathcal{F}}^0, \ c \in \mathbb{R} \right\}, \tag{4.35}$$
$$\widetilde{\mathcal{E}}^*(u, v) = \widetilde{\mathcal{E}}^{\rm ref}(u, v) \quad \text{for } u, v \in \widetilde{\mathcal{F}}^*,$$

where $\widetilde{\mathcal{E}}^{\rm ref}$ is given by (4.34). Note that $\widetilde{\mathcal{F}}^* \subset (\widetilde{\mathcal{F}}^0)^{\rm ref}$.

Theorem 4.11 *Assume that* **(D.2)** *holds and that the measure* $\widetilde{m}_0$ *is given by* (4.30).
(i) *The Dirichlet form* $(\widetilde{\mathcal{E}}^*, \widetilde{\mathcal{F}}^*)$ *of* $\widetilde{X}^*$ *on* $L^2(E^*; \widetilde{m}_0)$ *and its extended Dirichlet space* $\mathcal{F}_e^*$ *admit the expressions*

$$\widetilde{\mathcal{F}}^* = \mathcal{F}^* = \left\{ f = f_0 + c\, u_\alpha : \ f_0 \in \mathcal{F}^0, \ c \in \mathbb{R} \right\}, \tag{4.36}$$

$$\widetilde{\mathcal{F}}_e^* = \mathcal{F}_e^* = \left\{ f = f_0 + c\, \varphi : \ f_0 \in \mathcal{F}_e^0, \ c \in \mathbb{R} \right\}, \tag{4.37}$$

$$\widetilde{\mathcal{E}}^*(u, v) = \sum_{i=1}^{k} p_i\, \mathcal{E}^{{\rm ref},i}(u|_{E_{0i}}, u|_{E_{0i}}) \ \ \textit{for } u, v \in \widetilde{\mathcal{F}}^*. \tag{4.38}$$

Moreover, $f_0 \in \mathcal{F}^0$ *and* u_α *are* $\widetilde{\mathcal{E}}_\alpha^*$*-orthogonal for each* $\alpha > 0$, *and* $f_0 \in \mathcal{F}_e^0$ *and* φ *are* $\widetilde{\mathcal{E}}^*$*-orthogonal.*
(ii) *Assume further that condition* **(D.1)** *is satisfied. Let* $\widetilde{\mathcal{A}}^*$ *be the* $L^2(E^*; \widetilde{m}_0)$*-infinitesimal generator of* $\widetilde{X}^*$. *Then,* $f \in \mathcal{D}(\widetilde{\mathcal{A}}^*)$ *if and only if*

$$f|_{E_{0i}} \in \mathcal{D}(\mathcal{L}^i) \ \textit{for } 1 \le i \le k, \quad f \ \text{ admits an } X^0\text{-fine limit at } a, \quad \textit{and}$$
$$\sum_{i=1}^{k} p_i\, \mathcal{N}^i(f|_{E_{0i}}) = 0,$$

where, for $1 \le i \le k$, $\mathcal{L}^i$ *is defined by* (4.19) *with* $L^2(E_0, m_0)$, $\mathcal{F}^0$, $((\mathcal{F}^0)^{\rm ref}$ *and* $\mathcal{E}^{\rm ref})$ *being replaced by* $L^2(E_{0i}, m_0^i)$, $\mathcal{F}^{0i}$ *and* $((\mathcal{F}^{0i})^{\rm ref}, \mathcal{E}^{{\rm ref},i})$, *respectively, and* $\mathcal{N}^i$ *is defined by* (4.20) *with* $\mathcal{L}$, $\mathcal{E}^{\rm ref}$ *and* u_α *being replaced by* $\mathcal{L}^i$, $\mathcal{E}^{{\rm ref},i}$ *and* u_α^i, *respectively.*

Proof From (4.32), (4.33), (4.34) and (4.35), we obtain (4.36) and (4.38). Analogously, we get (4.37) from Theorem 4.4(iv). Taking (4.30) and (4.38) into account, we are led to (ii) from Theorem 4.8. □

5 Examples

5.1 Darning holes in the one-dimensional Brownian motion

Let $\mathbb{R}$ be the real line, I be its open subset and m be the Lebesgue measure on it. The restriction of m to I is denoted by m_0. We introduce function spaces by

$$\mathbb{G}(I) := \left\{ u : \text{ absolutely continuous on } I \text{ with } \int_I (u')^2 dx < \infty \right\},$$
$$\mathbb{H}^1_{0e}(I) := \{u \in \mathbb{G}(I) : \ u = 0 \ \text{ at the finite boundary points of } I\},$$
$$\mathbb{H}^1(I) := \mathbb{G}(I) \cap L^2(I; m_0) \quad \text{and} \quad \mathbb{H}^1_0(I) := \mathbb{H}^1_{0e}(I) \cap L^2(I; m_0),$$
$$\mathbf{D}^I(u, v) := \int_I u'(x)v'(x)m_0(dx) \quad \text{and} \quad (u, v) := \int_I u(x)v(x)m_0(dx).$$

Let $X^0(I)$ be the absorbed Brownian motion on I and $(\mathcal{E}^0, \mathcal{F}^0)$ be its Dirichlet form on $L^2(I; m_0)$. It holds then that

$$(\mathcal{E}^0, \mathcal{F}^0) = (\tfrac{1}{2}\mathcal{D}^I, \mathbb{H}^1_0(I)), \quad \mathcal{F}^0_e = \mathbb{H}^1_{0e}(I) \quad \text{and} \quad (\mathcal{F}^0)^{\text{ref}} = \mathbb{G}(I). \tag{5.1}$$

The linear operator $\mathcal{L}_I$ on $L^2(I; m_0)$ introduced by (4.19) reads as follows:

$$\begin{aligned} \mathcal{D}(\mathcal{L}_I) &= \Big\{ f \in \mathbb{H}^1(I) : \ f' \text{ has an absolutely continuous} \\ &\qquad \text{version with } f'' \in L^2(I; m_0) \Big\}, \\ \mathcal{L}_I f &= \frac{1}{2} f''. \end{aligned} \tag{5.2}$$

Moreover, an integration by parts gives the next description of the linear functional $\mathcal{N}_I(f)$ introduced by (4.20): for $f \in \mathcal{D}(\mathcal{L}_I)$,

$$\mathcal{N}_I(f) = \begin{cases} f'(b-) - f'(a+), & \text{when } I = (a, b); \\ f'(a-), & \text{when } I = (-\infty, a); \\ -f'(a+), & \text{when } I = (a, \infty). \end{cases} \tag{5.3}$$

In fact, the existence of the limit of f' at finite end points of I is clear from (5.1). When $I = (a, \infty)$,

$$\mathcal{N}_I(f) = \lim_{b \to \infty} f'(b)u_\alpha(b) - f'(a+)$$

and, if the first term of the right hand side does not vanish, then $f \notin \mathbb{G}(I)$.

We shall examine the one point extension of $X^0(I)$ by the darning in several cases of I.

(i) Reflected and circular Brownian motions

Let

$$I = (0, \infty), \quad E_0 = I, \quad E^* = [0, \infty) = I \cup \{0\}$$

and X^* be the extension of $X^0(I)$ to E^* of Theorem 4.1. Denote by $(\mathcal{E}^*, \mathcal{F}^*)$ the Dirichlet form of X^* on $L^2(E^*; m_0) (= L^2(E_0; m_0))$. Note that the point 0 is approachable by $X^0(I)$ with probability 1 and so $\varphi = 1$. Hence we conclude from Theorem 4.4 and (5.1) that

$$\mathcal{F}^* = \mathbb{H}^1(I) \quad \text{and} \quad \mathcal{E}^* = \frac{1}{2}\mathbf{D}^I \tag{5.4}$$

namely, X^* is the reflected Brownian motion of $E^* = [0, \infty)$. By Theorem 4.8 and (5.3), the L^2-generator $\mathcal{A}^*$ of X^* can be described as

$$\mathcal{D}(\mathcal{A}^*) = \{f \in \mathcal{D}(\mathcal{L}_I) : f'(0+) = 0\} \quad \text{and} \quad \mathcal{A}^* f(x) = \frac{1}{2} f''(x), \ x \in I, \tag{5.5}$$

where $\mathcal{D}(\mathcal{L})$ is defined by (5.2). We can also get the Skorohod equation for $X^* = (X_t^*, \mathbf{P}_x)$:

$$X_t^* = X_0^* + B_t + \ell_t, \quad t > 0, \quad \mathbf{P}_x\text{–a.s.}, \quad x \in [0, \infty),$$

where B_t is a Brownian motion with $B_0 = 0$ and ℓ_t is the positive continuous additive functional of X^* with Revuz measure $\delta_{\{0\}}$. See Theorem 5.2 below for a proof.

Next let $I = (0, 1)$, $E_0 = I$, $E^* = (0, 1) \cup \{0^*\}$, the one point compactification of I, and X^* be the extension of $X^0(I)$ to E^* as in Theorem 4.1. Let $(\mathcal{E}^*, \mathcal{F}^*)$ and $\mathcal{A}^*$ be the Dirichlet form and the L^2-generator of X^*. In the same way as above, we can conclude that

$$\mathcal{F}^* = \mathbb{H}_0^1(I) \cup \{\text{constant functions}\} \quad \text{and} \quad \mathcal{E}^* = \frac{1}{2}\mathbf{D}^I, \tag{5.6}$$

$$\mathcal{D}(\mathcal{A}^*) = \{f \in \mathcal{D}(\mathcal{L}_I) : f(0+) = f(1-), \ f'(0+) = f'(1-)\}, \quad \text{and}$$

$$\mathcal{A}^* f(x) = \frac{1}{2} f''(x) \quad \text{for } f \in \mathcal{D}(\mathcal{A})^* \text{ and } x \in I. \tag{5.7}$$

Consequently X^* is the Brownian motion on the circle E^*, which can be also obtained by wrapping the Brownian motion on $\mathbb{R}$ to $[0, 1)$ (more precisely, by modulo 1).

(ii) Skew Brownian motion

Let

$$E_0 = (-\infty, 0) \cup (0, \infty), \quad E^* = \mathbb{R},$$

and X^0 be the absorbed Brownian motion on E_0, namely X^0 is the subprocess on Brownian motion on $\mathbb{R}$ killed upon hitting Denote by $(\mathcal{E}^0, \mathcal{F}^0)$ the Dirichlet form of X^0 on $L^2(E_0; m_0)$.

Note that $\mathbb{R}^+ = (0, \infty)$ and $\mathbb{R}^- = (-\infty, 0)$ are two invariant sets for X^0. The restrictions of functions and measures on $\mathbb{R}$ to $\mathbb{R}^+, \ \mathbb{R}^-$ will be denoted by putting superscript $+$ and $-$, respectively. We can then rewrite the expression (5.1) as

$$\mathcal{F}_e^0 = \left\{u : u^\pm \in \mathbb{H}_{0e}^1(\mathbb{R}^\pm)\right\},$$

$$\mathcal{E}^0(u, v) = \frac{1}{2}\mathbf{D}^{\mathbb{R}^+}(u^+, v^+) + \frac{1}{2}\mathbf{D}^{\mathbb{R}^-}(u^-, v^-) \quad \text{for } u, v \in \mathcal{F}_e^0, \tag{5.8}$$

and we get

$$(\mathcal{F}^0)^{\rm ref} = \left\{u : u^\pm \in \mathbb{G}(\mathbb{R}^\pm)\right\},$$
$$\mathcal{E}^{\rm ref}(u, v) = \frac{1}{2}\mathbf{D}^{\mathbb{R}^+}(u^+, v^+) + \frac{1}{2}\mathbf{D}^{\mathbb{R}^-}(u^-, v^-) \quad \text{for } u, v \in (\mathcal{F}^0)^{\rm ref}. \tag{5.9}$$

Note that the functions in $(\mathcal{F}^0)^{\rm ref}$ may not be continuous at 0.

For any $p^+ > 0,\ p^- > 0$, let $\widetilde{m}_0$ be the measure on $\mathbb{R}$ defined by

$$\widetilde{m}_0(dx) = p^+ dx \ \text{ on } (0, \infty), \quad \widetilde{m}_0(dx) = p^- dx \ \text{ on } (-\infty, 0), \quad \text{and} \quad \widetilde{m}_0(\{0\}) = 0.$$

Then X^0 can be regarded as an $\widetilde{m}_0$-symmetric diffusion on $\mathbb{R}_0$ whose Dirichlet form $(\widetilde{\mathcal{E}}^0, \widetilde{\mathcal{F}}^0)$ on $L^2(\mathbb{R}_0, \widetilde{m}_0)$ is described as

$$\widetilde{\mathcal{F}}^0_e = \mathcal{F}^0_e, \quad \widetilde{\mathcal{E}}^0(u, v) = \frac{p^+}{2}\mathbf{D}^{\mathbb{R}^+}(u^+, v^+) + \frac{p^-}{2}\mathbf{D}^{\mathbb{R}^-}(u^-, v^-),\ u, v \in \widetilde{\mathcal{F}}^0_e,$$

and accordingly,

$$(\widetilde{\mathcal{F}}^0)^{\rm ref} = (\mathcal{F}^0)^{\rm ref}, \quad \text{and}$$
$$\widetilde{\mathcal{E}}^{\rm ref}(u, v) = \frac{p^+}{2}\mathbf{D}^{\mathbb{R}^+}(u^+, v^+) + \frac{p^-}{2}\mathbf{D}^{\mathbb{R}^-}(u^-, v^-) \quad \text{for } u, v \in (\widetilde{\mathcal{F}}^0)^{\rm ref}.$$

Let X^* be the m_0-symmetric extension of X^0 to $E^* = \mathbb{R}$ by Theorem 4.1, namely, X^* is constructed based on the entrance law $\{\mu_t, t > 0\}$ of X^0 from 0 specified by

$$\int_0^\infty \mu_t \, dt = m_0. \tag{5.10}$$

By Theorem 4.4, the Dirichlet form $(\mathcal{E}^*, \mathcal{F}^*)$ of X^* on $L^2(\mathbb{R}; m_0)(= L^2(E_0; m_0))$ is given by

$$\mathcal{F}^*_e = \{f = f_0 + c : f_0 \in \mathcal{F}^0\ c \in \mathbb{R}\} \quad \text{and} \quad \mathcal{E}^*(u, v) = \mathcal{E}^{\rm ref}(u, v) \quad \text{for } u, v \in \mathcal{F}^*_e. \tag{5.11}$$

In particular, we see from (5.11) that

$$\text{every } u \in \mathcal{F}^* \text{ is continuous at } 0. \tag{5.12}$$

We can conclude from (5.8), (5.9), (5.11) and (5.12) that

$$(\mathcal{F}^*_e, \mathcal{E}^*) = (\mathbb{G}(\mathbb{R}), \tfrac{1}{2}\mathbf{D}).$$

Hence X^* is nothing but the Brownian motion on $\mathbb{R}$.

On the other hand, let $\widetilde{X}^*$ be the $\widetilde{m}$-symmetric extension of X^0 in Theorem 4.10: namely, $\widetilde{X}^*$ is constructed based on the entrance law $\widetilde{\mu}_t$ from 0 specified by

$$\int_0^\infty \widetilde{\mu}_t dt = \widetilde{m}_0.$$

From (5.10) and the above, we have the relation

$$\widetilde{\mu}_t^+ = p^+\mu_t^+ \quad \text{and} \quad \widetilde{\mu}_t^- = p^-\mu_t^-.$$

By Theorem 4.11(i), the Dirichlet form $(\widetilde{\mathcal{E}}^*, \widetilde{\mathcal{F}}^*)$ of $\widetilde{X}^*$ on $L^2(\mathbb{R}; \widetilde{m}_0)$ is given by

$$\begin{aligned}&\widetilde{\mathcal{F}}^* = \mathcal{F}^* = \mathbb{H}^1(\mathbb{R}), \quad \text{and}\\ &\widetilde{\mathcal{E}}^*(u,v) = \widetilde{\mathcal{E}}^{\text{ref}}(u,v) = \frac{1}{2}p^+\mathbf{D}^{\mathbb{R}^+}(u^+,v^+) + \frac{1}{2}p_-\mathbf{D}^{\mathbb{R}^-}(u^-,v^-) \quad \text{for } u,v \in \mathbb{H}^1(\mathbb{R}).\end{aligned} \tag{5.13}$$

Let $\mathcal{A}^*$ be the infinitesimal generator of $\widetilde{X}^*$ on $L^2(\mathbb{R}; \widetilde{m}_0)$. We then see from Theorem 4.11(ii) and (5.3) that $f \in \mathcal{D}(\widetilde{\mathcal{A}}^*)$ if and only if

$$f^\pm \in \mathcal{D}(\mathcal{L}_{\mathbb{R}^\pm}) \quad \text{with} \quad f(0-) = f(0+) \text{ and } p^- f'(0-) = p^+ f'(0+) \tag{5.14}$$

and $\mathcal{A}^* f(x) = \frac{1}{2} f''(x),\ x \in E_0$. Here $\mathcal{D}(\mathcal{L}_{\mathbb{R}^\pm})$ is defined by (5.2).

Recall that a real-valued process Y is called *a skew Brownian motion* on $\mathbb{R}$ with parameter $\beta \in (-1, 1)$ if

$$Y_t = Y_0 + B_t + \beta L_t, \quad t \geq 0, \tag{5.15}$$

where B is Brownian motion on $\mathbb{R}$ and L is the symmetric local time of Y at 0, i.e.,

$$L_t = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{\{|Y_s| \leq \varepsilon\}} ds.$$

Theorem 5.1 *The process $\widetilde{X}^*$ is a skew Brownian motion with parameter $\frac{p^+ - p^-}{p^+ + p^-}$.*

Proof Suppose Y is a skew Brownian motion on $\mathbb{R}$ with parameter $\beta \in (-1, 1)$. Let $\gamma := \frac{1}{2} \log \frac{1+\beta}{1-\beta}$ and define

$$s(x) := \begin{cases} e^{-\gamma} x & \text{for } x < 0, \\ e^{\gamma} x & \text{for } x \geq 0. \end{cases} \quad \text{and} \quad \sigma(y) := \begin{cases} e^{-\gamma} & \text{for } y < 0, \\ e^{\gamma} & \text{for } y > 0. \end{cases}$$

It is proved in Harrison and Shepp [18] that Y is a skew Brownian motion of (5.15) if and only if $Z = s^{-1}(Y)$ is a continuous martingale with $\langle Z \rangle_t = \int_0^t \frac{1}{\sigma(Z_s)^2} ds$. Here s^{-1}

stands for the inverse function of s. Thus Z is the Brownian motion on $\mathbb{R}$ time-changed by Revuz measure $\sigma(x)^2dx$; or equivalently, the Dirichlet form for Z is $(\mathbf{D}^{\mathbb{R}}, \mathcal{F})$ in $L^2(\mathbb{R}, \sigma(x)^2dx)$. As $Y = s(Z)$, we conclude that Y is $\sigma(x)dx$-symmetric on $\mathbb{R}$ and its Dirichlet form $(\mathcal{E}^Y, \mathcal{F}^Y)$ in $L^2(\mathbb{R}, \sigma(x)dx)$ is given by

$$\mathcal{F}^Y = \mathcal{F}$$

$$\mathcal{E}^Y(u, v) = \frac{1}{2}\mathbf{D}^{\mathbb{R}}(u \circ s, v \circ s) = \frac{e^{\gamma}}{2}\mathbf{D}^{\mathbb{R}^+}(u, v) + \frac{e^{-\gamma}}{2}\mathbf{D}^{\mathbb{R}^-}(u, v) \quad \text{for } u, v \in \mathcal{F}.$$

If we take $e^{2\gamma} = p^+/p^-$, that is, take $\gamma = \frac{1}{2}\log\frac{p^+}{p^-}$, then $(p^+, p^-) = c\,(e^{\gamma}, e^{-\gamma})$ for some constant $c > 0$ and thus $\widetilde{X}^*$ has the same distribution as Y. Solve $\log\frac{1+\beta}{1-\beta} = 2\gamma = \log\frac{p^+}{p^-}$ for β, we have $\beta = \frac{p^+-p^-}{p^++p^-}$. This proves that $\widetilde{X}^*$ is skew Brownian motion on $\mathbb{R}$ with parameter $\frac{p^+-p^-}{p^++p^-}$. □

Theorem 5.2 *The local time L in (5.15) is the positive continuous additive functional of X having Revuz measure $(p^+ + p^-)\delta_0$.*

Proof In fact, one can derive another version of the Skorohod type equation (5.15) for $\widetilde{X}^*$ readily from the expression (5.13) of its Dirichlet form $(\widetilde{\mathcal{E}}^*, \widetilde{\mathcal{F}}^*)$ by using a decomposition theorem of strict additive functionals of $\widetilde{X}^*$ formulated in [12]. It is clear that this Dirichlet form on $L^2(\mathbb{R}; \widetilde{m})$ is a strongly local, recurrent and regular one for which each one point of $\mathbb{R}$ has a positive capacity. Therefore the associated $\widetilde{m}$-symmetric diffusion $\widetilde{X}^* = (\widetilde{X}^*_t, \widetilde{\mathbf{P}}^*_x)$ on $\mathbb{R}$ is conservative and its transition probability is absolutely continuous with respect to $\widetilde{m}$, so that general theorems in [12] are applicable.

Consider the coordinate function $\eta(x) = x$ on $\mathbb{R}$. Then $\eta \in \widetilde{\mathcal{F}}^*_{\rm loc} = \mathbb{H}^1_{\rm loc}(\mathbb{R})$ and its energy measure $\mu_{\langle\eta\rangle}$ admits the expression

$$\mu_{\langle\eta\rangle}(dx) = p^- I_{\mathbb{R}^-}(x)dx + p^+ I_{\mathbb{R}^+}(x)dx,$$

which is exactly the same as the underlying measure $\widetilde{m}$. Furthermore, we have

$$\widetilde{\mathcal{E}}^*(\eta, v) = p^- \int_{-\infty}^{0} v'(x)dx + p^+ \int_{0}^{\infty} v'(x)dx = -(p^+ - p^-)v(0) \quad \text{for } v \in C_0^\infty(\mathbb{R}).$$

Theorems 2.2 and 3.3 of [12] then lead us to the equation

$$\widetilde{X}^*_t = \widetilde{X}^*_0 + B_t + (p^+ - p^-)\ell_t, \quad \mathbf{P}^*_x - \text{a.s.}, \quad x \in \mathbb{R}, \tag{5.16}$$

where B_t is a Brownian motion with $B_0 = 0$ and ℓ_t is the positive continuous additive functional of $\widetilde{X}^*$ with Revuz measure δ_0. By the preceding theorem, we get the relationship $L = (p^+ + p^-)\ell$. □

(iii) One point skew extensions of $\mathbf{X^0}$ obtained by identifying multi-points

Choose any $k-1$ points

$$-\infty < a_1 < a_2 < \cdots < a_{k-1} < \infty$$

and let

$$I = \mathbb{R} \setminus \{a_1, a_2, \ldots, a_{k-1}\} = \bigcup_{i=1}^{k} I_i,$$

where

$$I_1 = (-\infty, a_1), \quad I_j = (a_j, a_{j+1}), \ 1 \le j \le k-1 \quad I_k = (a_{k-1}, \infty).$$

We consider the case that

$$E_0 = I, \quad E^* = I \cup \{a\},$$

where E^* is obtained from I by identifying the compact set $K = \bigcup_{i=1}^{k-1}\{a_i\}$ as one point a in the way described in Sect. 3. Measure m_0 is the restriction of the Lebesgue measure to I. The restrictions of functions and measures on I to the interval I_i will be designated by using the superscript ^{i}.

Let X^0 be the absorbed Brownian motion on I and $\mathbf{p} = (p_1, p_2, \ldots, p_k)$ be a k-vector with positive entries. X^0 is then symmetric with respect to the measure

$$\widetilde{m}_0 = \sum_{i=1}^{k} p_i \, m_0^i,$$

so that we can construct its $\widetilde{m}_0$-symmetric extension $\widetilde{X}^*$ to E^* according to Theorem 4.10. The Dirichlet form $(\widetilde{\mathcal{E}}^*, \widetilde{\mathcal{F}}^*)$ on $L^2(E^*; \widetilde{m}_0)$ of $\widetilde{X}^*$ then admits the description

$$\widetilde{\mathcal{F}}^* = \{f \in \mathbb{H}^1(\mathbb{R}) : f(a_1) = f(a_2) = \cdots = f(a_{k-1})\}, \quad \text{and}$$
$$\widetilde{\mathcal{E}}^*(f, g) = \sum_{i=1}^{k} \frac{1}{2}\, p_i \, \mathbf{D}^{I_i}(f^i, g^i) \quad \text{for } f, g \in \widetilde{\mathcal{F}}^*. \tag{5.17}$$

Let $\mathcal{A}^*$ be the L^2-infinitesimal generator of $\widetilde{X}^*$ on $L^2(\mathbb{R}; \widetilde{m}_0)$. Then $f \in \mathcal{D}(\widetilde{\mathcal{A}}^*)$ if and only if

$$f|_{I_i} \in \mathcal{D}(\mathcal{L}_{I_i}) \quad \text{for } 1 \le i \le k \quad \text{with} \quad f(a_1\pm) = f(a_2\pm) = \cdots = f(a_{k-1}\pm),$$
$$\text{and} \quad p_1 f'(a_1+) + \sum_{j=2}^{k-1} p_j \left(f'(a_j-) - f'(a_{j-1}+)\right) - p_k f'(a_{k-1}-) = 0. \tag{5.18}$$

Both (5.17) and (5.18) can be shown in the same way as in the previous cases by using Theorem 4.11 and (5.3).

5.2 Diffusions on half lines merging at one point

First of all, we recall the absorbed diffusion X^0 on the open half line $I = (0, \infty)$ with generator (1.5) considered at the end of Sect. 1. Let $(\mathcal{E}^0, \mathcal{F}^0)$ and $((\mathcal{F}^0)^{\text{ref}}, \mathcal{E}^{\text{ref}})$ be the Dirichlet form of X^0 on $L^2(I; m)$ and its reflected Dirichlet space, respectively. Note that the latter is given by (1.7) and (1.8). We assume that the boundary 0 is regular but ∞ is non-regular. Then the former is given by (1.9), namely, $f \in \mathcal{F}^0$ if and only if $f \in (\mathcal{F}^0)^{\text{ref}}_a$ and $f(0+) = 0$. Therefore, the condition **(D.1)** in Sect. 4 is satisfied for $K = \{0\}$. The linear operator $\mathcal{L}$ is equal to (1.5) with the domain $\mathcal{D}(\mathcal{L})$ being given by (1.10). Let us compute the flux $\mathcal{N}(f)$ for $f \in \mathcal{D}(\mathcal{L})$ defined by (4.22).

Lemma 5.3 *We have*

$$\mathcal{N}(f) = -\frac{df}{ds}(0+) \quad \textit{for } f \in \mathcal{D}(\mathcal{L}).$$

Proof For $f \in \mathcal{D}(\mathcal{L})$, the integration by parts gives

$$\int_0^x \frac{df}{ds}\frac{du_\alpha}{ds} ds + \int_0^x d\frac{df}{ds} u_\alpha = \frac{df}{ds}(x)u_\alpha(x) - \frac{df}{ds}(0+).$$

Since the left hand side converges to $\mathcal{N}(f)$ as $x \to \infty$, the finite limit

$$c = \lim_{x\to\infty} u_\alpha(x) \cdot \frac{df}{ds}(x)$$

exists and hence it suffices to prove $c = 0$.

Since ∞ is assumed to be non-regular, either m or s diverges near ∞. Suppose m diverges near ∞. Then $\lim_{x\to\infty} u_\alpha(x) = 0$ because u_α is non-increasing in x and m-integrable. If c were not 0, then $\frac{df}{ds}$ diverges near ∞ violating the property that $f \in (\mathcal{F}^0)^{\text{ref}}$. Next suppose s diverges near ∞. Then the same property of f implies $\lim_{x\to\infty} \frac{df}{ds}(x) = 0$ and we get $c = 0$. □

Keeping the above observation in mind, we now consider a finite number of disjoint rays ℓ_i, $i = 1, \ldots, k$, on $\mathbb{R}^2$ merging at a point $a \in \mathbb{R}^2$. Each ray ℓ_i is homeomorphic to the open half line $(0, \infty)$ and the point a is the boundary of each ray at 0-side. We put

$$E_0 = \sum_{i=1}^k \ell_i, \quad E = E_0 \cup \{a\}.$$

E is endowed with the induced topology as a subset of $\mathbb{R}^2$.

Let m be a positive Radon measure on E such that $\mathrm{Supp}[m] = E$ and $m(\{a\}) = 0$. The restriction of m to ℓ_i is denoted by m^i. For any function g on E_0, its restriction to ℓ_i will be denoted by g^i. We consider a diffusion process $X^0 = \{X_t^0, \zeta^0, P_x^0\}$ on E_0 such that its restriction $X^{0,i}$ to each open half line $\ell_i \sim (0, \infty)$ is the absorbing diffusion governed by the speed measure m^i and a canonical scale, say s^i, which is assumed to satisfy

$$s^i(0+) > -\infty, \quad 1 \leq i \leq k.$$

Since $m^i((0, 1)) < \infty,\ 1 \leq i \leq k$, 0 is a regular boundary for each $X^{0,i},\ 1 \leq i \leq k$. As was explained in the last part of Sect. 1, each $X^{0,i}$ then satisfies condition (1.5). We shall also assume that ∞ is non-regular for each $X^{0,i},\ 1 \leq i \leq k$.

Therefore X^0 meets all conditions A.1, A.2, A.3 and A.4 imposed in [14, Theorem 4.1], which guarantees the construction by a darning of a unique m-symmetric diffusion X on E with no sojourn nor killing at a extending X^0 (here we are not a priori given a process on E whose part on E_0 is X^0, so that Theorem 3.1 of Sect. 3 is not applicable to X^0).

Denote by $(\mathcal{E}^{0,i}.\mathcal{F}^{0,i})$ the Dirichlet form of $X^{0,i}$ on $L^2(\ell_i; m^i)$ and by $((\mathcal{F}^{0,i})^{\mathrm{ref}}, \mathcal{E}^{\mathrm{ref},i})$ the reflected Dirichlet space of $\mathcal{F}^{0,i},\ 1 \leq i \leq k$. From (1.6) and (1.7), we then have

$$(\mathcal{F}^{0,i})^{\mathrm{ref}} = \{v : v \text{ is absolutely continuous with respect to } s^i \text{ and } \mathcal{E}^{\mathrm{ref},i}(u, u) < \infty\},$$

where

$$\mathcal{E}^{\mathrm{ref},i}(v, v) = \int_0^\infty \left(\frac{dv(x)}{ds^i(x)}\right)^2 ds^i(x), \tag{5.19}$$

$$\mathcal{F}^{0,i} = \left\{v \in (\mathcal{F}^{0,i})^{\mathrm{ref}} \cap L^2((0, \infty); m^i) : \ v(0+) = 0\right\} \tag{5.20}$$

and

$$\mathcal{E}^{0,i}(v_1, v_2) = \mathcal{E}^{\mathrm{ref},i}(v_1, v_2) \quad \text{for } v_1, v_2 \in \mathcal{F}^{0,i}.$$

Now let $(\mathcal{E}, \mathcal{F})$ be the Dirichlet form on $L^2(E : m)$ of the extended diffusion X. Since X^0 has the properties **(D.1), (D.2)** of Sect. 4 with $E_{0i} = \ell_i,\ 1 \leq i \leq k$, we have from (4.31) and (4.32)

$$\mathcal{F} = \{f = f_0 + c\,u_\alpha : \ f_0^i \in \mathcal{F}^{0,i},\ 1 \leq i \leq k,\ c \in \mathbb{R}\}(\subset \mathcal{F}^{\mathrm{ref}})$$

$$\mathcal{E}(u, v) = \sum_{i=1}^k \mathcal{E}^{\mathrm{ref},i}(u^i,\ v^i) \quad \text{for } u, v \in \mathcal{F},$$

where $\mathcal{E}^{\mathrm{ref},i}$ and $\mathcal{F}^{0,i}$ are specified by (5.19) and (5.20), respectively.

The entrance law $\{\mu_t, t > 0\}$ from a for X is the sum of its restriction μ_t^i to ℓ_i which is describable as

$$\mu_t^i(f)dt = \mathbf{P}^{0,i}_{f \cdot m^i}\left(X^{0,i}_{\zeta^{0,i}-} = 0 \text{ and } \zeta^{0,i} \in dt\right). \tag{5.21}$$

We next choose any k-vector $\mathbf{p} = (p_1, \ldots, p_k)$ with positive entries and define a new measure $\widetilde{m}$ on E_0 by

$$\widetilde{m}^i = p_i m^i \quad 1 \leq i \leq k,$$

which is extended to E by setting $\widetilde{m}(\{a\}) = 0$. Since X^0 is also $\widetilde{m}$-symmetric, we can construct a unique $\widetilde{m}$-symmetric diffusion $\widetilde{X}$ on E with no sojourn nor killing at a which extends X^0. By virtue of Theorem 4.11(i), the Dirichlet form $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ on $L^2(E; \widetilde{m})$ of $\widetilde{X}$ can be described as follows:

$$\widetilde{\mathcal{F}} = \mathcal{F} = \left\{ f = f_0 + c\, u_\alpha :\ f_0^i \in \mathcal{F}^{0,i} \text{ for } 1 \leq i \leq k, \text{ and } c \in \mathbb{R} \right\},$$
$$\widetilde{\mathcal{F}}_e = \mathcal{F}_e = \left\{ f = f_0 + c\, \varphi :\ f_0^i \in \mathcal{F}_e^{0,i} \text{ for } 1 \leq i \leq k, \text{ and } c \in \mathbb{R} \right\},$$
$$\widetilde{\mathcal{E}}(u, v) = \sum_{i=1}^k p_i\ \mathcal{E}^{\text{ref},i}(u^i, v^i) \quad \text{for } u, v \in \widetilde{\mathcal{F}}_e.$$

Observe that f_0 and u_α are $\widetilde{\mathcal{E}}_\alpha$-orthogonal for each $\alpha > 0$, and f_0 and φ are $\widetilde{\mathcal{E}}$-orthogonal.

Let $\widetilde{\mathcal{A}}$ be the $L^2(E; \widetilde{m}_0)$-infinitesimal generator of $\widetilde{X}$. Combining Theorem 4.11(ii) with (1.10), (1.12) and Lemma 5.3, we can see that $f \in \mathcal{D}(\widetilde{\mathcal{A}})$ if and only if the following conditions are satisfied:

$$f^i \in (\mathcal{F}^{0,i})_a^{\text{ref}},\ \frac{df^i}{ds^i} \text{ is absolutely continuous with respect to } m^i,$$
$$\mathcal{L}^i f^i = \frac{d}{dm^i}\frac{d}{ds^i} f^i \in L^2(\ell_i; m^i), \quad 1 \leq i \leq k, \tag{5.22}$$
$$-\infty < f^1(0+) = \cdots = f^k(0+) < \infty, \quad \sum_{i=1}^k p_i \frac{df^i}{ds^i}(0+) = 0. \tag{5.23}$$

We have in this case

$$\widetilde{\mathcal{A}} f(x) = \mathcal{L}^i f^i(x), \quad \text{if } x \in \ell_i, \quad 1 \leq i \leq k.$$

The entrance law $\{\widetilde{\mu}_t, t > 0\}$ from a for $\widetilde{X}$ is given by

$$\widetilde{\mu}_t^i = p_i \mu_t^i, \quad 1 \leq i \leq k,$$

where $\{\mu_t^i, t > 0\}$ is given by (5.21).

Clearly the example in Sect. 5.1 may be considered as a special case of the present one with $k = 2$ and $E = \mathbb{R}$.

5.3 Multidimensional Brownian motions

Let E be an open subset of the Euclidean n-space $\mathbb{R}^n$ and K be a compact subset of E. We let $E_0 = E \setminus K$. We consider the absorbed Brownian motions X and X^0 on E and E_0, respectively. They are symmetric with respect to the Lebesgue measure and X^0 is the part process of X on E_0.

We assume that K is non-polar. Then clearly X and X^0 satisfy all conditions in Theorem 4.1 and hence a unique m_0-symmetric diffusion X^* extending X^0 to $E^* = E_0 \cup \{a\}$ can be constructed by darning the hole K. Here E^* is the one-point extension of E_0 by regarding the set K as a one point a and m_0 is the Lebesgue measure on E_0 extended to E by setting $m_0(\{a\}) = 0$.

We consider the space

$$\mathbb{G}(E_0) = \left\{ u \in L^1_{\text{loc}}(E_0) : \frac{\partial u}{\partial x_i} \in L^2(E_0),\ 1 \le i \le n \right\}$$

and define

$$\mathbf{D}(u, v) = \int_{E_0} \nabla u(x) \cdot \nabla v(x) dx, \quad u, v \in \mathbb{G}(E_0).$$

The completion of $C_0^\infty(E_0)$ in $(\mathbb{G}(E_0), \mathbf{D})$ is denoted by $\mathbb{H}^1_{0,e}(E_0)$. We further let

$$\mathbb{H}^1(E_0) = \mathbb{G} \cap L^2(E_0), \quad \mathbb{H}^1_0 = \mathbb{H}^1_{0,e}(E_0) \cap L^2(E_0).$$

Then the Dirichlet form of X^0 on $L^2(E_0)$ equals $(\frac{1}{2}\mathbf{D}, \mathbb{H}^1_0(E_0))$. The extended and reflected Dirichlet spaces of the latter are $(\mathbb{H}^1_{0,e}(E_0),\ \frac{1}{2}\mathbf{D})$ and $(\mathbb{G}(E_0),\ \frac{1}{2}\mathbf{D})$, respectively (cf. [3]).

We denote by φ and u_α the hitting probability and α-order hitting probability of the set K for X, respectively. In terms of the Brownian motion on $\mathbb{R}^n$, they are the hitting and α-order hitting probabilities of K before leaving the set E. The linear operator $\mathcal{L}$ on $L^2(E_0)$ specified by (4.19) and the flux $\mathcal{N}(f)$ specified by (4.20) are

$$\mathcal{L} = \tfrac{1}{2}\Delta \quad \text{with} \quad \mathcal{D}(\mathcal{L}) = \left\{ f \in \mathbb{H}^1(E_0) : \Delta f \in L^2(E_0) \right\},$$
$$\mathcal{N}(f) = \tfrac{1}{2}\mathbf{D}(f, u_\alpha) + \frac{1}{2}(\Delta f, u_\alpha) \quad \text{for } f \in \mathcal{D}(\mathcal{L}).$$

By Theorem 4.4, the Dirichlet form $(\mathcal{E}^*, \mathcal{F}^*)$ of X^* on $L^2(E^*; m_0)$ and its extended Dirichlet space $\mathcal{F}^*_e$ can be expressed as follows:

$$\mathcal{F}^* = \{f = f_0 + cu_\alpha : f_0 \in \mathbb{H}^1_0(E_0),\ c \in \mathbb{R}\}\ (\subset \mathbb{H}^1(E_0)),$$
$$\mathcal{F}^*_e = \{f = f_0 + c\varphi : f_0 \in \mathbb{H}^1_{0,e}(E_0),\ c \in \mathbb{R}\}\ (\subset \mathbb{G}(E_0)),$$
$$\mathcal{E}^*(u, v) = \frac{1}{2}\mathbf{D}(u, v) \quad \text{for } u, v \in \mathcal{F}^*_e.$$

We know that u_α (resp. φ) is $\mathcal{E}^*_\alpha$(resp. $\mathcal{E}^*$)-orthogonal to the space $\mathbb{H}^1_0(E_0)$(resp. $\mathbb{H}^1_{0,e}(E_0)$). By Theorem 4.8, the generator $\mathcal{A}^*$ of X^* on $L^2(E^*; m_0)$ can be characterized as

$$f \in \mathcal{D}(\mathcal{A}^*) \Longleftrightarrow f \in \mathcal{D}(\mathcal{L}),\ f \text{ admits } X^0\text{-fine limit at } a \text{ and } \mathcal{N}(f) = 0.$$
$$\mathcal{A}^* f = \frac{1}{2}\Delta, \quad f \in \mathcal{D}(\mathcal{A}^*).$$

Note that $(\mathcal{E}^*, \mathcal{F}^*)$ is a quasi-regular Dirichlet form on $L^2(E^*; m_0)$ but not a regular Dirichlet form unless every point of ∂K is a regular boundary point of E_0 with respect to the Dirichlet problem for $(\alpha - \frac{1}{2}\Delta)$ on E_0. Therefore we can not construct X^* by using the theory of the regular Dirichlet form in general.

We next consider the case where the closed set K is the complement of a bounded open set $E_0 \subset \mathbb{R}^n$. In this case, $E^* = E_0 \cup \{a\}$ is just the one-point compactification of E_0. The symmetric diffusion X^* extending the absorbed Brownian motion $X^0 = (X^0_t, \zeta^0, \mathbf{P}^0_x)$ on E_0 to E^* has the Dirichlet form $(\mathcal{E}^*, \mathcal{F}^*)$ on $L^2(E^*; m_0)$ expressible as

$$\mathcal{F}^* = \mathbb{H}^1_0(E_0) + \left\{\text{constant functions on } E^*\right\},$$
$$\mathcal{E}^*(w_1, w_2) = \frac{1}{2}\mathbf{D}(f_1, f_2) \quad \text{for } w_i = f_i + c_i \text{ with } f_i \in \mathbb{H}^1_0(E_0) \text{ and } c_i \in \mathbb{R},\ i=1,2,$$

which is a regular, strongly local and irreducible recurrent Dirichlet form as has been studied in [14, Sect. 3]. Hence we can construct the symmetric diffusion X^* on E^* by a direct use of the Dirichlet form theory in this case. The L^2-generator of X^* can be characterized exactly in the same way as the preceding case. However, on account of Lemma 4.9, the flux $\mathcal{N}(f)$ taking part in the lateral condition now has the expression

$$\mathcal{N}(f) = \frac{1}{2}\int_{E_0} \Delta f(x)dx, \quad f \in \mathcal{D}(\mathcal{L}),$$

which can be rewritten by the Green–Gauss formula as

$$-\lim_{j\to\infty} \frac{1}{2}\int_{\partial D_j} \nabla f(\xi)\cdot \mathbf{n_j}(\xi)\sigma_j(d\xi)$$

provided that $f \in C_b^1(E_0)$, where $\{D_j, j \geq 1\}$ is a sequence of open sets with smooth boundaries that increases to E_0, $\mathbf{n_j}$, and σ_j denote the inward normal vector and the surface element for the surface ∂D_j of D_j. In this sense, $\mathcal{N}(f)$ may be interpreted as the flux of the vector field ∇f at a or into K.

Suppose E_0 is the union of a finite number of disjoint bounded open sets $E_{01}, \ldots, E_{0k}$. We choose any k-vector $\mathbf{p} = (p_1, \ldots, p_k)$ and define a measure $\widetilde{m}_0$ on E_0 by

$$\widetilde{m}_0 = \sum_{i=1}^{k} p_i \cdot m_{0i},$$

where m_{0i} denotes the restriction of the Lebesgue measure to E_{0i}. We extend $\widetilde{m}_0$ to E^* by setting $\widetilde{m}_0(\{a\}) = 0$. Then X^0 is still symmetric with respect to $\widetilde{m}_0$ and we can construct a unique $\widetilde{m}_0$-symmetric diffusion $\widetilde{X}^*$ extending X^0 to E^* either by darning the hole K or by using the corresponding Dirichlet form. The entrance law $\widetilde{\mu}_t$ taking part of the darning admits the expression

$$\widetilde{\mu}_t(B)dt = \sum_{i=1}^{k} p_i \int_{B \cap E_{0i}} \mathbf{P}_x^0(\zeta^0 \in dt)dx, \quad B \in \mathcal{B}(E_0)$$

in view of (4.3) and (4.20).

5.4 Multidimensional censored stable processes

In this section, we consider a case where X^0 is of pure jump type and admits no killings inside E_0. A typical example of such a process is a censored stable process on an Euclidean open set studied in [2].

Let D be an open *n-set* in $\mathbb{R}^n$, that is, there exists a constant $C_1 > 0$ such that

$$m(B(x, r)) \geq C_1 r^n \quad \text{for all } x \in D \text{ and } 0 < r \leq 1.$$

Here m is the Lebesgue measure on $\mathbb{R}^n$, $B(x, r) := \{y \in \mathbb{R}^n : |x - y| < r\}$ and $|\cdot|$ is the Euclidean metric in $\mathbb{R}^n$. Note that bounded Lipschitz domains in $\mathbb{R}^n$ are open n-set and any open n-set with a closed subset having zero Lebesgue measure removed is still an n-set. Fix $0 < \alpha < 2$ and an n-set D (which can be disconnected) in $\mathbb{R}^n$. Define

$$W^{\alpha/2,2}(D) := \left\{ u \in L^2(D; dx) : \int_{D \times D} \frac{(u(x) - u(y))^2}{|x - y|^{n+\alpha}} dxdy < \infty \right\},$$

$$\mathcal{E}(u, v) := \mathcal{A}_{n,\alpha} \int_{D \times D} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+\alpha}} dxdy \quad \text{for } u, v \in \mathcal{F},$$

with $\mathcal{A}_{n,\alpha} = \frac{\alpha 2^{\alpha-1}\Gamma(\frac{\alpha+n}{2})}{\pi^{n/2}\Gamma(1-\frac{\alpha}{2})}$. When $D = \mathbb{R}^n$, $(\mathcal{E}, W^{\alpha/2,2}(\mathbb{R}^n))$ is just the Dirichlet form on $L^2(\mathbb{R}^n, dx)$ of the symmetric α-stable process on $\mathbb{R}^n$.

We refer the reader to [2] for the following facts. The bilinear form $(\mathcal{E}, W^{\alpha/2,2}(D))$ is a regular irreducible Dirichlet form on $L^2(\overline{D}; 1_D(x)dx)$ and the associated Hunt process X on $\overline{D}$ may be called a *reflected α-stable process*. It is shown in [9] that X has Hölder continuous transition density functions with respect to the Lebesgue measure dx on $\overline{D}$ and therefore X can be refined to start from every point in $\overline{D}$. X admits no killing inside $\overline{D}$. Further, X admits no jump from D to ∂D nor from ∂D to ∂D.

The subprocess $X^0 = (X_t^0, \mathbf{P}_x^0, \zeta^0)$ of X killed upon hitting ∂D is called the *censored α-stable process* in D, which has been studied in details in [2]. The process X^0 is symmetric with respect to the Lebesgue measure and its Dirichlet form on $L^2(D, dx)$ is given by $(\mathcal{E}, W_0^{\alpha/2,2}(D))$, where $W_0^{\alpha/2,2}(D)$ is the closure of $C_c^\infty(D)$ in $\mathcal{F}$ with respect to $\mathcal{E}_1 := \mathcal{E} + (\cdot,\cdot)_{L^2(D,dx)}$. Note that the censored stable process X^0 has no killings inside D. The extended Dirichlet form of X^0 is given by $(\mathcal{E}, W_{0,e}^{\alpha/2,2}(D))$, where $W_{0,e}^{\alpha/2,2}(D)$ is the $\mathcal{E}$-closure of $C_c^\infty(\mathbb{R}^n)$.

Let $\tau_D := \inf\{t > 0 : \ X_t \notin D\}$. Note that for $\beta > 0$, $u_\beta(x) := \mathbf{E}_x\left[e^{-\beta\tau_D}\right]$ is a β-harmonic function of X^0 and so it is continuous on D (see [2, (3.8)]). Any bounded measurable function f on D is extended to $\overline{D}$ by defining $f(x) = 0$ on ∂D. By [9], $G_\alpha f(x) := \mathbf{E}_x\left[\int_0^\infty e^{-\beta t} f(X_t)dt\right]$ is a continuous function on $\overline{D}$. Applying strong Markov property of X at its first exit time τ_D from D, we have for $G_\beta^0 f(x) := \mathbf{E}_x\left[\int_0^{\tau_D} e^{-\beta t} f(X_t)dt\right]$,

$$G_\beta^0 f(x) = G_\beta f(x) - \mathbf{E}_x\left[e^{-\beta\tau_D} G_\beta f(X_{\tau_D})\right] \quad \text{for } x \in D.$$

Since $x \mapsto \mathbf{E}_x\left[e^{-\beta\tau_D} G_\beta f(X_{\tau_D})\right]$ is a β-harmonic function of X^0 and thus it is continuous on D, we conclude that $G_\beta^0 f$ is continuous on D. Hence the condition (**C.2**) in Sect. 3 is always satisfied for censored α-stable process X^0 in any open n-set D. In view of [13, Sect. 5.3], a Lévy system of X^0 is given by $(N(x, dy), dt)$ with

$$N(x, dy) = 2\mathcal{A}_{n,\alpha}\, |x - y|^{-(n+\alpha)} dy$$

and the condition (**C.3**) of Sect. 3 is clearly satisfied.

Note that if D_1 is an open subset of $\overline{D}$, then X and its subprocess killed upon leaving D_1 have the same class of m-polar sets in D_1. If a closed set $\Gamma \subset \partial D$ Hausdorff measure when $n \geq 2$ and is non-empty when $n = 1$, then by [2, Theorem 2.5 and Remark 2.2(i)]

$$\varphi_\Gamma(x) := \mathbf{P}_x(\sigma_\Gamma < \sigma_{\partial D\setminus\Gamma}) > 0, \quad \text{for every } x \in D \tag{5.24}$$

if and only if $\alpha > n - d$ when $n \geq 2$ and $\alpha > 1$ when $n = 1$.

In the following $D \subset \mathbb{R}^n$ is a proper open n-set, Γ is a closed subset of ∂D that satisfies the Hausdorff dimensional condition preceding (5.24) with $\alpha > n - d$ when $n \geq 2$ and $\alpha > 1$ when $n = 1$. Let $D^* = D \cup \{a\}$ be the topological space obtained from $D \cup \Gamma$ by regarding Γ as the one point $\{a\}$ in the way prescribed in Sect. 3. We consider the extensions of the censored stable process X^0 to D^* in the following three cases separately.

(i) D is an open n-set, $\Gamma = \partial D$, ∂D is compact, and $\alpha \in (n-d, n)$. When D is bounded, D^* is just the one point compactification of D. We now apply Theorem 4.1 to the case that $E = \overline{D}$, $K = \partial D$. By the above mentioned properties of the reflected stable process X on $\overline{D}$, it clearly satisfies conditions **(B.1)**, **(B.3)** and **(B.5)** of Sect. 3. Since ∂D is compact, the first half of **(B.2)** is also clear. Note that $\varphi(x) := \varphi_{\partial D}(x) = 1$ on D with D is bounded, and $0 < p_{\partial D} < 1$ on D when D is unbounded with compact boundary. Hence the second half of **(B.2)** is also satisfied. By Theorem 4.1, we can construct a unique symmetric extension X^* on D^* of X^0 by darning the hole ∂D. Let $u_1 := \mathbf{E}_x\left[e^{-\tau_D}\right]$. It follows from Theorem 4.4 that the Dirichlet form $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ and its extended Dirichlet form $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}}_e)$ is given by

$$\widetilde{\mathcal{F}} = \left\{ f = f_0 + cu_1 : \ f_0 \in W_0^{\alpha/2,\,2}(D) \text{ and } c \in \mathbb{R} \right\},$$
$$\widetilde{\mathcal{F}}_e = \left\{ f = f_0 + c : \ f_0 \in W_{0,e}^{\alpha/2,\,2}(D) \text{ and } c \in \mathbb{R} \right\},$$
$$\widetilde{\mathcal{E}}(f, g) = \mathcal{A}_{n,\alpha} \int_{D\times D} \frac{(f(x) - f(y))(g(x) - g(y))}{|x-y|^{n+\alpha}}\, dxdy \quad \text{for } f, g \in \widetilde{\mathcal{F}}_e,$$

(ii) D is an n-open set having disconnected boundary ∂D. A prototype is a bounded domain D with one or several holes in its interior. Suppose that $\partial D = \Gamma \cup \Gamma_2$, where Γ and Γ_2 are non-trivial disjoint open subsets of ∂D, with Γ being compact and satisfying the Hausdorff dimensional condition preceding (5.24) and $\alpha \in (n-d, n)$. In this case, $0 < \varphi_\Gamma(x) \le 1$ for $x \in D$. We apply Theorem 4.1 to the case that $E = D \cup \Gamma$, $K = \Gamma$. Let $X^{(1)}$ be the subprocess of X killed upon hitting Γ_2. $X^{(1)}$ lives on the state space $D \cup \Gamma$ and X^0 can be regarded as its part process on D. Since the Dirichlet form of $X^{(1)}$ is regular as the part of the regular Dirichlet form of X on the open set $\overline{D} \setminus \Gamma_2$ (cf. [13, Sect. 4.4]), $X^{(1)}$ satisfies condition **(B.5)** of Sect. 3. In fact, the Dirichlet form of $X^{(1)}$ is given by $(\mathcal{E}, W_{\Gamma_2}^{\alpha/2,\,2}(D))$, where

$$W_{\Gamma_2}^{\alpha/2,\,2}(D) := \left\{ u \in W^{\alpha/2,\,2}(D) : \ u = 0 \ \mathcal{E}\text{-q.e. on } \Gamma_2 \right\} = \overline{C_c^\infty(\overline{D} \setminus \Gamma_2)}.$$

Other conditions of Theorem 4.1 are also readily verifiable for $X^{(1)}$ as in the case of **(i)**. Hence we can obtain the unique symmetric extension X^* on D^* of X^0 by darning the hole Γ. Let $\varphi(x) := \varphi_\Gamma = \mathbf{P}_x\left(\sigma_\Gamma < \sigma_{\Gamma_2}\right)$ and $u_1(x) := \mathbf{E}_x\left[e^{-\sigma_\Gamma};\ \sigma_\Gamma < \sigma_{\Gamma_2}\right]$. It follows from Theorem 4.4 that the Dirichlet form $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ and its extended Dirichlet form $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}}_e)$ is given by

$$\widetilde{\mathcal{F}} = \left\{ f = f_0 + cu_1 : \ f_0 \in W_0^{\alpha/2,2}(D) \text{ and } c \in \mathbb{R} \right\},$$
$$\widetilde{\mathcal{F}}_e = \left\{ f = f_0 + c\varphi : \ f_0 \in W_{0,e}^{\alpha/2,2}(D) \text{ and } c \in \mathbb{R} \right\},$$
$$\widetilde{\mathcal{E}}(f,g) = \mathcal{A}_{n,\alpha} \int_{D\times D} \frac{(f(x)-f(y))(g(x)-g(y))}{|x-y|^{n+\alpha}} dxdy \quad \text{for } f, g \in \widetilde{\mathcal{F}}_e,$$

(iii) $\alpha > 1 = n$, $D = (0,\infty)$ and $\Gamma = \{0\}$. In this case $\varphi(x) = \varphi_\Gamma(x) = 1$. $D^* = [0,\infty)$. Just as in **(i)**, X^* can be constructed from X^0 by darning the hole $\{0\}$ and X^* coincides with the reflected stable process X on $[0,\infty)$ we started with. This can be seen as follows. Let $u_1(x) = \mathbf{E}_x\left[e^{-\sigma_0}\right]$. It follows from Theorem 4.4 that the Dirichlet form $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ and its extended Dirichlet form $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}}_e)$ is given by

$$\widetilde{\mathcal{F}} = \left\{ f = f_0 + cu_1 : \ f_0 \in W_0^{\alpha/2,2}(D) \text{ and } c \in \mathbb{R} \right\} = W^{\alpha/2,2}(D),$$
$$\widetilde{\mathcal{F}}_e = \left\{ f = f_0 + c : \ f_0 \in W_{0,e}^{\alpha/2,2}(D) \text{ and } c \in \mathbb{R} \right\}$$
$$= \left\{ f \in L^1_{\text{loc}}(\mathbb{R}) : \int_{\mathbb{R}_+\times\mathbb{R}_+} \frac{(f(x)-f(y))^2}{|x-y|^{1+\alpha}} dxdy < \infty \right\}$$
$$\widetilde{\mathcal{E}}(f,g) = \mathcal{A}_{n,\alpha} \int_{\mathbb{R}_+\times\mathbb{R}_+} \frac{(f(x)-f(y))(g(x)-g(y))}{|x-y|^{n+\alpha}} dxdy \quad \text{for } f, g \in \widetilde{\mathcal{F}}_e,$$

Note that given an open n-set with disconnected boundary, extensions in case (i) and (ii) can be different. For example for $D = \{x \in \mathbb{R}^n : 1 < |x| < 2\}$ with $\Gamma := \{x \in \mathbb{R}^n : |x| = 1\}$, the extension process X^* in case (ii) is transient and gets "birth" only when X^0 approaches Γ, while in case (i), the extension process is conservative and gets "birth" when X^0 approaches ∂D.

5.5 Multidimensional non-symmetric diffusions

In this subsection, we apply Theorem 3.1 to give an example of one-point extensions of non-symmetric diffusions in Euclidean domains. This example is mentioned in Sect. 6.2 of [8], where it is promised that details will be given somewhere else.

Let D be a proper domain in $\mathbb{R}^n$ and m be the Lebesgue measure on D. Assume that ∂D is bounded and regular for Brownian motion, or, equivalently, for $\frac{1}{2}\Delta$. Let

$$\mathcal{L} = \frac{1}{2}\nabla\cdot(a\nabla) + b\cdot\nabla + q$$
$$= \frac{1}{2}\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial}{\partial x_j}\Big) + \sum_{i=1}^n b_i \frac{\partial}{\partial x_i} + q,$$

where $a : \mathbb{R}^n \to \mathbb{R}^d \otimes \mathbb{R}^n$ is a measurable, symmetric $(n \times n)$-matrix-valued function which satisfies the uniform elliptic condition

$$\lambda^{-1} I_{n\times n} \leq a(\cdot) \leq \lambda I_{n\times n}$$

for some $\lambda \geq 1$, $b = (b_1, \ldots, b_n) : \mathbb{R}^n \to \mathbb{R}^n$ are measurable functions which could be singular such that $|b|^2 \in \mathbf{K}(\mathbb{R}^n)$ and q is a non-positive measurable function in $\mathbf{K}(\mathbb{R}^n)$ vanishing in a neighborhood of ∂D. Here $\mathbf{K}(\mathbb{R}^n)$ denotes the Kato class functions on $\mathbb{R}^n$. We refer the reader to [10] for its definition. We only mention here that $L^p(\mathbb{R}^n, dx) \subset \mathbf{K}(\mathbb{R}^n)$ for $p > n/2$.

Let $\widehat{q} =: q + \sum_{i=1}^n \frac{\partial b_i}{\partial x_i}$. We assume that $\widehat{q}$ satisfies the condition that

$$\widehat{q} \in \mathbf{K}(\mathbb{R}^n), \quad \widehat{q} \leq 0 \ \text{ on } \ \mathbb{R}^n \quad \text{ and } \quad \widehat{q} = 0 \text{ in a neighborhood of } \partial D.$$

Under the above condition, the Dirichlet form $(\mathcal{E}, \mathcal{F})$ generated by $(C_c^\infty(\mathbb{R}^n), \mathcal{L})$ is regular on $\mathbb{R}^n$ and satisfies the (generalized) sector condition. Let X be the diffusion in $\mathbb{R}^n$ associated with $(\mathcal{E}, \mathcal{F})$, which can start from every point in $\mathbb{R}^n$ (see [10]). It is clear that X has a weak dual diffusion $\widehat{X}$ in $\mathbb{R}^n$ with respect to the Lebesgue measure m on $\mathbb{R}^n$ whose generator is

$$\widehat{\mathcal{L}} = \frac{1}{2} \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\Big(a_{ij} \frac{\partial}{\partial x_j}\Big) - \sum_{i=1}^n b_i \frac{\partial}{\partial x_i} + \widehat{q},$$

the dual operator of $\mathcal{L}$ on $\mathbb{R}^n$. As $(\mathcal{F}, \mathcal{E})$ satisfies the sector condition, it follows from [27] that every semi-polar is m-polar for X; that is, the condition **(B.4)** of Sect. 3 is satisfied.

Let X^0 and $\widetilde{X}^0$ be the subprocess of X and $\widehat{X}$, respectively, killed upon leaving D.

Let Γ be a closed subset of ∂D that is non-polar with respect to X and $\widehat{X}$. Let $D^* = D \cup \{a\}$ be the topological space obtained from $D \cup \Gamma$ by identifying Γ as the one-point $\{a\}$ in a way described in §3. See §5.4 for three possible scenario for Γ.

Observe that conditions **(B.1), (B.2), (B.3)** and **(B.5)** as well as their dual conditions of Sect. 3 are trivially satisfied. The conditions **(C.2), (C.4)** and their dual ones of Sect. 3 are satisfied by [10, Lemma 5.7 and Theorem 5.11]. Thus we can apply Theorem 3.1 to get a weak duality preserving diffusion extension X^* of X^0 to $D^* := D \cup \{a\}$.

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6 Appendix: inside killings allowed

Let E be a locally compact separable metric space and E_Δ be its one point compactification. When E is compact, Δ is added as an isolated point. Fix a non-isolated point $a \in E$ and put $E_0 = E \setminus \{a\}$. Let m_0 be a σ-finite measure on E_0 with $\mathrm{Supp}[m_0] = E_0$. The measure m_0 is extended to a measure m on E by setting $m(\{a\}) = 0$.

When studying the one-point extension of a pair of Markov processes X^0 and $\widehat{X}^0$ on E_0 in weak duality, it is assumed in Sects. 4 and 5 of Chen–Fukushima–Ying [8] that

X^0 and $\widehat{X}^0$ admit no killings inside E_0. In fact this condition can be much weakened. In this Appendix, we indicate how the results in [8] can be extended to allow killings of X^0 and $\widehat{X}^0$ inside E_0 and what modifications should be made to allow this extension, which broadens the applicability of [8] considerably.

First, we note that results Sects. 2 and 3 of [8] hold for a pair of processes in weak duality, which *may have* killings inside E_0. We also note that results in [7] hold without no killings assumption.

In Sect. 4 of [8], it is assumed that X^0 and $\widehat{X}^0$ admits no killings inside E_0. But this assumption can be dropped as follows. Replace the condition **(A.2)** in [8, Sect. 4] by
(A.2)$'$ X^0 and $\widehat{X}^0$ are approachable to $\{a\}$:

$$\mathbf{P}^0_x\left(\zeta^0<\infty,\ X^0_{\zeta^0-}=a\right)>0 \ \text{ and } \ \widehat{\mathbf{P}}^0_x\left(\widehat{\zeta}^0<\infty,\ \widehat{X}^0_{\widehat{\zeta}^0-}=a\right)>0 \quad \text{for every } x\in E_0, \tag{6.1}$$

Here for a Borel set $B\subset E$, the notation "$X^0_{\zeta^0-}\in B$" means that the left limit of X^0_t at $t=\zeta^0$ exists under the topology of E and takes values in $B\subset E$. We use the same convention for $\widehat{X}$.

However we need to add the following condition **(5)** to the list of conditions **(1)–(4)** for extension process X and $\widehat{X}$ appeared before Proposition 4.1 in [8].
(5) X and $\widehat{X}$ admit no jumps from E_0 to a.

When X^0 and $\widehat{X}^0$ have no killings inside E_0, condition **(5)** is a consequence of the assumptions, as is proved in Proposition 4.1(ii) of [8]. When we allow X^0 and $\widehat{X}^0$ to have killings inside E_0, we have to impose **(5)**. In this case, we remove (ii) from Proposition 4.2 of [8].

We can also weaken the assumption that "X^0 and $\widehat{X}^0$ admit no killings inside E_0" in Sect. 5 of [8]. Below are the change of the condition we can make and the corresponding modifications of the definition and statement we need to make.

Replace condition (**A.2**) of [8, Sect. 5] by
(A.2a)] X^0 and $\widehat{X}^0$ satisfy, for every $x\in E_0$,

$$\mathbf{P}^0_x\left(\zeta^0<\infty,\ X^0_{\zeta^0-}=a\right)>0,$$

$$\mathbf{P}^0_x\left(\zeta^0<\infty,\ X^0_{\zeta^0-}\in\{a\}\cup(E_\Delta\setminus U_0)\right)=\mathbf{P}^0_x(\zeta^0<\infty), \tag{6.2}$$

$$\widehat{\mathbf{P}}^0_x\left(\widehat{\zeta}^0<\infty,\ \widehat{X}^0_{\widehat{\zeta}^0-}=a\right)>0,$$

$$\widehat{\mathbf{P}}^0_x\left(\widehat{\zeta}^0<\infty,\ \widehat{X}^0_{\widehat{\zeta}^0-}\in\{a\}\cup(E_\Delta\setminus U_0)\right)=\widehat{\mathbf{P}}^0_x(\widehat{\zeta}^0<\infty), \tag{6.3}$$

for some neighborhood U_0 of a in E.
Here, as in Sect. 4 of [8], for a Borel set $B\subset \mathbf{E}_\Delta$, the notation "$X^0_{\zeta^0-}\in B$" means that the left limit of $t\mapsto X^0_t$ at $t=\zeta^0$ exists under the topology of E_Δ and takes values in B.

Accordingly we need to replace the definition of the spaces W in (5.11) and W_a in (5.23) of [8] by

$$W=\left\{w\in W':\ \text{if}\ \ \zeta(w)<\infty\ \text{then}\ w(\zeta(w)-):=\lim_{t\uparrow\zeta(\omega)} w(t)\in\{a\}\cup(E_\Delta\setminus U_0)\right\},\tag{6.4}$$

and

$$\begin{aligned}W_a=\{w:\ &\text{a cadlag function from } [0,\zeta(w)) \text{ to } E \text{ for some } \zeta(w)\in(0,\infty]\\ &\text{with } w(0)=a,\ w(t)\in E_0 \text{ for } t\in(0,\zeta(w)) \text{ and } w(\zeta(w)-)\\ &\in\{a\}\cup(E_\Delta\setminus U_0) \text{ if } \zeta(w)<\infty\},\end{aligned}\tag{6.5}$$

respectively.

We also need to replace Theorem 5.15(i) of [8] by the following statement:

(i) X is a right process on E. Its sample path $\{X_t,\ 0\le t<\zeta\}$ is cadlag on $[0,\infty)$, continuous when $X_t=a$ and satisfies

$$X_{\zeta-}\in\{a\}\cup(E_\Delta\setminus U_0)\quad\text{when}\quad \zeta<\infty$$

and it admits no jumps from E_0 into a.

According to the above change of the condition, we replace the inclusion (5.22) of [8] by

$$\big\{\zeta<\infty,\ w(\zeta-)\in E_\Delta\setminus U_0\big\}\subset\{\tau_U<\zeta\}$$

holding for a neighborhood U of a with $\overline{U}\subset U_0$, and then the proof of Lemma 5.5 of [8] goes through. The same modification of the proof works for Lemma 5.11 of [8]. All other arguments in Sect. 5 of [8] leading to Theorem 5.15 remain true with no change.

The condition **(A.2)**$'$ preceding Theorem 5.17 of [8] can be also replaced by the following weaker condition:

(A.2a)$'$ For every $x\in E_0$,

$$\mathbf{P}^0_x\left(\zeta^0<\infty, X^0_{\zeta^0-}=a\right)>0,\quad \mathbf{P}^0_x\left(X^0_{\zeta^0-}\in\{a\}\cup(E_\Delta\setminus U_0)\right)=1,\tag{6.6}$$

$$\widehat{\mathbf{P}}^0_x\left(\widehat{\zeta}^0<\infty, \widehat{X}^0_{\widehat{\zeta}^0-}=a\right)>0,\quad \widehat{\mathbf{P}}^0_x\left(\widehat{X}^0_{\widehat{\zeta}^0-}\in\{a\}\cup(E_\Delta\setminus U_0)\right)=1.\tag{6.7}$$

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