Flux and lateral conditions for symmetric Markov processes

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Abstract

The purpose of this paper is to give an affirmative answer at infinitesimal generator level to the 40 years old Feller's boundary problem for symmetric Markov processes with general quasi-closed boundaries. For this, we introduce a new notion of flux functional, which can be intrinsically defined via the minimal process X^0 in the interior. We then use it to characterize the L^2-infinitesimal generator of a symmetric process that extends X^0. Special attention is paid to the case when the boundary consists of countable many points possessing no accumulation points.

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1 Introduction

Let E be a Lusin space and m a σ-finite measure on it. Throughout this paper, we let X be an irreducible m-symmetric right process on E, F be a non-m-polar, quasi-closed subset of E and X^0 be the subprocess of X killed upon leaving $E_0 := E \setminus F$. The subprocess X^0 is then symmetric with respect to the measure $m_0 := m|_{E_0}$. We assume that

$$X \text{ admits no jumps from } E_0 \text{ to } F. \tag{1.1}$$

One can view X as a general symmetric extension of X^0 from E_0 to $E = E_0 \cup F$. A natural question then arises:

$$\text{How can we characterize } X \text{ through quantities that are intrinsic to } X^0? \tag{1.2}$$

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This question belongs to the *boundary problem of Markov processes*, dating back to W. Feller [9] where the problem was raised after his discovery of the most general boundary conditions for the one-dimensional diffusions [8]. We studied this problem (1.2) in [3] at the infinitesimal generator level when F is a single point and in [4] at the resolvent level when F is a countable set. In particular, in Section 4 of [3], we have answered question (1.2) in the special case that F consists of only one point a by characterizing the L^2-generator of X by means of a lateral condition described in terms of *flux* of X^0 at the point a. We have seen in [3, §4] that the results in [5] enable us to define the notion of the flux using the *reflected Dirichlet space* of X^0, which is intrinsically determined by X^0 under the condition (1.1).

The aim of the present paper is to give an answer to question (1.2) at the generator level for the general case where F is a quasi-closed set. In describing the lateral condition on the L^2-generator $\mathcal{A}$ of X, we will introduce a new notion of the *flux functional* defined in terms of the reflected Dirichlet space of X^0.

In §2 and §3 of this paper, notions of reflected and active reflected Dirichlet spaces for X^0 will be reviewed and their relations to the Dirichlet space of X will be investigated by reformulating and further extending the related results in [5] and [16]. In particular, we shall present an explicit description of the orthogonal decomposition of the active reflected Dirichlet space with respect to the α-order form.

In §4, we shall formulate lateral conditions involving the flux functional to characterize the domain $\mathcal{D}(\mathcal{A})$ of the L^2-generator of X. They only involve the quantities intrinsic to X^0, the restrictions to F of jumping and killing measures of X and the restriction $\mathcal{C}_F$ of a core $\mathcal{C}$ of $(\mathcal{F}, \mathcal{E})$ to F. We shall show in §5 that, when F is a a countable set so that each point in F can be separated from the rest of points in F by a quasi-open set, we can take as $\mathcal{C}_F$ the space of functions on F with finite support, making the lateral conditions to characterize $\mathcal{A}$ completely intrinsic.

Two examples of the multidimensional Brownian motion are given in section 6 to show that the notion of flux introduced in this paper is a genuine extension of the classical one.

We remark here that in [4] we have given an answer to (1.2) at the resolvent level for a Markov process X having a weak dual $\widehat{X}$ and for a countable F. Indeed, we have represented in [4, §2] the resolvent of X explicitly in terms of the Feller measures of X^0 (typical intrinsic quantities for X^0) and conversely constructed in [4, §3] a duality preserving extensions of X^0 and its dual by a method of darning countable holes. When X^0 is symmetric, the constructed process is symmetric and we can apply to the latter the present characterization of its L^2-generator in terms of the flux. See [4, §4.1] for such an example of the extension.

In sections 4 and 5 of [3], one point skew extensions of X^0 are formulated and examined by changing the symmetrizing measure m_0 by multiplying a different positive constant on each irreducible component of X^0. An the end of section 6 of the present paper, we shall apply our characterization in terms of the flux functional of section 4 to a skew extension X of a one-dimensional absorbed Brownian motion X^0 on $\mathbb{R} \setminus F$ where F is a countable set possessing an accumulation point. The countably many point skew extension X will also be constructed from the one-dimensional Brownian motion by repeating the one-point skew darning discussed in [3, §4].

2 Reflected Dirichlet space of X^0

Let $(\mathcal{E},\mathcal{F})$ be the Dirichlet form on $L^2(E,m)$ associated with the symmetric right process X satisfying the conditions stated in §1. Then $(\mathcal{E},\mathcal{F})$ is irreducible and quasi-regular. In view of the quasi-homeomorphism method in [6], without loss of generality, we may and do assume that E is a locally compact separable metric space, m is a positive Radon measure on E with $\text{supp}[m]=E$, $(\mathcal{E},\mathcal{F})$ is an irreducible regular symmetric Dirichlet form in $L^2(E,m)$, and $X=(X_t,\mathbf{P}_x,\zeta)$ is an m-symmetric Hunt process associated with $(\mathcal{E},\mathcal{F})$. We will use $(\mathcal{E},\mathcal{F}_e)$ to denote the extended Dirichlet space of $(\mathcal{E},\mathcal{F})$ and $\mathcal{E}_1:=\mathcal{E}+(\cdot\,,\,\cdot)_{L^2(E,m)}$. The expectation with respect to the probability measure $\mathbf{P}_x$ will be denoted as $\mathbf{E}_x$. Throughout this paper, we use the convention that any function defined on E is extended to $E_\partial:=E\cup\{\partial\}$ by taking value 0 at the cemetery point ∂.

For reader's convenience, let's recall the following definitions from [15] and [11].

Definition 2.1 (i)An increasing sequence of closed sets $\{F_n\}_{n\geq 1}$ of E is an *$\mathcal{E}$-nest* if and only if $\cup_{n\geq 1}\mathcal{F}_{F_n}$ is $\mathcal{E}_1$-dense in $\mathcal{F}$, where $\mathcal{E}_1=\mathcal{E}+(\ ,\ )_{L^2(E,m)}$ and

$$\mathcal{F}_{F_n}:=\{u\in\mathcal{F}:\ u=0\ m\text{-a.e. on } E\setminus F_n.$$

(ii) A subset $N\subset E$ is *$\mathcal{E}$-polar* if and only if there is an $\mathcal{E}$-nest $\{F_n\}_{n\geq 1}$ such that $N\subset\cap_{n\geq 1}(E\backslash F_n)$.

(iii) A function f on E is said *$\mathcal{E}$-quasi-continuous* if there is an $\mathcal{E}$-nest $\{F_n\}_{n\geq 1}$ such that $f|_{F_n}$ is continuous on F_n for each $n\geq 1$, which is denoted in abbreviation by $f\in C(\{F_n\})$.

(iv) A statement depending on $x\in A$ is said to hold *$\mathcal{E}$-quasi-everywhere* ($\mathcal{E}$-q.e. in abbreviation) on A if there is an $\mathcal{E}$-polar set $N\subset A$ such that the statement is true for every $x\in A\setminus N$.

(v) A subset $A\subset E$ is said to be quasi-open (quasi-closed) if there is an $\mathcal{E}$–nest $\{F_k\}_{k\geq 1}$ such that $F_k\cap A$ is relatively open (relatively closed, respectively) in F_k for each $k\geq 1$.

It is known (cf. [11] that a set $A\subset E$ is $\mathcal{E}$-polar if and only of $\text{Cap}(A)=0$, where Cap is the 1-capacity associated with the Dirichlet form $(\mathcal{E},\mathcal{F})$. It is also known that It is known that every element u in $\mathcal{F}_e$ admits a quasi-continuous version. We assume throughout this section that functions in $\mathcal{F}_e$ are always represented by their quasi-continuous versions. In the sequel, the abbreviations CAF, PCAF and MAF stands for "continuous additive functional", "positive continuous additive functional" and "martingale additive functional", respectively, whose definitions can be found in [11].

Since the quasi-closedness is invariant under the quasi-homeomorphism, we may further assume that the set F is a quasi-closed subset of E having positive 1-capacity with respect to $(\mathcal{E},\mathcal{F})$ and satisfying condition (1.1). Put $E_0=E\setminus F$ and let $X^0=(X^0_t,\mathbf{P}^0_x,\zeta^0)$ be the subprocess of X killed upon leaving E_0, which is known to be an m_0-symmetric special standard Markov process on E_0, where $m_0:=m|_{E_0}$. The Dirichlet form of X^0 on $L^2(E_0;m_0)$ is denoted by $(\mathcal{E}^0,\mathcal{F}^0)$ which is described as

$$\mathcal{F}^0=\{u\in\mathcal{F}:\ u=0\ \text{ q.e. on } F\}\qquad\text{and}\qquad \mathcal{E}^0=\mathcal{E}\big|_{\mathcal{F}^0\times\mathcal{F}^0}.$$

This Dirichlet form $(\mathcal{E}^0,\mathcal{F}^0)$ is known to be quasi-regular and transient ([5, Lemma 2.2]). When restricted on E_0, the $\mathcal{E}^0$-notions coincide with $\mathcal{E}$-notions. For example, a set $A\subset E_0$ is $\mathcal{E}$-polar if and only if it is $\mathcal{E}^0$-polar. See [5, Lemma 2.2] for details. In the sequel, for simplicity, we will drop the prefix "$\mathcal{E}$-" or "$\mathcal{E}^0$-" from $\mathcal{E}$-q.e. or $\mathcal{E}^0$-q.e., etc., when it is clear from the context.

One can view X as a most general symmetric extension of X^0. To characterize the L^2-generator of X in terms of the quantities intrinsic to X^0, it is useful to consider a universal extension of $(\mathcal{E}^0, \mathcal{F}^0)$ called the reflected Dirichlet space.

The notion of the reflected Dirichlet space of a transient regular Dirichlet space $\mathcal{F}^0$ was first introduced by M. L. Silverstein in [16] and [17] in two different ways, which were later on made precise and shown to be equivalent in [2] by the first author of the present paper. The first way is to add to $\mathcal{F}^0$ *the space of all harmonic functions on* E_0 *having finite Dirichlet integrals* by using the equilibrium measures ([17] and [2]) or the energy functional ([5] and [3]), while the second way is to consider *the space of all functions on* E_0 *with finite Dirichlet integrals* by using the energy measures of $u \in \mathcal{F}_{\rm loc}$ (see [16] and [2]). Note that the results in [2] and [16]-[17] are applicable here for our quasi-regular Dirichlet form $(\mathcal{E}, \mathcal{F}^0)$ due to its quasi-homeomorphism (see [6]) to a transient regular Dirichlet form on a locally compact metric space.

We now recall the definitions, with some improvements over those in [5] and [3]. In particular, we will not apply a quasi-homeomorphism to X^0 and $(\mathcal{E}^0, \mathcal{F}^0)$ rendering them into a Hunt process and a regular Dirichlet form as we have done in [3].

We first introduce some notions related to the standard process $X^0 = \{X^0_t, \mathbf{P}^0_x, \zeta^0\}$ on E_0. We will use the convention that $X^0_\infty = \partial$ and any function f on E_0 is extended to $E_0 \cup \{\partial\}$ by setting $f(\partial) = 0$. Let $(\Omega, \{\mathcal{G}^0_t\}_{0\le t\le\infty})$ be the filtered sample space of X^0 where $\mathcal{G}^0_\infty$ (resp. $\mathcal{G}^0_t$) is the σ-algebra generated by $\{X^0_s : 0 \le s < \infty\}$ (resp. $\{X^0_s : 0 \le s \le t\}$.) For a probability measure μ on $E_0 \cup \partial$, we denote by $\mathcal{G}^\mu_\infty$ (resp. $\mathcal{G}^\mu_t$) the $\mathbf{P}^0_\mu$-completion of $\mathcal{G}^0_\infty$ (resp. $\mathcal{G}^0_t$ in $\mathcal{G}^\mu_\infty$,)

A nearly Borel set $A \subset E_0$ is called X^0-invariant if $\mathbf{P}^0_x(\Omega_A) = 1$ for every $x \in A$, where

$$\Omega_A = \left\{\omega \in \Omega : X^0_t(\omega) \in A \text{ and } X^0_{t-}(\omega) \in A \text{ for every } t \in [0, \zeta^0)\right\}.$$

Then the restriction $X^0|_A$ defined in a natural way is a standard process on A. The minimum augmented admissible filtration $\{\mathcal{G}^A_t\}_{0\le t\le\infty}$ for this standard process can be described as follows (cf.[11]):

$$\mathcal{G}^A_t = \bigcap_{\mu\in\mathcal{P}(A_\partial)} \mathcal{G}^\mu_t \cap \Omega_A, \qquad 0 \le t \le \infty,$$

where $\mathcal{P}(A_\partial)$ denotes the family of all probability measures carried by $A_\partial := A \cup \{\partial\}$. When $A = E_0$, we write $\mathcal{G}_t$ for $\mathcal{G}^{E_0}_t$.

We say a random variable Φ on Ω is $X^0|_A$-measurable if the restriction $\Phi|_{\Omega_A}$ is measurable with respect to $\mathcal{G}^A_\infty$. The random variable Φ needs not to be defined on $\Omega \setminus \Omega_A$ in this case. Recall that a nearly Borel set $N \subset E_0$ is called X^0-properly exceptional if $E_0 \setminus N$ is X^0-invariant and $m(N) = 0$.

The following result is needed in our definition of terminal random variable and is known to the experts. For completeness, we give a proof here by using quasi-homeomorphism [6] between m_0 symmetric Markov process $(X^0, \mathbf{P}^0_x, \zeta^0)$ and symmetric Hunt process on a locally compact separable metric space.

Lemma 2.2 *Let $A := \{\zeta^0 < \infty$ and $X^0_{\zeta^0-}$ exits and takes value in $E_0\}$. Define*

$$\zeta_i^0(\omega) := \begin{cases} \zeta^0(\omega) & \text{if } \omega \in A \\ \infty & \text{if } \omega \notin A \end{cases} \qquad \text{and} \qquad \zeta_p^0(\omega) := \begin{cases} \zeta^0(\omega) & \text{if } \omega \notin A \\ \infty & \text{if } \omega \in A \end{cases}.$$

Then for q.e $x \in E_0$, $\mathbf{P}^0_x$-a.s., ζ_i^0 is a totally inaccessible stopping time with respect to $\{\mathcal{G}_t, t \geq 0\}$ and ζ_p is a predictable stopping time with respect to $\{\mathcal{G}_t, t \geq 0\}$. The stopping times ζ_i^0 and ζ_p^0 are called, respectively, the totally inaccessible part and the predictable part of ζ^0.

Proof. Since the process $(X^0, \mathbf{P}^0_x, \zeta^0)$ is m_0 symmetric on E_0, by [15] its associated Dirichlet form $(\mathcal{E}^0, \mathcal{F}^0)$ is quasi-regular on E_0. Thus by [6], $(\mathcal{E}^0, \mathcal{F}^0)$ and the process $(X^0, \mathbf{P}^0_x, \zeta^0)$ is quasi-homeomorphic to a regular Dirichlet form $(\widetilde{\mathcal{E}}^0, \widetilde{\mathcal{F}}^0)$ on a locally compact separable metric space $\widetilde{E}_0$ and its associated Hunt process $(\widetilde{X}^0, \widetilde{\mathbf{P}}^0_x, \widetilde{\zeta}^0)$ on $\widetilde{E}_0$. In fact, if we let Ψ be the quasi-homeomorphism from E_0 to $\widetilde{E}_0$ constructed in [6], then X^0 can be realized as the pullback image of $\widetilde{X}^0$ under Ψ, that is $X^0_t = \Psi^{-1}(\widetilde{X}^0_t)$ for $t < \widetilde{\zeta}^0$ and $\zeta^0 = \widetilde{\zeta}^0$. Since $\widetilde{X}^0$ is a Hunt process in $\widetilde{E}_0$, $\widetilde{X}^0_{\widetilde{\zeta}^0-} \in \widetilde{E}_0 \cup \{\widetilde{\partial}\}$, where $\widetilde{\partial}$ is a one-point compactification of $\widetilde{E}_0$. Define $\widetilde{A} := \left\{\widetilde{\zeta}^0 < \infty \text{ and } X^0_{\widetilde{\zeta}^0-} \in \widetilde{E}_0\right\}$. Clearly

$$\widetilde{\zeta}_i^0 := \widetilde{\zeta}^0 \cdot \mathbf{1}_{\widetilde{A}} + \infty \cdot \mathbf{1}_{\widetilde{A}^c} \qquad \text{and} \qquad \widetilde{\zeta}_p^0 := \widetilde{\zeta}^0 \cdot \mathbf{1}_{\widetilde{A}^c} + \infty \cdot \mathbf{1}_{\widetilde{A}}$$

are the totally inaccessible and predictable parts of $\widetilde{\zeta}^0$, respectively. By quasi-homeomorphism Ψ, it follows that

$$\zeta_i^0 = \zeta^0 \cdot \mathbf{1}_A + \infty \cdot \mathbf{1}_{A^c} \qquad \text{and} \qquad \zeta_p^0 = \zeta^0 \cdot \mathbf{1}_{A^c} + \infty \cdot \mathbf{1}_A$$

are the totally inaccessible and predictable parts of ζ^0, respectively. □

Definition 2.3 (1) We call a random variable $\Phi = \Phi(\omega)$ on Ω a *terminal random variable* of X^0 if there exists an X^0-properly exceptional set $N \subset E_0$ such that

(i) Φ is $X^0|_{E_0 \setminus N}$-measurable;

(ii) for every $\omega \in \Omega_{E_0 \setminus N}$, $\Phi(\theta_t \omega) = \Phi(\omega)$ for every $t < \zeta^0(\omega)$, where θ_t is the shift operator on Ω; and

(iii) $\{\Phi = 0\} \supset \{\zeta_i^0 < \infty\}$, where ζ_i^0 is the inaccessible part of the lifetime ζ^0.

where θ_t is the shift operator on Ω and σ^0_B denotes the hitting time of X^0_t for a set $B \subset E_0$.

(2) A function f on E_0 is called X^0-*harmonic* if, for any quasi open subset D with compact closure in E_0,

$$\mathbf{E}^0_x\left[\left|f(X^0_{\sigma^0_{E_0 \setminus D}})\right|\right] < \infty \quad \text{and} \quad f(x) = \mathbf{E}^0_x\left[f(X^0_{\sigma^0_{E_0 \setminus D}})\right] \qquad \text{for q.e. } x \in E_0,$$

where $\sigma^0_{E_0 \setminus D} := \inf\{t > 0 : X^0_t \in E_0 \setminus D\}$ denotes the first time of $E_0 \setminus D$ by X^0.

Note that X^0 is the subprocess of the Hunt process X killed upon leaving E_0. In view of the proof of Lemma 2.2, condition (iii) of the definition above for a terminal random variable Φ of X^0 implies that for any compact set $K \subset E_0$, $\{\Phi \neq 0\} \subset \left\{\sigma^0_{E_0 \setminus K} < \infty\right\}$ $\mathbf{P}_x$-a.s. on $\Omega_{E_0 \setminus N}$ for any $x \in E_0 \setminus N$.

Denote by $(\mathcal{F}^0_e, \mathcal{E}^0)$ the extended Dirichlet space of $(\mathcal{E}^0, \mathcal{F}^0)$.

Lemma 2.4 (i) *Let Φ be a terminal random variable with $\mathbf{E}^0_x[|\Phi|] < \infty$ for q.e. $x \in E_0$. Then*

$$h(x) := \mathbf{E}^0_x[\Phi], \qquad x \in E_0. \tag{2.1}$$

is X^0-harmonic in E_0. Moreover, for any $\mathcal{E}^0$-nest $\{A_k\}$ consisting of compact sets,

$$\lim_{k\to\infty} h(X^0_{\sigma^0_k}) = \Phi \qquad \mathbf{P}^0_x\text{-a.s. and in } L^1(\mathbf{P}^0_x) \text{ for q.e. } x \in E_0,$$

where $\sigma^0_k = \sigma^0_{E_0 \setminus A_k}$.

(ii) *For any $f \in \mathcal{F}^0_e$ and for stopping times $\{\sigma^0_k, k \geq 1\}$ as in* (i)*,*

$$\lim_{k\to\infty} f(X^0_{\sigma^0_k}) = 0 \qquad \text{in } L^1(\mathbf{P}^0_x) \text{ and in probability } (\mathbf{P}^0_x) \text{ for q.e. } x \in E_0.$$

Proof. (i) That h is X^0-harmonic in E_0 follows immediately from the definition of a terminal random variable and the assumption (1.1) for X. Suppose $\{A_k, k \geq 1\}$ is an $\mathcal{E}^0$-nest consisting of compact sets. Let $\widetilde{\sigma}^0_k = \sigma^0_k \wedge \zeta^0$. By Lemma 2.2 (i) of [5], we know that, for q.e. $x \in E_0$,

$$\lim_{k\to\infty} \widetilde{\sigma}^0_k = \zeta^0 \qquad \mathbf{P}_x\text{-a.s.} \tag{2.2}$$

Let $N \subset E_0$ be a properly exceptional set such that the conditions (i), (ii) and (iii) for the terminal random variable Φ of X^0 and property (2.2) hold on $E_0 \setminus N$ and Φ is $\mathbf{P}_x$-integrability for every $x \in E_0 \setminus N$. By setting $A = E_0 \setminus N$, we then have, for every $x \in A$ and k, $\mathbf{P}^0_x$-a.s.

$$h(X_{\sigma^0_k}) = \mathbf{1}_{\{\sigma^0_k < \zeta^0\}} \mathbf{E}^0_{X_{\widetilde{\sigma}^0_k}}[\Phi] = \mathbf{E}^0_x\left[\Phi \circ \theta_{\widetilde{\sigma}^0_k} \cdot \mathbf{1}_{\{\sigma^0_k < \zeta^0\}} \middle| \mathcal{G}^A_{\widetilde{\sigma}^0_k}\right] = \mathbf{E}^0_x\left[\mathbf{1}_{\{\sigma^0_k < \zeta^0\}} \Phi \mid \mathcal{G}^A_{\widetilde{\sigma}^0_k}\right] = \mathbf{E}^0_x\left[\Phi \mid \mathcal{G}^A_{\widetilde{\sigma}^0_k}\right],$$

By letting $k \to \infty$, we get for any $x \in A$

$$\lim_{k\to\infty} h(X_{\sigma^0_k}) = \mathbf{E}^0_x\left[\Phi \,\middle|\mathcal{G}^A_\infty\right] = \Phi \qquad \mathbf{P}^0_x\text{−a.s. and in } L^1(\mathbf{P}^0_x)$$

because $\sigma\left(\cup_{k=1}^\infty \mathcal{G}^A_{\widetilde{\sigma}^0_k}\right) = \mathcal{G}^A_\infty$ by virtue of (2.2).
(ii) In view of the definition of $\mathcal{E}^0$-nest $\{A_k\}$, the proof is the same as that for [3, Lemma 4.2]. □

We denote by $\{P^0_t, t \geq 0\}$ the transition semigroup of X^0. For functions u, v on E_0, we let $(u, v) := \int_{E_0} u(x)v(x)m_0(dx)$. The X^0-energy functional of an X^0-excessive measure η and an X^0-excessive function u is defined by

$$L^{(0)}(\eta, u) := \lim_{t\downarrow 0} \frac{1}{t}\langle \eta - \eta P^0_t, u\rangle. \tag{2.3}$$

The next lemma can be shown in the same way as the proof of [3, Lemma 4.3] (see also [2, Theorem 1.8] and [16]).

Lemma 2.5 *Let Φ be a terminal random variable with $\mathbf{E}^0_x\left[\Phi^2\right] < \infty$ for q.e. $x \in E_0$. Let $h(x)$ be the function defined by* (2.1) *and define*

$$g(x) = \mathbf{E}^0_x\left[\Phi^2\right] - h(x)^2, \qquad x \in E_0, \tag{2.4}$$

$$Mh(t) = h(X^0_t)\mathbf{1}_{\{t<\zeta^0\}} + \Phi\mathbf{1}_{\{t\geq\zeta^0\}} - h(X^0_0) \qquad t \geq 0.$$

Then g is X^0-excessive, $\{Mh(t)\}_{t\geq 0}$ is a $\mathbf{P}^0_x$-square integrable, uniformly integrable martingale additive functional of X^0 and

$$g(x) = P^0_t g(x) + \mathbf{E}^0_x\left[(Mh(t))^2\right], \qquad t \geq 0, \qquad \text{q.e. } x \in E_0.$$

In particular

$$\frac{1}{2}L^{(0)}(m_0, g) = e(Mh),$$

where $L^{(0)}$ denotes the energy functional of an X^0-excessive measure and an X^0-excessive function defined by (2.3) and $e(A)$ denotes the energy of an additive functional A defined in [11, §5.2].

Now let

$$\mathbf{N} = \{\Phi : \ \text{terminal random variable with } \mathbf{E}^0_x[\Phi^2] < \infty \text{ for q.e. } x \in E_0 \text{ and } L^{(0)}(m_0, g) < \infty\}, \tag{2.5}$$

where g is defined by (2.4) for Φ. The *reflected Dirichlet space* $((\mathcal{F}^0)^{\mathrm{ref}}, \mathcal{E}^{\mathrm{ref}})$ *of* $(\mathcal{F}^0, \mathcal{E}^0)$ are then defined as follows:

$$(\mathcal{F}^0)^{\mathrm{ref}} = \mathcal{F}^0_e + \mathbf{HN}, \tag{2.6}$$

where

$$\mathbf{HN} = \{h : \ h(x) = \mathbf{E}_x[\Phi] \ \text{ for q.e. } x \in E_0 \text{ with } \ \Phi \in \mathbf{N}\}. \tag{2.7}$$

For $f = f_0 + h \in (\mathcal{F}^0)^{\mathrm{ref}}$, where $f_0 \in \mathcal{F}^0_e$ and $h = \mathbf{E}_\cdot[\Phi]$ with $\Phi \in \mathbf{N}$, we let

$$\mathcal{E}^{\mathrm{ref}}(f, f) = \mathcal{E}^0(f_0, f_0) + \tfrac{1}{2}L^{(0)}(m_0, g), \tag{2.8}$$

for g defined by (2.4) for Φ.

We note that the space $\mathcal{F}^0_e \cap \mathbf{HN}$ consists only of the zero function because of Lemma 2.4 and hence the above definition makes sense. (2.8) gives a pre-Hilbertian norm on $(\mathcal{F}^0)^{\mathrm{ref}}$ and, by the polarization, we get $\mathcal{E}^{\mathrm{ref}}(h, f_0) = 0$ for functions h, f_0 as above. On account of Lemma 2.5 and [2, Theorem 1.8], it is also clear that the above definition of the reflected Dirichlet space coincides with the one given in [2] when $(\mathcal{E}^0, \mathcal{F}^0)$ is regular and transient.

The quadratic form $((\mathcal{F}^0)^{\mathrm{ref}}, \mathcal{E}^{\mathrm{ref}})$ is called the *reflected Dirichlet space of* X^0. In the rest of this section, we shall state its basic relationship to the extended regular Dirichlet space $(\mathcal{F}_e, \mathcal{E})$ of the m symmetric Hunt process $X = (X_t, \mathbf{P}_x, \zeta)$ on E by reorganizing the results obtained in [5]. To this end, we need the following notion of Feller measures. The cemetery for X is designated by ∂ and any numerical function φ on E is extended to $E_\partial := E \cup \{\partial\}$ by setting $\varphi(\partial) = 0$ by convention.

Define for $\varphi \in \mathcal{B}(F)_b$,

$$\mathbf{H}\varphi(x) := \mathbf{E}_x\left[\varphi(X_{\sigma_F});\ \sigma_F < \infty\right] \qquad \text{for } x \in E,$$

and $q(x) := 1 - \mathbf{H}1(x) = \mathbf{P}_x(\sigma_F = \infty)$. For φ, $psi \in \mathcal{B}(F)_b^+$, define

$$U(\varphi \otimes \psi) := L^0((\mathbf{H}\varphi) \cdot m_0, \mathbf{H}\psi) \quad \text{and} \quad V(\varphi) := L^0((\mathbf{H}\varphi) \cdot m_0, q). \tag{2.9}$$

Here L^0 is the X^0-energy functional introduced by (2.3).

By Lemma 2.3(i) of [5], U is a symmetric measure on $F \times F$, which will be called the *Feller measure* for F. The measure V on F will be called the *supplementary Feller measure* for F. Note that under condition (1.1), these Feller measures are intrinsically defined by X^0. Indeed we have the identity for $\varphi \in \mathcal{B}(F)_b^+$

$$\mathbf{H}\varphi(x) = \mathbf{E}_x^0 \left[\varphi(X^0_{\zeta^0-});\ X^0_{\zeta^0-} \in F,\ \zeta^0 < \infty \right], \qquad x \in E_0. \tag{2.10}$$

To see this, let $\sigma'_F = \inf\{t > 0 : X_{t-} \in F\}$. Then, in view of [1, p 59], we have $\mathbf{P}_x(\sigma'_F < \sigma_F) = 0$ for any $x \in E$, while (1.1) implies that $\mathbf{P}_x(\sigma_F < \sigma'_F) = 0$ for any $x \in E_0$. Hence (2.10) follows. Clearly (2.10) holds for q.e. $x \in E_0$ for any nearly Borel function φ on F with $\mathbf{H}|\varphi|(x) < \infty$ for q.e. $x \in E$.

We also need to consider the Lévy system $(N(x, dy), H)$ for the m-symmetric Hunt process X on E. The Revuz measure of the PCAF H of X will be denoted as μ_H. We define

$$J(dx, dy) = N(x, dy)\mu_H(dx) \quad \text{and} \quad \kappa(dx) = N(x, \partial)\mu_H(dx) \tag{2.11}$$

as the jumping measure and the killing measure of X (or, equivalently, of $(\mathcal{E}, \mathcal{F})$). We point out that the jumping measure J defined here is twice the jumping measure J defined in [5].

In the following, we will use d to denote the diagonal of $E \times E$. For $u \in \mathcal{F}_e$, $\mu_{\langle u \rangle}$ and $\mu^c_{\langle u \rangle}$ denote the Revuz measures of the PCAFs $\langle M^u \rangle$ and $\langle M^{u,c} \rangle$ of X, respectively. Here M^u is the MAF of X in the Fukushima's decomposition:

$$u(X_t) - u(X_0) = M_t^u + N_t^u, \qquad t \geq 0,$$

where N^u is the CAF of X having zero energy, and $M^{u,c}$ is the continuous martingale part of M^u. Recall that every member u in $\mathcal{F}_e$ is represented by its quasi-continuous version. So in particular, $\mathbf{H}u(x) := \mathbf{E}_x[u(X_{\sigma_F})]$ is well defined for q.e. $x \in E_0$.

Theorem 2.6 *We have the inclusions*

$$\{ (\mathbf{H}u)|_{E_0} : u \in \mathcal{F}_e \} \subset \mathbf{HN} \qquad \text{and} \qquad \mathcal{F}_e|_{E_0} \subset (\mathcal{F}^0)^{\text{ref}}. \tag{2.12}$$

For $u, v \in \mathcal{F}_e$, it holds that

$$\begin{aligned} \mathcal{E}(u, v) &= \mathcal{E}^{\text{ref}}(u|_{E_0}, v|_{E_0}) + \frac{1}{2}\mu^c_{\langle \mathbf{H}u, \mathbf{H}v \rangle}(F) \\ &\quad + \frac{1}{2}\int_{F \times F \setminus d} (u(\xi) - u(\eta))(v(\xi) - v(\eta)) J(d\xi, d\eta) + \int_F u(\xi)v(\xi)\kappa(d\xi), \end{aligned} \tag{2.13}$$

and

$$\mathcal{E}^{\text{ref}}(u|_{E_0}, v|_{E_0}) = \mathcal{E}^0(u_0, v_0) + \frac{1}{2}\int_{F \times F \setminus d} (u(\xi) - u(\eta))(v(\xi) - v(\eta)) U(d\xi, d\eta) + \int_F u(\xi)v(\xi)V(d\xi), \tag{2.14}$$

where $u_0 := u - \mathbf{H}u$ and $v_0 := v - \mathbf{H}v$.

Proof. This theorem follows from Theorem 2.7 and Theorem 3.4 of [5]. Though it is assumed that X has no killings inside E_0 for [5, Theorem 3.4], the present much weaker condition (1.1) suffices. In fact, we get from (2.10) the expression

$$\mathbf{H}\varphi(x) = \mathbf{E}^0_x[\Phi] \qquad \text{for} \quad \Phi = \varphi(X_{\zeta^0-})\mathbf{1}_{\{X_{\zeta^0-}\in F,\ \zeta^0<\infty\}} \quad \text{q.e. } x \in E_0, \tag{2.15}$$

for any nearly Borel function φ on F with $\mathbf{H}|\varphi|(x) < \infty$ q.e. Clearly Φ is a terminal random variable of X^0. The rest of the proof of Theorem 3.4 in [5] remains valid with no change.

By [5, (3.14)] combined with the proof of [5, Theorem 2.7], we have for $u, v \in \mathcal{F}_e$,

$$\begin{aligned} & \mathcal{E}(u,v) - \mathcal{E}^{\mathrm{ref}}(u|_{E_0}, v|_{E_0}) \\ = \; & \frac{1}{2}\mu_{\langle \mathbf{H}u, \mathbf{H}v\rangle}(F) + \frac{1}{2}\int_F u(\xi)v(\xi)\kappa(d\xi) \\ = \; & \frac{1}{2}\mu^c_{\langle \mathbf{H}u, \mathbf{H}v\rangle}(F) + \frac{1}{2}\int_{F\times F\setminus d}(u(\xi)-u(\eta))(v(\xi)-v(\eta))J(d\xi,d\eta) + \int_F u(\xi)v(\xi)\kappa(d\xi). \end{aligned}$$

This proves (2.13). Note that the jumping measure J introduced in [5] is one half of the one in (2.11) of this paper. By [5, Theorems 2.7 and 2.11], we have for $u, v \in \mathcal{F}_e$,

$$\begin{aligned} \mathcal{E}(u,v) &= \mathcal{E}(u_0,v_0) + \mathcal{E}(\mathbf{H}u, \mathbf{H}v) \\ &= \mathcal{E}(u_0,v_0) + \frac{1}{2}\mu^c_{\langle \mathbf{H}u, \mathbf{H}v\rangle}(F) + \frac{1}{2}\int_{F\times F\setminus d}(u(\xi)-u(\eta))(v(\xi)-v(\eta))(U+J)(d\xi,d\eta) \\ &\quad + \int_F u(\xi)v(\xi)(V+\kappa)(d\xi). \end{aligned}$$

This together with (2.13) yields (2.14). □

3 Orthogonal decomposition of active reflected Dirichlet space

Recall that $(\mathcal{F}, \mathcal{E})$ and $(\mathcal{E}^0, \mathcal{F}^0)$ are the Dirichlet forms of X on $L^2(E;m)$ and X^0 on $L^2(E_0, m_0)$, respectively, while the reflected Dirichlet space $((\mathcal{F}^0)^{\mathrm{ref}}, \mathcal{E}^{\mathrm{ref}})$ of $(\mathcal{E}^0, \mathcal{F}^0)$ is introduced by (2.6) and (2.8). In [2], the *active reflected Dirichlet space* $(\mathcal{F}^0)^{\mathrm{ref}}_a$ of $(\mathcal{E}^0, \mathcal{F}^0)$ is defined to be

$$(\mathcal{F}^0)^{\mathrm{ref}}_a := (\mathcal{F}^0)^{\mathrm{ref}} \cap L^2(E_0, m_0), \tag{3.1}$$

and it is shown in [2, Theorem 3.10] that $(\mathcal{E}^{\mathrm{ref}}, (\mathcal{F}^0)^{\mathrm{ref}}_a)$ is actually a Dirichlet form on $L^2(E_0; m_0)$. Since $\mathcal{F} = \mathcal{F}_e \cap L^2(E;m)$, we deduce from Theorem 2.6 that

$$\mathcal{F}\big|_{E_0} \subset (\mathcal{F}^0)^{\mathrm{ref}}_a. \tag{3.2}$$

For $\alpha > 0$ and a terminal random variable Φ of X^0, we put

$$\mathbf{H}^0\Phi(x) := \mathbf{E}^0_x[\Phi] \quad \text{and} \quad \mathbf{H}^0_\alpha\Phi(x) := \mathbf{E}^0_x\left[e^{-\alpha\zeta^0}\Phi\right], \qquad x \in E,$$

whenever the expectations make sense. We also define for terminal random variables Φ, Ψ of X^0,

$$U_\alpha(\Phi, \Psi) := \alpha(\mathbf{H}^0_\alpha\Phi, \mathbf{H}^0\Psi).$$

Here and in what follows, we denote by (u, v) the integral $\int_{E_0} u(x)v(x)m_0(dx)$.

Lemma 3.1 *If Φ is a non-negative terminal random variable of X^0 with $\mathbf{H}^0\Phi(x) < \infty$, then*

$$\mathbf{H}^0\Phi(x) - \mathbf{H}^0_\alpha\Phi(x) = \alpha G^0_0\mathbf{H}^0_\alpha\Phi(x),$$
$$\mathbf{H}^0_\beta\Phi(x) - \mathbf{H}^0_\alpha\Phi(x) = (\alpha-\beta)G^0_\alpha\mathbf{H}^0_\beta\Phi(x) \qquad \text{for } \alpha,\ \beta > 0.$$

If Φ is a non-negative terminal random variable of X^0, then $U_\alpha(\Phi,\Phi)$ is increasing in $\alpha > 0$ and $\frac{1}{\alpha}U_\alpha(\Phi,\Phi)$ is decreasing in $\alpha > 0$.

Proof. We have

$$\begin{aligned}
\mathbf{H}^0\Phi(x) - \mathbf{H}^0_\alpha\Phi(x) &= \alpha\mathbf{E}^0_x\left[\int_0^{\zeta^0} e^{-\alpha(\zeta^0-s)}ds\cdot\Phi\right] \\
&= \alpha\mathbf{E}^0_x\left[\int_0^\infty e^{-\alpha\zeta^0(\theta_s)}\Phi(\theta_s\omega)\mathbf{1}_{\{s<\zeta^0\}}ds\right] \\
&= \alpha\mathbf{E}^0_x\left[\int_0^\infty \mathbf{E}_{X^0_s}(e^{-\alpha\zeta^0}\Phi)\mathbf{1}_{\{s<\zeta^0\}}ds\right] \\
&= \alpha G^0_0\mathbf{H}^0_\alpha\Phi.
\end{aligned}$$

The second identity can be obtained similarly. Suppose $U_\beta(\Phi,\Phi) < \infty$ for $\beta > 0$ and a non-negative terminal random variable Φ. Then it follows from the second identity and the symmetry of G^0_α that, for $\beta > \alpha$,

$$U_\beta(\Phi,\Phi) - U_\alpha(\Phi,\Phi) = (\beta-\alpha)\left((\mathbf{H}^0_\beta\Phi, \mathbf{H}^0\Phi) - (\mathbf{H}^0_\beta\Phi, \alpha G^0_\alpha\mathbf{H}^0\Phi)\right),$$

which is non-negative because $\alpha G^0_\alpha\mathbf{H}^0\Phi \le \mathbf{H}^0\Phi$. The decreasing property of $\frac{1}{\alpha}U_\alpha(\Phi,\Phi)$ is obvious. □

We introduce a subspace of terminal random variables by

$$\mathbf{N}_1 = \{\Phi \in \mathbf{N}:\ U_1(|\Phi|,|\Phi|) < \infty\}. \tag{3.3}$$

Proposition 3.2 *A terminal random variable Φ of X^0 is in $\mathbf{N}_1$ if and only if $\mathbf{H}^0_\alpha|\Phi| \in (\mathcal{F}^0)^{\text{ref}}_a$ for some (and hence for all) $\alpha > 0$ and*

$$\Phi = \mathbf{1}_{\{\zeta^0<\infty\}}\Phi \qquad \mathbf{P}_x\text{-a.e. for q.e. } x \in E. \tag{3.4}$$

In this case

$$\mathbf{H}^0_\alpha\Phi = \mathbf{H}^0\Phi - \alpha G^0_0\mathbf{H}^0_\alpha\Phi \tag{3.5}$$

represents the unique decomposition of $\mathbf{H}^0_\alpha\Phi$ as a sum of elements of $\mathbf{HN}$ and $\mathcal{F}^0_e$. Furthermore

$$\mathcal{E}^{\text{ref}}_\alpha(\mathbf{H}^0_\alpha\Phi, \mathbf{H}^0_\alpha\Phi) = \mathcal{E}^{\text{ref}}(\mathbf{H}^0\Phi, \mathbf{H}^0\Phi) + U_\alpha(\Phi.\Phi). \tag{3.6}$$

where $\mathcal{E}^{\text{ref}}_\alpha(u,v) = \mathcal{E}^{\text{ref}}(u,v) + \alpha(u,v)$ for $u, v \in (\mathcal{F}^0)^{\text{ref}}_a$.

Proof. It suffices to prove this Proposition for non-negative terminal random variable Φ of X^0. By Lemma 3.1,

$$\mathbf{H}^0\Phi = \alpha G_0^0\mathbf{H}_\alpha^0\Phi + \mathbf{H}_\alpha^0\Phi.$$

Suppose $\Phi \in \mathbf{N}_1$. The above identity implies not only (3.5) but also

$$\alpha(\mathbf{H}_\alpha^0\Phi, \mathbf{H}_\alpha^0\Phi) + \alpha^2(\mathbf{H}_\alpha^0\Phi, G_0^0\mathbf{H}_\alpha^0\Phi) = U_\alpha(\Phi, \Phi). \tag{3.7}$$

Since $U_\alpha(\Phi,\Phi) < \infty$, this means first that $\mathbf{H}_\alpha^0\Phi \in L^2(E_0; m_0)$ and secondly that $\mathbf{H}_\alpha^0\Phi$ is of finite energy integral with respect to G_0^0 and accordingly $G_0^0\mathbf{H}_\alpha^0\Phi \in \mathcal{F}_e^0$. Hence $\mathbf{H}_\alpha\Phi \in (\mathcal{F}^0)_a^{\rm ref}$ by (3.5). Furthermore, (3.5) and (3.7) yield (3.6).

We next prove (3.4). To this end, consider an $\mathcal{E}^0$-nest $\{A_k\}$ and let σ_k^0 be the hitting time of $E_0 \setminus A_k$ for X^0. By Lemma 2.4 (i),

$$\lim_{k\to\infty} (\mathbf{H}^0\Phi)(X_{\sigma_k^0}) = \Phi \qquad \mathbf{P}_x^0\text{−a.s.} \quad \text{for q.e. } x \in E_0.$$

Since the function $\mathbf{H}_\alpha^0\Phi$ is in $L^2(E_0; m_0)$ and is α-excessive relative to X^0, it is finite q.e. on E_0. So by an analogous argument to that for Lemma 2.4(i),

$$\lim_{k\to\infty} e^{-\alpha\sigma_k^0}(\mathbf{H}_\alpha^0\Phi)(X_{\sigma_k^0}) = e^{-\alpha\zeta^0}\Phi \qquad \mathbf{P}_x^0\text{−a.s.} \quad \text{for q.e. } x \in E_0,$$

and, consequently, in view of (2.2)

$$\lim_{k\to\infty} (\mathbf{H}_\alpha^0\Phi)(X_{\sigma_k^0}) = \lim_{k\to\infty} e^{-\alpha(\zeta^0-\sigma_k^0)}\Phi\mathbf{1}_{\{\zeta^0<\infty \text{ and } \sigma_k^0<\infty\}} = \Phi\mathbf{1}_{\{\zeta^0<\infty\}} \qquad \mathbf{P}_x^0\text{−a.s.} \ \text{ for q.e. } x \in E_0.$$

Identity (3.5) and Lemma 2.4(ii) then yield (3.4).

Conversely, suppose that a non-negative terminal random variable Φ of X^0 satisfies (3.4) and $\mathbf{H}_\alpha^0\Phi \in (\mathcal{F}^0)_a^{\rm ref}$. Then,

$$H_\alpha^0\Phi = \mathbf{H}^0\Psi + f_0 \tag{3.8}$$

for some $\Psi \in \mathbf{N}$ and $f \in \mathcal{F}_0$. In the similar manner as in the preceding paragraph where we have used (3.5), we can draw conclusion from (3.8) that

$$\Psi = \mathbf{1}_{\{\zeta^0<\infty\}}\,\Phi \qquad \mathbf{P}_x\text{-a.s. for q.e. } x \in E_0.$$

Therefore we get $\Psi = \Phi$ by our assumption (3.4).

Now the identity (3.8) together with Lemma 3.1 yields $f_0 = G_0^0\mathbf{H}_\alpha^0\Phi \in \mathcal{F}_e^0$. We note that if $G_0^0 u \in \mathcal{F}_e^0$ for some non-negative measurable function u on E_0, then

$$\mathcal{E}^0(G_0^0u, G_0^0u) = (u, G_0^0u) < \infty \quad \text{and} \quad \mathcal{E}^0(G_0^0u, w) = (u, w) \qquad \text{for every } w \in \mathcal{F}_e^0. \tag{3.9}$$

The above can be proved by approximate u by functions $u_n = \mathbf{1}_{\{g\ge 1/n\}}\,(u\wedge n)$, where g is a reference function for the transient Dirichlet form $(\mathcal{E}^0, \mathcal{F}^0)$ (cf. [11, Theorem 1.5.4].) Therefore we have

$$(\mathbf{H}_\alpha^0\Phi, G_0^0\mathbf{H}_\alpha^0\Phi) < \infty.$$

This combined with (3.7) shows that $U_\alpha(\Phi,\Phi) < \infty$, namely, $\Phi \in \mathbf{N}_1$. □

Theorem 3.3 *Define the subspace* $\mathbf{N}_1$ *of* $\mathbf{N}$ *by* (3.3). *Then for every* $\alpha > 0$,

$$(\mathcal{F}^0)_a^{\rm ref} = \mathbf{H}_\alpha^0 \mathbf{N}_1 + \mathcal{F}^0 = \{\mathbf{H}_\alpha^0 \Phi + f_0 : \Phi \in \mathbf{N}_1,\ f_0 \in \mathcal{F}^0\}. \tag{3.10}$$

The above decomposition is an $\mathcal{E}_\alpha^{\rm ref}$*-orthogonal decomposition. Further we have the following expression of the* α*-order form* $\mathcal{E}_\alpha^{\rm ref}$ *of an element* $f = \mathbf{H}_\alpha^0 \Phi + f_0$, $\Phi \in \mathbf{N}_1,\ f_0 \in \mathcal{F}^0$, *of* $(\mathcal{F}^0)_a^{\rm ref}$*:*

$$\mathcal{E}_\alpha^{\rm ref}(f,f) = \mathcal{E}^{\rm ref}(\mathbf{H}^0\Phi, \mathbf{H}^0\Phi) + U_\alpha(\Phi,\Phi) + \mathcal{E}_\alpha^0(f_0,f_0). \tag{3.11}$$

Proof. By the equivalent definition of reflected Dirichlet space as the space of all functions on E_0 with finite Dirichlet integrals (see [2, Theorem 3.9]), we see immediately that the active Dirichlet space $(\mathcal{E}_\alpha^{\rm ref}, (\mathcal{F}^0)_a^{\rm ref})$ is in fact the reflected Dirichlet space of $(\mathcal{E}_\alpha^0, \mathcal{F}^0)$. Note that $(\mathcal{E}_\alpha^0, \mathcal{F}^0)$ is the Dirichlet space of the α-subprocess Y of X^0, and $Y_t = X^0_{t\wedge T}$ where T is an exponential random variable with mean $1/\alpha$ that is independent of X^0. Thus for every $u \in (\mathcal{F}^0)_a^{\rm ref}$, by the definition (2.6) for $(\mathcal{E}_\alpha^0, \mathcal{F}^0)$ and Lemma 2.4, u has the following $\mathcal{E}_\alpha^{\rm ref}$-orthogonal decomposition

$$u(x) = f_0 + \mathbf{E}_x^0\left[\lim_{k\to\infty} u(Y_{\sigma_{E_0\setminus A_k}})\right], \qquad x \in E_0, \tag{3.12}$$

where $f_0 \in \mathcal{F}^0$, $\{A_k, k \geq 1\}$ is an $\mathcal{E}^0$-nest consisting of compact subsets of E_0 and

$$\sigma_{E_0\setminus A_k} := \inf\{t > 0 : Y_t \in E_0 \setminus A_k\}.$$

On the other hand, as $u \in (\mathcal{F}^0)_a^{\rm ref} \subset (\mathcal{F}^0)^{\rm ref}$, by (2.6) and Lemma 2.4,

$$\Phi := \lim_{k\to\infty} u(X^0_{\sigma^0_{E_0\setminus A_k}})$$

is a terminal random variable of X^0. Therefore, in view of (2.2),

$$\lim_{k\to\infty} u(Y_{\sigma_{E_0\setminus A_k}}) = \lim_{k\to\infty} u(X^0_{\sigma^0_{E_0\setminus A_k}\wedge T})\mathbf{1}_{\{\zeta^0<T\}} = \Phi\,\mathbf{1}_{\{\zeta^0<T\}} = \Phi_1\,\mathbf{1}_{\{\zeta^0<T\}},$$

where $\Phi_1 = \Phi\,\mathbf{1}_{\{\zeta^0<\infty\}}$. So (3.12) becomes

$$u(x) = f_0(x) + \mathbf{E}_x^0\left[\Phi_1\,\mathbf{1}_{\{\zeta^0<T\}}\right] = f_0(x) + \mathbf{E}_x^0\left[e^{-\alpha\zeta^0}\Phi_1\right] = f_0(x) + \mathbf{H}_\alpha^0\Phi_1(x).$$

Applying the above argument to $|u|$ in place of u, we deduce from Proposition 3.2 that $\Phi_1 \in \mathbf{N}_1$. This proves that $(\mathcal{F}^0)_a^{\rm ref} \subset \mathcal{F}^0 + \mathbf{H}_\alpha \mathbf{N}_1$. The other direction $\mathbf{H}_\alpha^0\mathbf{N}_1 + \mathcal{F}^0 \subset (\mathcal{F}^0)^{\rm ref} \cap L^2(E_0, m_0) = (\mathcal{F}^0)_a^{\rm ref}$ is obvious. This proves (3.10).

The $\mathcal{E}_\alpha^{\rm ref}$-orthogonality of the decomposition (3.10) can be also seen by (3.5) and (3.9):

$$\mathcal{E}_\alpha^{\rm ref}(\mathbf{H}_\alpha^0\Phi, f_0) = -\alpha\mathcal{E}^0(G_0^0\mathbf{H}_\alpha^0\Phi, f_0) + \alpha(\mathbf{H}_\alpha^0\Phi, f_0) = 0, \quad \Phi \in \mathbf{N}_1,\ f_0 \in \mathcal{F}^0.$$

(3.11) then follows from (3.6). □

Remark 3.4 See [16, Theorem 14.5] for a related result.

In relation to the given process X on E, we let

$$\mathbf{H}_\alpha\varphi(x) = \mathbf{E}_x\left[e^{-\alpha\sigma_F}\varphi(X_{\sigma_F})\right], \quad x \in E, \tag{3.13}$$

for $\alpha > 0$ and for any nearly Borel numerical function on F with $\mathbf{H}_\alpha|\varphi|(x) < \infty$ for q.e. $x \in E$. Note that $\mathbf{H}_{0+}\varphi$ coincides with $\mathbf{H}\varphi$ introduced in the preceding section. Analogously to (2.15), we then have for $x \in E_0$,

$$\mathbf{H}_\alpha\varphi(x) = \mathbf{H}^0_\alpha\Phi(x) \qquad \text{for } \Phi = \varphi(X_{\zeta^0-})\mathbf{1}_{\{X_{\zeta^0-}\in F,\ \zeta^0<\infty\}}. \tag{3.14}$$

This Φ is obviously a terminal random variable.

For $u \in \mathcal{F}$ and $\alpha > 0$, we have the following $\mathcal{E}_\alpha$-orthogonal decomposition

$$u = \mathbf{H}_\alpha u + f_0 \qquad \text{with } f_0 \in \mathcal{F}^0.$$

Since $\mathbf{H}_\alpha u\big|_{E_0} \in (\mathcal{F}^0)^{\text{ref}}_a$ by (3.2), we conclude from (3.14) that $\mathbf{H}_\alpha u\big|_{E_0} \in \mathbf{H}^0_\alpha\mathbf{N}_1$ and $u \in (\mathcal{F}^0)^{\text{ref}}_a$. The following theorem now follows from Theorem 2.6 and (3.11).

Theorem 3.5 *Fix $\alpha > 0$. We have the inclusions*

$$\{(\mathbf{H}_\alpha u)|_{E_0} : u \in \mathcal{F}\} \subset \mathbf{H}^0_\alpha\mathbf{N}_1 \qquad \text{and} \qquad \mathcal{F}|_{E_0} \subset (\mathcal{F}^0)^{\text{ref}}_a. \tag{3.15}$$

For $u, v \in \mathcal{F}$, it holds that

$$\begin{aligned}\mathcal{E}_\alpha(u,v) &= \mathcal{E}^{\text{ref}}_\alpha(u|_{E_0}, v|_{E_0}) + \frac{1}{2}\mu^c_{\langle\mathbf{H}u,\mathbf{H}v\rangle}(F) \\ &\quad + \frac{1}{2}\int_{F\times F\setminus d}(u(\xi)-u(\eta))(v(\xi)-v(\eta))J(d\xi,d\eta) + \int_F u(\xi)v(\xi)\kappa(d\xi),\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}\mathcal{E}^{\text{ref}}_\alpha(u|_{E_0}, v|_{E_0}) &= \mathcal{E}^0_\alpha(u_0,v_0) + \mathcal{E}^{\text{ref}}(\mathbf{H}_\alpha u|_{E_0}, \mathbf{H}_\alpha v|_{E_0}) \\ &= \mathcal{E}^0_\alpha(u_0,v_0) + \frac{1}{2}\int_{F\times F\setminus d}(u(\xi)-u(\eta))(v(\xi)-v(\eta))U(d\xi,d\eta) \\ &\quad + \int_F u(\xi)v(\xi)V(d\xi) + U_\alpha(u,v)\end{aligned} \tag{3.17}$$

where $u_0 := u - \mathbf{H}_\alpha u$ and $v_0 := v - \mathbf{H}_\alpha v$ and $U_\alpha(\varphi,\psi) = \alpha(\mathbf{H}_\alpha\varphi, \mathbf{H}\psi)$ for $\varphi,\psi \in \mathcal{B}_+(F)$.

4 Flux and lateral condition

A nearly Borel measurable function f on E_0 is said to have an X^0-*fine limit function* on F if there exists a nearly Borel measurable function ψ on F such that

$$\mathbf{P}^0_x\left(\lim_{t\uparrow\zeta^0} f(X^0_t) = \psi(X^0_{\zeta^0-}) \,\middle|\, \zeta^0 < \infty \text{ and } X^0_{\zeta^0-} \in F\right) = 1 \quad \text{for q.e. } x \in E_0.$$

In this case, we write ψ as γf and call γf the X^0-fine limit function of f on F.

Lemma 4.1 (i) *If f_0 is the restriction to E_0 of an X-q.e. finely continuous function f on E, then f_0 admits an X^0-fine limit function $f|_F$ on F.*

(ii) *If $f \in \mathcal{F}_e$, then f admits an X^0-fine limit function $f|_F$ on F. If $f \in \mathcal{F}_e^0$, the f admits zero X^0-fine limit function on F.*

Proof. (i) By [11, Theorem 4.2.2],

$$\mathbf{P}_x\left(\lim_{t'\uparrow t} f(X_{t'}) = f(X_{t-}) \ \text{ for every } \ t \in [0,\zeta)\right) = 1 \qquad \text{for q.e.} \quad x \in X.$$

The assertion (i) follows from our assumption (1.1) and the remark following (2.10).

(ii) The first assertion is immediate from (i). The second follows from the fact that any function in $\mathcal{F}_e^0$ is a quasi-continuous function in $\mathcal{F}_e$ vanishing q.e. on F. □

Let us put

$$\mathcal{H}_F = \left\{\psi \in \mathcal{B}(F) : \mathbf{H}_\alpha|\psi| \in (\mathcal{F}^0)_a^{\text{ref}}\right\}. \tag{4.1}$$

In view of Proposition 3.2, (3.14) and (3.15), this space is independent of $\alpha > 0$ and

$$\mathbf{H}_\alpha \mathcal{H}_F \subset \mathbf{H}_\alpha^0 \mathbf{N}_1 \subset (\mathcal{F}^0)_a^{\text{ref}} \qquad \text{and} \qquad \gamma(\mathcal{F}) \subset \mathcal{H}_F. \tag{4.2}$$

Consider the following two conditions:

$$\text{If } f \in (\mathcal{F}^0)_a^{\text{ref}} \text{ admits an } X^0\text{-fine limit function 0 on } F\text{, then } f \in \mathcal{F}^0. \tag{4.3}$$

$$\mathbf{P}_x^0\left(X_{\zeta^0-} \in F \;\middle|\; \zeta^0 < \infty\right) = 1 \qquad \text{for q.e. } x \in E_0. \tag{4.4}$$

These are assumptions imposed on F in relation to the process X^0. For instance, when X^0 is the absorbed Brownian motion on the interval $(0,1)$, then $\mathcal{F}^0 = H_0^1(0,1)$ and $(\mathcal{F}^0)_a^{\text{ref}} = H^1(0,1)$. Condition (4.4) is satisfied if $F = \{0,1\}$ (in which case X is the reflected Brownian motion on $[0,1]$) but it is not satisfied when $F = \{0\}$ (in which case X is the Brownian motion on $[0,1)$ reflected at 0 and absorbed at 1). On the other hand, when X^0 is a diffusion on $(0,\infty)$ for which 0 is regular and ∞ is non-regular, then condition (4.3) is fulfilled if $F = \{0\}$ because it is known that $\mathcal{F}^0 = \{f \in (\mathcal{F})_a^{\text{ref}} : \ f(0+) = 0\}$ in this case.

Lemma 4.2 (i) *For $\psi \in \mathcal{H}_F$, the function $\mathbf{H}_\alpha\psi$ has X^0-fine limit function ψ on F for any $\alpha > 0$.*

(ii) *Condition* (4.3) *is implied by condition* (4.4).

Proof. For $\Phi \in \mathbf{N}$, let $h(x) := \mathbf{E}_x[\Phi]$ with $x \in E_0$. By Lemma 2.5, $\{Mh(t)\}_{t\geq 0}$ is a $\mathbf{P}_x^0$-square integrable martingale for q.e. $x \in E_0$. Combining this with Lemma 2.4(i) and (2.2), we have

$$\lim_{t\uparrow\zeta^0} h(X_t^0) = \Phi \qquad \mathbf{P}_x^0\text{-a.s. on } \{\zeta^0 < \infty\} \quad \text{for q.e } x \in E_0. \tag{4.5}$$

(i) Let $\psi \in \mathcal{H}_F$. By (3.5), (3.14) and Lemma 4.1(ii),

$$\lim_{t\uparrow\zeta^0} \mathbf{H}_\alpha\psi(X_t^0) = (\psi\cdot\mathbf{1}_F)(X^0_{\zeta^0-}) \qquad \mathbf{P}_x^0\text{-a.s. on } \{\zeta^0<\infty\} \quad \text{for q.e } x\in E_0,$$

which implies that $\mathbf{H}_\alpha\psi$ admits ψ as an X^0-fine limit function on F.

(ii) Suppose that a function $f\in(\mathcal{F}^0)^{\rm ref}_a$ admits an X^0-fine limit function 0 on F. By Theorem 3.3 and (3.5), we can decompose f as $f(x)=\mathbf{E}_x^0[\varPhi]+f_0(x)$ with $\varPhi\in\mathbf{N}_1$ and $f_0\in\mathcal{F}_e^0$. By Lemma 4.1(ii), $\gamma f_0=0$ and so $\varPhi\cdot\mathbf{1}_{\{X_{\zeta^0-}\in F,\zeta^0<\infty\}}=0$ by (4.5). (3.4) and condition (4.4) yield that $\varPhi=0$ and consequently $f=f_0\in\mathcal{F}_e^0\cap L^2(E_0;m_0)=\mathcal{F}^0$. □

Let us introduce a linear operator $\mathcal{L}$ on $L^2(E_0;m_0)$ specified by the following:

$$f\in\mathcal{D}(\mathcal{L}) \quad\text{with}\quad \mathcal{L}f=g\ (\in L^2(E_0;m_0)),$$

if and only if

$$f\in(\mathcal{F}^0)^{\rm ref}_a \quad\text{with}\quad \mathcal{E}^{\rm ref}(f,v)=-(g,v)\quad\text{for every}\ \ v\in\mathcal{F}^0. \tag{4.6}$$

Lemma 4.3 *Assume condition* (4.3) *holds. Suppose* $u\in\mathcal{D}(\mathcal{L})$ *having an* X*-fine limit function* $\gamma u\in\mathcal{H}_F$ *on* F *and*

$$\mathcal{L}u=\alpha u \qquad \textit{for some } \alpha>0.$$

Then $u=\mathbf{H}_\alpha(\gamma u)$ *on* E_0

Proof. By (4.6), $u\in(\mathcal{F}^0)^{\rm ref}_a$ and $\mathcal{E}^{\rm ref}_\alpha(u,w)=0$ for any $w\in\mathcal{F}^0$. Define $\psi:=\gamma u$ and $u_0:=u-\mathbf{H}_\alpha\psi$. Then $u_0\in(\mathcal{F}^0)^{\rm ref}_a$ by (4.2) and $\gamma(u_0)=\psi-\psi=0$ by Lemma 4.2. Hence by assumption (4.3), $u_0\in\mathcal{F}^0$.

Since by (4.2) $\mathbf{H}_\alpha\psi\in\mathbf{H}^0_\alpha\mathbf{N}_1$, we have by Theorem 3.3 that

$$\mathcal{E}^{\rm ref}_\alpha(\mathbf{H}_\alpha\psi,w)=0 \qquad \text{for every } w\in\mathcal{F}^0.$$

It follows then

$$\mathcal{E}^{\rm ref}_\alpha(u_0,w)=\mathcal{E}^{\rm ref}_\alpha(u-\mathbf{H}_\alpha\psi,w)=0 \qquad \text{for every } w\in\mathcal{F}^0.$$

Taking $w=u_0$ yields $\mathcal{E}^{\rm ref}_\alpha(u_0,u_0)=0$ and therefore $u=\mathbf{H}_\alpha\psi=\mathbf{H}_\alpha(\gamma u)$. □

For $f\in\mathcal{D}(\mathcal{L})$ and $\psi\in\mathcal{H}_F$, define

$$\mathcal{N}(f)(\psi):=\mathcal{E}^{\rm ref}(f,\mathbf{H}_\alpha\psi)+(\mathcal{L}f,\mathbf{H}_\alpha\psi)_{L^2(E_0,m_0)},\quad \alpha>0. \tag{4.7}$$

Note that for α and $\beta>0$, $\mathbf{H}_\alpha\psi-\mathbf{H}_\beta\psi\in\mathcal{F}^0$ by Lemma 3.1 and (3.14). Hence $\mathcal{N}(f)(\psi)$ defined by (4.7) is independent of the choice of $\alpha>0$ in view of (4.6). We call $\mathcal{N}(f)$ the *flux functional* of f being regarded as a linear functional on the space $\mathcal{H}_F$.

In the remaining of this paper, we assume that

$$m(F)=0 \qquad\text{and}\qquad \mu^c_{\langle\mathbf{H}u\rangle}(F)=0\quad\text{for every } u\in\mathcal{F}. \tag{4.8}$$

Denote by $\mathcal{A}$ the L^2-infinitesimal generator of X. That is, $\mathcal{A}$ is the self adjoint operator on $L^2(E;m)$ $(= L^2(E_0;m_0))$ such that

$$f \in \mathcal{D}(\mathcal{A}) \text{ with } \mathcal{A}f = g \ \text{ if and only if } \ f \in \mathcal{F} \text{ with } \mathcal{E}(f,v) = -(g,v) \text{ for every } v \in \mathcal{F}. \tag{4.9}$$

Recall the operator $\mathcal{L}$ is defined by (4.6). We see from Theorem 2.6 that $\mathcal{L}$ is an extension of $\mathcal{A}$ in the sense that

$$\mathcal{D}(\mathcal{A}) \subset \mathcal{D}(\mathcal{L}) \quad \text{and} \quad \mathcal{A}f = \mathcal{L}f \ \text{ for } f \in \mathcal{D}(\mathcal{A}). \tag{4.10}$$

We aim at formulating a *lateral condition* that gives a characterization of a function in $\mathcal{D}(\mathcal{L})$ to be in $\mathcal{D}(\mathcal{A})$. To this end, we need to introduce a function space on F on account of Theorem 3.5.

For a Borel function φ on F, we let

$$\mathcal{E}^F(\varphi,\varphi) := \frac{1}{2}\int_{F\times F\setminus d} (\varphi(\xi)-\varphi(\eta))^2 (J+U)(d\xi,d\eta) + \int_F \varphi(\xi)^2(\kappa+V)(d\xi). \tag{4.11}$$

We define a function space $\mathcal{G}_F$ on F by

$$\mathcal{G}_F = \left\{\text{Borel function } \varphi \text{ on } F: \ \mathcal{E}^F(\varphi,\varphi) + U_1(|\varphi|,|\varphi|) < \infty\right\}. \tag{4.12}$$

The inner product $\mathcal{E}^F$ is well defined on the function space $\mathcal{G}_F$ by polarization. For each $\alpha > 0$, we define

$$\mathcal{E}^{[F,\alpha]}(\varphi,\psi) := \mathcal{E}^F(\varphi,\psi) + U_\alpha(\varphi,\psi), \qquad \varphi,\ \psi \in \mathcal{G}_F. \tag{4.13}$$

Under the present condition (4.8), Theorem 3.5 implies that

$$\gamma(\mathcal{F}) \subset \mathcal{G}_F \qquad \text{and} \qquad \mathcal{E}_\alpha(\mathbf{H}_\alpha\varphi, \mathbf{H}_\alpha\psi) = \mathcal{E}^{[F,\alpha]}(\varphi,\psi) \quad \text{for } \varphi,\ \psi \in \gamma(\mathcal{F}). \tag{4.14}$$

Let $\mathcal{C}$ be a core of the regular Dirichlet space $(\mathcal{E},\mathcal{F})$: $\mathcal{C}$ is a subset of $\mathcal{F}\cap C_c(E)$ which is dense in $(\mathcal{F},\mathcal{E}_1)$ and in $(C_c(E), \|\cdot\|_\infty)$, where $C_c(E)$ denotes the collection of continuous functions on E with compact support. Denote by $\mathcal{C}_F$ the space of functions in $\mathcal{C}$ restricted to F.

It follows from (3.16)-(3.17) that $(\mathbf{H}_1(\gamma\mathcal{F}),\mathcal{E}_1)$ is a closed subspace of the Dirichlet space $(\mathcal{F},\mathcal{E}_1)$ spanned by $\mathbf{H}_1(\mathcal{C}_F)$. Therefore (4.14) implies that $\gamma(\mathcal{F})$ is a closed subspace of $(\mathcal{G}_F,\mathcal{E}^{[F,1]})$ spanned by $\mathcal{C}_F$. In other words, if we define

$$\mathcal{G}'_F := \text{ the closure of } \mathcal{C}_F \text{ in } (\mathcal{G}_F,\mathcal{E}^{[F,1]}), \tag{4.15}$$

then

$$\gamma(\mathcal{F}) = \mathcal{G}'_F. \tag{4.16}$$

The identity (4.16) means that the trace space $\gamma(\mathcal{F})$ is completely described by the Feller measures U, V, U_1, the restrictions to F of the jumping measure J and the killing measure κ of X, and the function space $\mathcal{C}_F$.

Now we are in a position to present our main theorem.

Theorem 4.4 *Assume that condition* (4.8) *holds.*

(i) *Suppose* $f \in \mathcal{D}(\mathcal{A})$. *Then* $f \in \mathcal{D}(\mathcal{L})$ *and* f *satisfies the lateral conditions that*

$$f \text{ admits an } X^0\text{-fine limit function } \gamma f \in \mathcal{G}'_F, \tag{4.17}$$

and for every $\psi \in \mathcal{G}'_F$,

$$\mathcal{N}(f)(\psi) + \tfrac{1}{2}\int_{F\times F\setminus d}((\gamma f)(\xi) - (\gamma f)(\eta)(\psi(\xi) - \psi(\eta))J(d\xi, d\eta) + \int_F (\gamma f)(\xi)\psi(\xi)\kappa(d\xi) = 0. \tag{4.18}$$

(ii) *Assume the condition* (4.3) *holds. If* $f \in \mathcal{D}(\mathcal{L})$ *satisfies the lateral conditions* (4.17) *and* (4.18) *for every* $\psi \in \mathcal{C}_F$, *then* $f \in \mathcal{D}(\mathcal{A})$.

Proof. (i). Suppose $f \in \mathcal{D}(\mathcal{A})$. Then for $\alpha > 0$, $f = G_\alpha g$ with $g = (\alpha - \mathcal{A})f \in L^2(E; m_0)$. Here G_α denotes the α-resolvent of X. Since $f \in \mathcal{F}$, f admits the X^0-fine limit function $\gamma f = f|_F \in \gamma(\mathcal{F})$ by Lemma 4.1. For $\psi \in \gamma(\mathcal{F})$, $\mathbf{H}_\alpha\psi \in \mathcal{F}$. Since $\mathcal{L}$ is an extension of $\mathcal{A}$, we have from (4.9)

$$\mathcal{E}(f, \mathbf{H}_\alpha\psi) + (\mathcal{L}f, \mathbf{H}_\alpha\psi) = 0,$$

whose left hand side coincides with the left hand side of (4.18) in view of (4.7) and (2.13). We can then replace $\gamma(\mathcal{F})$ with $\mathcal{G}'_F$ by (4.16).

(ii). Suppose that $f \in \mathcal{D}(\mathcal{L})$ satisfies (4.17) and (4.18). Let $g = (\alpha - \mathcal{L})f$ and $f_0 := f - G_\alpha g$. Then by (4.2), (4.10) and (4.15), f_0 admits an X^0-fine limit function $\gamma f_0 \in \gamma(\mathcal{F}) \subset \mathcal{H}_F$ and $(\alpha - \mathcal{L})f_0 = 0$. Consequently, $f_0 = \mathbf{H}_\alpha(\gamma f_0) \in \mathcal{F}$ by virtue of Lemma 4.3.

As $f_0 \in \mathcal{D}(\mathcal{L})$ and $(\alpha - \mathcal{L})f_0 = 0$,

$$\mathcal{N}(f_0)(\psi) = \mathcal{E}^{\text{ref}}(f_0, \mathbf{H}_\alpha\psi) + (\mathcal{L}f_0, \mathbf{H}_\alpha\psi) = \mathcal{E}^{\text{ref}}_\alpha(f_0, \mathbf{H}_\alpha\psi)$$

for every $\psi \in \mathcal{H}_F$. On the other hand, we have by (i) that $G_\alpha g \in \mathcal{D}(\mathcal{A})$ satisfies the equation (4.18) and so does f_0. It follows then for every $\varphi \in \mathcal{C}_F$,

$$\begin{aligned}&\mathcal{E}^{\text{ref}}_\alpha(\mathbf{H}_\alpha(\gamma f_0), \mathbf{H}_\alpha\varphi)\\ &+\frac{1}{2}\int_{F\times F\setminus d}(\gamma f_0(\xi) - \gamma f_0(\eta))(\varphi(\xi) - \varphi(\eta))J(d\xi, d\eta) + \int_F \gamma f_0(\xi)\varphi(\xi)\kappa(d\xi) = 0.\end{aligned}$$

Since $f_0 \in \mathcal{F}$, we see by (3.16) that the above identity is equivalent to $\mathcal{E}_\alpha(\mathbf{H}_\alpha(\gamma f_0), \mathbf{H}_\alpha\varphi) = 0$ for every $\varphi \in \mathcal{C}_F$, which extends to every $\varphi \in \gamma(\mathcal{F})$ since $\mathcal{C}$ is a core of $(\mathcal{E}, \mathcal{F})$. Taking $\varphi = \gamma f_0$, we obtain $\mathbf{H}_\alpha(\gamma f_0) = 0$ and, consequently, $f = G_\alpha g \in \mathcal{D}(\mathcal{A})$. □

Remark 4.5 (i) We note that the space $\mathcal{G}'_F$ is contained in the L^2-space is contained in $L^2(F; \nu)$, where ν is a measure on F defined by

$$\int_F \psi(\eta)\nu(d\eta) = U_1(1, \psi) \qquad \text{for } \psi \in \mathcal{B}_+(F).$$

In fact, $U_\alpha(\varphi, \psi)$ is known to increase to $U(\varphi, \psi)$ as $\alpha \uparrow \infty$ for $\varphi,\ \psi \in \mathcal{B}_+(F)$, and we get from (4.11) the following inequality

$$\mathcal{E}^{[F,\mathbf{1}]}(\varphi, \varphi) \geq \frac{1}{2}\int_{F\times F\setminus d}(\varphi(\xi) - \varphi(\eta))^2 U_1(d\xi, d\eta) + U_1(\varphi, \varphi) \geq \int_F \varphi(\xi)^2\nu(d\xi). \tag{4.19}$$

(ii) The descriptions (4.17) and (4.18) are given in terms of the quantities intrinsic to X^0, the restrictions to F of jumping and killing measures of X and the restriction $\mathcal{C}_F$ of a core $\mathcal{C}$ of $(\mathcal{F},\mathcal{E})$ to F. There are many cases where $\mathcal{C}_F$ can be chosen in a universal way not depending on the space $(\mathcal{E},\mathcal{F})$.

We shall verify in the next section that, when F is a locally finite countable subset of E, we can choose as $\mathcal{C}_F$ the space $\mathcal{B}_c(F)$ of functions on F vanishing except on a finite number of points.

(iii) If the Dirichlet form $(\mathcal{E}^0,\mathcal{F}^0)$ satisfies the Poincaré inequality in the sense that there exists a constant $C>0$ with

$$(u,u) \le C\,\mathcal{E}^0(u,u) \qquad \text{for every } u \in \mathcal{F}^0,$$

then for $\psi \in \mathcal{H}_F$,

$$\mathcal{N}(f)(\psi) = \mathcal{E}^{\rm ref}(f,\mathbf{H}\psi) + (\mathcal{L}f,\mathbf{H}\psi) \qquad \text{for } f \in \mathcal{D}(\mathcal{L}).$$

The proof is quite analogous to the one for [3, Lemma 4.9] and is omitted. □

5 When boundary set F is countable

In this section, we assume that the quasi-closed set F is countable, that is, $F=\{a_1,a_2,\cdots\}$, and F is *locally finite* in the (generalized) sense that for every $x \in F$, there is a quasi-open set U_x containing x such that $U_x \cap (F\setminus\{x\})=\emptyset$. Note that this (generalized) notion of locally finiteness is invariant under the quasi-homeomorphism of Dirichlet forms and is an extension of the classical notion of locally finiteness on locally compact metric spaces.

Recall $E_0 := E\setminus F$. We assume that the irreducible m-symmetric Hunt process $X=(X_t,\zeta,\mathbf{P}_x)$ on E satisfies the condition (1.1), the Dirichlet form $(\mathcal{E},\mathcal{F})$ of X is regular, $m(F)=0$ and further each point a_i has positive capacity:

$$\mathrm{Cap}(a_i) > 0 \qquad \text{for each } i \ge 1,$$

where Cap denotes the 1-capacity for $(\mathcal{E},\mathcal{F})$.

Lemma 5.1 *Under the above condition, we have $\mu^c_{\langle \mathbf{H}u\rangle}(F)=0$ for every $u \in \mathcal{F}$.*

Proof. By [5, Lemma 2.8], there is a positive smooth Radon measure μ with quasi-support F. Let A^μ be the PCAF of X with Revuz measure μ and $\{\tau_t, t\ge 0\}$ be the right inverse of A^μ. Then the time-changed process $Y := \{X_{\tau_t}, t \ge 0\}$ is an μ-symmetric Markov process on F whose Dirichlet form $(\check{\mathcal{E}},\check{\mathcal{F}})$ is quasi-regular on F. It is known (cf. [11, Theorem 6.2.1]) that the extended Dirichlet space $(\check{\mathcal{E}},\check{\mathcal{F}}_e)$ of $(\check{\mathcal{E}},\check{\mathcal{F}})$ is given by $\check{\mathcal{F}}_e=\mathcal{F}_e|_F$ and $\check{\mathcal{E}}(f,g)=\mathcal{E}(\mathbf{H}f,\mathbf{H}g)$ for $f,g\in\check{\mathcal{F}}_e$. It is proved in Corollary 2.9 and Theorems 2.10-2.11 of [5] that the strongly local part $\check{\mathcal{E}}^c$ in the Beurling-Deny decomposition of $(\check{\mathcal{E}},\check{\mathcal{F}}_e)$ is given by

$$\check{\mathcal{E}}^c(f,f)=\frac{1}{2}\mu^c_{\langle \mathbf{H}f\rangle}(F) \qquad \text{for } f \in \check{\mathcal{F}}_e.$$

Note that by the equivalent characterization of $\mathcal{E}$-nest (see [11, Lemma 5.1.6], $\{F_k \cap F, k \geq 1\}$ is an $\check{\mathcal{E}}$-nest whenever $\{F_k, k \geq 1\}$ is an $\mathcal{E}$-nest. So by the definition of quasi-openness and the (generalized) locally finiteness condition on F, we see that each point $a_i \in F$ is (relatively) quasi-open in F with respect to the Dirichlet form $(\check{\mathcal{E}}, \check{\mathcal{F}})$. It follows from Lemma 5.3.3 and Theorem 4.6.1 of [11] that $\check{\mathcal{E}}^c(f, f) = 0$ for every $f \in \check{\mathcal{F}}_e$. This in particular implies that $\mu^c_{\langle \mathbf{H}u \rangle}(F) = 0$ for every $u \in \mathcal{F}$. □

Lemma 5.1 says that the condition (4.8) is satisfied. For $i \neq j$, let

$$U^{ij} := U(\{a_i\}, \{a_j\}) \qquad \text{and} \qquad U^{ij}_\alpha := U_\alpha(\{a_i\}, \{a_j\})$$

and $V^i := V(\{a_i\})$.

Define for $x \in E$ and $i \geq 1$,

$$\varphi^{(i)}(x) := \mathbf{P}_x(\sigma_F < \infty,\ X_{\sigma_F} = a_i) \quad \text{and} \quad u^{(i)}_\alpha(x) := \mathbf{E}_x\left[e^{-\alpha \sigma_F};\ X_{\sigma_F} = a_i\right], \tag{5.1}$$

which are positive m_0-a.e. by the irreducibility assumption on X. As X admits no jumps from E_0 to F, we have as before

$$\varphi^{(i)}(x) = \mathbf{P}^0_x\left(\zeta^0 < \infty \text{ and } X^0_{\zeta^0 -} = a_i\right) \quad \text{and} \quad u^{(i)}_\alpha(x) = \mathbf{E}^0_x\left[e^{-\alpha\zeta^0}; X^0_{\zeta^0 -} = a_i\right]. \tag{5.2}$$

Since $\varphi^{(i)} = \mathbf{H}\mathbf{1}_{\{a_i\}}$ and $u^{(i)}_\alpha = \mathbf{H}_\alpha \mathbf{1}_{\{a_i\}}$, we see that

$$U^{ij} = L^0(\varphi^{(i)} \cdot m_0, \varphi^{(j)}), \quad V^i = L^0(\varphi^{(i)} \cdot m_0, 1 - \varphi^{(j)}) \quad \text{and} \quad U^{ij}_\alpha = \alpha(u^{(i)}_\alpha, \varphi^{(j)})$$

We notice that $\varphi^{(i)}$, $u^{(i)}_\alpha$ admit the expressions

$$\varphi^{(i)}(x) = \frac{1}{v(a_i)} \mathbf{H}_{0+} v(x) \quad \text{and} \quad u^{(i)}_\alpha(x) = \frac{1}{v(a_i)} \mathbf{H}_\alpha v(x) \qquad \text{for } x \in E,$$

where

$$v(x) := \mathbf{E}_x\left[\int_0^{\sigma_{F \setminus a_i}} e^{-t} f(X_s) ds\right], \quad x \in E,$$

for a strictly positive bounded m-integrable function f on E. Since $v \in \mathcal{F}$ and $v(a_i) > 0$ by the locally finiteness of F, we have $\varphi^{(i)} \in \mathcal{F}_e$ and $u^{(i)}_\alpha \in \mathcal{F}$ (cf. [11]).

Let $\mathcal{B}_c(F)$ be the space of functions on F that take value 0 except on a finite many points; in other words, $\mathcal{B}_c(F)$ is the of the linear space spanned by $\{u^{(i)}_1, i \geq 1\}$. Note that $\mathcal{B}_c(F) \subset \gamma(\mathcal{F})$ as $\mathbf{1}_{\{a_i\}} = u^{(i)}_1|_F$. Let $\mathcal{C} = \mathcal{F} \cap C_c(E)$, which is a core of the regular Dirichlet space $(\mathcal{F}, \mathcal{E})$. For any choice of compact set $K \subset E$, one can find a function in $\mathcal{C}$ which is supported by K (cf. [11, Lemma 1.4.2]). Due to the locally finiteness assumption on F, we therefore have

$$\mathcal{C}_F = \mathcal{B}_c(F). \tag{5.3}$$

Recall the definition of the function space $\mathcal{G}_F$ in (4.12). Under the current assumption on F in this section, the inner product $\mathcal{E}^{[F,1]}$ on the space $\mathcal{G}_F$ takes the following specific form:

$$\mathcal{E}^{[F,1]}(\psi,\psi) = \frac{1}{2}\sum_{i,j\ge 1: i\neq i}(\psi(a_i)-\psi(a_j))^2(U^{ij}+J_{ij}) + \sum_{i\ge 1}\psi(a_i)^2(V^i+\kappa_i) + \sum_{i,j\ge 1}\psi(a_i)\psi(a_j)U_1^{ij}.$$

where $J_{ij} := J(\{a_i\},\{a_j\})$ and $\kappa_i := \kappa(\{a_i\})$. As (4.19), we also have the bound

$$\mathcal{E}^{[F,1]}(\psi,\psi) \ge \sum_{i\ge 1}\psi(a_i)^2 U_1 1(i) \quad \text{with} \qquad U_1 1(i) = \sum_{j\ge 1} U_1^{ij}.$$

Since $U_1 1(i) > 0$ for every $i \ge 1$, we conclude from this that the space $\mathcal{G}_F$ is a Hilbert space with inner product $\mathcal{E}^{[F,1]}$. Hence the space $\mathcal{G}'_F$ defined by (4.15) is now described as

$$\mathcal{G}'_F = \text{the closed subspace of } (\mathcal{G}_F,\ \mathcal{E}^{[F,1]}) \ \text{ spanned by } \mathcal{B}_c(F). \tag{5.4}$$

We can define for $f \in \mathcal{D}(\mathcal{L})$, the flux of f at a_i by

$$\mathcal{N}(f)(a_i) := \mathcal{N}(f)(\mathbf{1}_{\{a_i\}}) = \mathcal{E}^{\mathrm{ref}}(f, u_\alpha^{(i)}) + (\mathcal{L}f, u_\alpha^{(i)}).$$

Theorem 4.4 now reads as follows:

Theorem 5.2

(i) *If $f \in \mathcal{D}(\mathcal{A})$, then $f \in \mathcal{D}(\mathcal{L})$ and f satisfies the lateral conditions that*

$$f \text{ admits an } X^0\text{-fine limit function } \gamma f \in \mathcal{G}'_F, \tag{5.5}$$

for the space $\mathcal{G}'_F$ specified by (5.4) *and*

$$\mathcal{N}(f)(a_i) + \sum_{j\ge 1, j\neq i}((\gamma f)(a_i) - (\gamma f)(a_j))J_{ij} + (\gamma f)(a_i)\kappa_i = 0 \quad \textit{for every } i \ge 1. \tag{5.6}$$

(ii) *Assume the condition* (4.3) *holds. If $f \in \mathcal{D}(\mathcal{L})$ satisfies the lateral conditions* (5.5) *and* (5.6), *then $f \in \mathcal{D}(\mathcal{A})$.*

Finally let us consider a special case when X admits no jumps from F to F nor killing at F; that is,

$$J_{ij} = 0 \ \text{ for } \ i,j \ge 1,\ i \neq j \qquad \text{and} \qquad \kappa_i = 0 \ \text{ for } i \ge 1. \tag{5.7}$$

The process X is then uniquely determined by its part process X^0 on E_0 because, in view of [4, Theorem 2.6], the resolvent G_α of X is described by the resolvent G_α^0 of X^0 and $\varphi^{(i)}$, $u_\alpha^{(i)}$, the quantities completely determined by X^0.

Under the above condition (5.7), it follows immediately from Theorem 2.6 and (5.4) the following simple description of the Dirichlet form $(\mathcal{E},\mathcal{F})$ of X in terms of the active reflected Dirichlet space $((\mathcal{F}^0)_a^{\mathrm{ref}}, \mathcal{E}^{\mathrm{ref}})$ of the Dirichlet form $(\mathcal{E}^0,\mathcal{F}^0)$ of X^0.

Theorem 5.3 *Assume condition* (5.7) *holds.*
(i) $\mathcal{F}_e\big|_{E_0} \subset (\mathcal{F}^0)^{\rm ref}$ and $\mathcal{E}(u,v) = \mathcal{E}^{\rm ref}(u\big|_{E_0}, v\big|_{E_0})$ *for* $u, v \in \mathcal{F}_e$.
(ii) *For* $\alpha > 0$*, let*

$$\mathcal{H}_\alpha := \left\{ \sum_{i=1}^{\infty} c_i\, u_\alpha^{(i)} :\ c_i \in \mathbb{R} \ \ \textit{for} \ \ i \geq 1 \right\},$$

where the infinity sum is assumed to be convergent in the space $((\mathcal{F}^0)^{\rm ref}_a, \mathcal{E}^{\rm ref}_\alpha)$*. Then* $\mathcal{H}_\alpha$ *is a closed subspace of* $(\mathcal{F}^0)^{\rm ref}_a$ *and moreover the space* $(\mathcal{F}, \mathcal{E}_\alpha)$ *is the subspace of* $((\mathcal{F}^0)^{\rm ref}_a, \mathcal{E}^{\rm ref}_\alpha)$ *expressible as an* $\mathcal{E}_\alpha$*-sum*

$$\mathcal{F}\big|_{E_0} = \mathcal{F}^0 \oplus \mathcal{H}_\alpha.$$

6 Examples: flux for absorbed Brownian motions

In the first half of this section, we show that when X^0 is an absorbed Brownian motion in a domain in $\mathbb{R}^n$, the flux appearing in the last sections is a genuine extension of the classical notion. For an open set $U \subset \mathbb{R}^n$, we let

$$H^1(U) = \left\{ u \in L^2(U) : \frac{\partial u}{\partial x_i} \in L^2(U) \ \ \text{for } 1 \leq i \leq n \right\}$$

and define

$$\mathbf{D}^U(u,v) = \int_U \nabla u(x) \cdot \nabla v(x) dx \qquad \text{for } u, v \in H^1(U).$$

$C^1_c(U)$ will denote the space of continuously differentiable functions on U with compact support. The completion of $C^1_c(U)$ in $(H^1(U)$ with metric $\mathbf{D}^U(u,u) + (u,u)_{L^2(U)}$ is denoted by $H^1_0(U)$.

Let D be a bounded C^2-smooth domain in $\mathbb{R}^n$ and K a closed subset of ∂D. Let $X = (X_t, \mathbf{P}_x)$ be the (normally) reflected Brownian motion on $\overline{D}$ killed upon hitting K. Let $E = \overline{D} \setminus K$, $F = \partial D \setminus K$, and m the Lebesgue measure on E. The subprocess X^0 of X killed upon leaving $E_0 := E \setminus F = D$ is just the absorbed Brownian motion in D. Let $(\mathcal{E}, \mathcal{F})$ and $(\mathcal{E}^0, \mathcal{F}^0)$ denote the Dirichlet forms of X and X^0, respectively. It is well-known that

$$(\mathcal{E}^0,\ \mathcal{F}^0) = \left(\tfrac{1}{2}\mathbf{D}^D,\ H^1_0(D)\right).$$

The active reflected Dirichlet space $(\mathcal{E}^{\rm ref}, (\mathcal{F}^0)^{\rm ref}_a)$ is

$$\left(\mathcal{E}^{\rm ref},\ (\mathcal{F}^0)^{\rm ref}_a\right) = \left(\tfrac{1}{2}\mathbf{D}^D,\ H^1(D)\right).$$

The active reflected Dirichlet space $(\mathcal{E}^{\rm ref}, (\mathcal{F}^0)^{\rm ref}_a)$ is a regular Dirichlet space on $\overline{D}$ and its associated process is the classical (normally) reflected Brownian motion on $\overline{D}$. The Dirichlet form $(\mathcal{E}, \mathcal{F})$ of X is given by

$$\mathcal{F} := \left\{ u \in H^1(D) :\ u = 0 \ \ \mathcal{E}^{\rm ref}\text{-q.e. on } K \right\} \qquad \text{and} \qquad \mathcal{E} = \frac{1}{2}\mathbf{D}^D.$$

Let $\mathcal{A}$ denote the L^2-infinitesimal generator of X. The linear operator $\mathcal{L}$ on $L^2(D)$ specified by (3.5) is

$$\mathcal{L} = \tfrac{1}{2}\Delta \qquad \text{with} \qquad \mathcal{D}(\mathcal{L}) = \left\{ u \in H^1(D) : \Delta u \in L^2(D) \right\}.$$

The Dirichlet form $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on E. Let $\mathcal{C}$ be a core of it and let $\mathcal{C}_F = \mathcal{C}|_F$.

Lemma 6.1 *Under the above condition, the flux functional $\mathcal{N}(f))(\psi)$ for $f \in \mathcal{D}(\mathcal{L})$ and $\psi \in \mathcal{C}_F$ has the form*

$$\mathcal{N}(f)(\psi) = \frac{1}{2}\int_F \psi(x)\frac{\partial f}{\partial \mathbf{n}}(x)\sigma(dx),$$

where $\mathbf{n}(x)$ is the unit inward normal vector of ∂D at $x \in \partial D$ and $\sigma(dx)$ is the Lebesgue surface measure on ∂D.

Proof. By definition (4.7), for $f \in \mathcal{D}(\mathcal{L})$ and $\psi \in \check{\mathcal{C}}$,

$$\begin{aligned} \mathcal{N}(f)(\psi) &= \frac{1}{2}\int_D \nabla f(x)\cdot\nabla \mathbf{H}_1\psi(x)dx + \frac{1}{2}(\Delta f, \mathbf{H}_1\psi)_{L^2(D)} \\ &= \frac{1}{2}\int_{\partial D}\frac{\partial f}{\partial \mathbf{n}}(x)\mathbf{H}_1\psi(x)\sigma(dx) \\ &= \frac{1}{2}\int_{\partial D}\psi(x)\frac{\partial f}{\partial \mathbf{n}}(x)\sigma(dx) \\ &= \frac{1}{2}\int_F \psi(x)\frac{\partial f}{\partial \mathbf{n}}(x)\sigma(dx), \end{aligned}$$

where in the second equality, we used Gauss-Green formula on D, and the last identity is due to the fact that $\psi = 0$ on $\partial D \setminus F$ for $\psi \in \mathcal{C}_F$. □

Next let $X = (X_t, \mathbf{P}_x)$ be the Brownian motion on $\mathbb{R}^n$, K be a closed subset of $\mathbb{R}^n$ expressible as a disjoint union of compact subsets K_i, $i \geq 1$, which are locally finite, and $X^0 = (X^0_t, \zeta^0, \mathbf{P}^0_x)$ be the absorbed Brownian motion on $E_0 = \mathbb{R}^n \setminus K$ obtained from X by killing upon the hitting time σ_K. We assume that each set K_i is not polar. We can then apply Theorem 3.1 of [4] to X and produce an extension X^* of X^0 to the space $E^* = E_0 \cup \{a_1, a_2, \cdots\}$ obtained from $\mathbb{R}^n$ by regarding each compact set K_i as a one point a_i, $i \geq 1$.

Since X is symmetric with respect to the Lebesgue measure, we can use Theorem 5.2 in characterizing the extension X^*. Here we show that the flux $\mathcal{N}(f)(a_i)$ appearing there is actually an generalization of the classical notion of the *flux of the vector field ∇f through the surface ∂K_i* (cf [12]).

Notice that the Dirichlet form of X^0 on $L^2(E_0)$ equals $(\frac{1}{2}\mathbf{D}^{E_0}, H^1_0(E_0))$. The active reflected Dirichlet space of the latter is $(\frac{1}{2}\mathbf{D}^{E_0}, W^{1,2}(E_0))$ (cf. [2]).

For $i \geq 1$, we let

$$u^{(i)}_\alpha(x) := \mathbf{E}_x\left[e^{-\alpha\sigma_K}; X_{\sigma_K} \in K_i\right] = \mathbf{E}^0_x\left[e^{-\alpha\zeta^0}; X^0_{\zeta^0-} \in K_i\right].$$

The linear operator $\mathcal{L}$ on $L^2(E_0)$ specified by (3.5) and the flux $\mathcal{N}(f)(a_i)$ at a_i specified by (3.6) are

$$\begin{aligned} &\mathcal{L} = \tfrac{1}{2}\Delta \qquad \text{with} \qquad \mathcal{D}(\mathcal{L}) = \left\{f \in H^1(E_0) : \Delta f \in L^2(E_0)\right\}, \\ &\mathcal{N}(f)(a_i) = \tfrac{1}{2}\mathbf{D}(f, u^{(i)}_\alpha) + \frac{1}{2}(\Delta f, u^{(i)}_\alpha) \qquad \text{for } f \in \mathcal{D}(\mathcal{L}) \text{ and } i \geq 1. \end{aligned}$$

Lemma 6.2 *Suppose that the disjoint compact sets $\{K_i, i \geq 1\}$ are locally finite in the sense that there is only finite many of them intersects with each ball in $\mathbb{R}^n$. Assume that each ∂K_i is a C^1-class hypersurface. Then, for each $i \geq 1$ and for any C^2-smooth function f on $\mathbb{R}^n$ with compact support,*

$$\mathcal{N}(f)(a_i) = -\frac{1}{2}\int_{\partial K_i} \frac{\partial f}{\partial \mathbf{n}}(\xi)\sigma(d\xi),$$

where $\mathbf{n}$ *denotes the outward normal for* ∂K_i *and* σ *is the surface measure on* ∂K_i.

Proof. We prove for the case that $i = 1$. Let $f \in C_c^2(\mathbb{R}^2)$ and let $r > 0$ large enough so that the support of f is strictly contained in ball $B_r := B(0, r)$. Since $u_\alpha^{(1)}$ takes value 1 on ∂K_1 and 0 on ∂K_i for $i \geq 2$, we have by the Gauss-Green formula

$$\begin{aligned}
\mathcal{N}_r(f)(a_1) &= \frac{1}{2}\int_{B_r \setminus K} \nabla f(x) \cdot \nabla u_\alpha^{(1)}(x)dx + \frac{1}{2}\int_{B_r \setminus K} \Delta f(x)\, u_\alpha^{(1)}(x)dx \\
&= -\frac{1}{2}\int_{\partial K_1} \frac{\partial f}{\partial \mathbf{n}}(\xi)\sigma(d\xi).
\end{aligned}$$

This proves the lemma. □

In the second half of this section, we investigate the extensions of a specific one dimensional absorbed Brownian motion and their characterizations.

Let $a_0 = 0$ and $\{a_n\}_{n\geq 1}$ be a sequence of positive numbers strictly decreasing to 0. Set

$$F := \{a_n\}_{n\geq 0}, \qquad I_0 := (-\infty, 0), \qquad I_1 := (a_1, \infty), \qquad I_n := (a_n, a_{n-1}) \quad \text{for } n \geq 2$$

and $E_0 := \mathbb{R} \setminus F = \cup_{n=0}^\infty I_n$. The Lebesgue measure on $\mathbb{R}$ is denoted by m.

Let X^0 be the absorbed Brownian motion on E_0, namely the Brownian motion being killed upon hitting the set F. Since $a_0 = 0$ is an accumulation point in F, we can not use Theorem 5.2 nor Theorem 5.3 in characterizing extensions of X^0 to $\mathbb{R}$. Instead we shall utilize Theorem 3.5 and Theorem 4.4.

Proposition 6.3 *Let* X *be an* m*-symmetric diffusion process on* $\mathbb{R}$ *extending* X^0 *that has no killings at* F*. Assume that the Dirichlet form* $(\mathcal{E}, \mathcal{F})$ *of* X *on* $L^2(\mathbb{R}; m)$ *has* $C_c^1(\mathbb{R})$ *as its core. Then* X *coincides in law with the Brownian motion on* $\mathbb{R}$.

Proof. We will use Theorem 3.5 to the characterization of $(\mathcal{E}, \mathcal{F})$. First we verify that condition (4.8) is fulfilled, namely,

$$\mu_{\langle \mathbf{H}u \rangle}(\{a_i\}) = 0 \qquad \text{for every } i \geq 0 \text{ and for any } \ u \in \mathcal{F}. \tag{6.1}$$

When $i \geq 1$, (6.1) can be shown just as in the proof of Lemma 5.1 (as a special easier case). To show (6.1) for $i = 0$, it suffices to prove $\mu_{\langle u \rangle}(\{0\}) = 0$ for any $u \in \mathcal{F}$, because this is then true for any $u \in \mathcal{F}_e$ and in particular for $\mathbf{H}u$ with $u \in \mathcal{F}$.

Suppose $\mu_{\langle u \rangle}(\{0\}) > 0$ for some $u \in \mathcal{F}$. Since $(\mathcal{E}, \mathcal{F})$ is local, for any $\psi \in C_c^1(\mathbb{R})$, we have $\psi(u) \in \mathcal{F}$ and

$$\mathcal{E}(\psi(u), \psi(u)) \geq \psi'(0)^2 \mu_{\langle u \rangle}(\{0\}). \tag{6.2}$$

Take $\phi \in C_c^1(\mathbb{R})$ so that $\phi(0) = 0$, $\phi'(0) = 1$ and $\|\phi'\|_\infty = 1$. For $n \geq 1$, define $\psi_n(t) = n^{-1}\phi(nt)$. Clearly, $\psi_n \in C_c^1(\mathbb{R})$ with $\psi_n'(0) = 1$, $\|\psi_n'\|_\infty = \|\phi'\|_\infty = 1$ and $\lim_{n\to\infty} \psi_n = 0$. Since $\psi_n(u)$ is a normal contraction of u, we have (cf. $(\mathcal{E}.4)'$ of [11, page 5])

$$\mathcal{E}(\psi_n(u),\, \psi_n(u)) \leq \mathcal{E}(u,u) \qquad \text{for every } n \geq 1.$$

Moreover, by the mean-value theorem $|\psi_n(u)| \leq \|\psi_n'\|_\infty |u| = |u|$, we see that $\psi_n(u)$ converges in $L^2(\mathbb{R}, m)$ to 0. Thus by the Banach-Saks theorem (see [15, Lemma I.2.12]), there exists a subsequence $\{n_k, k \geq 1\}$ such that $w_j := j^{-1}\sum_{k=1}^j \psi_{n_k}(u)$ converges to some w in $(\mathcal{F}, \mathcal{E}_1)$. As $w_j \to 0$ in $L^2(\mathbb{R}, m)$, $w = 0$. On the other hand, since $w_j = \left(j^{-1}\sum_{k=1}^j \psi_{n_k}\right)(u)$, we have by (6.2)

$$\mathcal{E}(w_j, w_j) \geq \mu_{\langle u\rangle}(\{0\}) > 0 \qquad \text{for every } j \geq 1.$$

This is a contradiction, as we just showed that $\lim_{j\to\infty} \mathcal{E}(w_j, w_j) = 0$. Therefore (6.1) is true.

Denote by $(\mathcal{E}^0, \mathcal{F}^0)$ and $(\mathcal{E}^{\rm ref}, (\mathcal{F}^0)^{\rm ref}_a)$ the Dirichlet form of X^0 on $L^2(E_0; m_0)$ and its active reflected Dirichlet space, respectively. For a function f on $\mathbb{R}$ and $n \geq 0$, we let $f_n := f|_{I_n}$. Then

$$\begin{aligned} \mathcal{F}^0 &= \left\{ f \in L^2(\mathbb{R}; m) : \ f_n \in H_0^1(I_n) \text{ for every } n \geq 0 \right\}, \\ (\mathcal{F}^0)^{\rm ref}_a &= \left\{ f \in L^2(\mathbb{R}; m) : \ f_n \in H^1(I_n) \text{ for every } n \geq 0 \right\}, \\ \mathcal{E}^{\rm ref}(f,f) &= \sum_{n=0}^\infty \frac{1}{2}\mathbf{D}^{I_n}(f_n, f_n) \qquad \text{for } f \in (\mathcal{F}^0)^{\rm ref}_a. \end{aligned} \tag{6.3}$$

By virtue of Theorem 3.5, (6.1) and the assumption, we have

$$\mathcal{F}\big|_{E_0} \subset (\mathcal{F}^0)^{\rm ref}_a \qquad \text{and} \qquad \mathcal{E}_1(u,u) = \mathcal{E}_1^{\rm ref}(u\big|_{E_0}, u\big|_{E_0}) \quad \text{for } u \in \mathcal{F}.$$

Since $\mathcal{F}$ is the $\mathcal{E}_1$-closure of $C_c^1(\mathbb{R})$, we get from (6.3) the desired conclusion

$$\mathcal{F} = H^1(\mathbb{R}) \qquad \text{and} \qquad \mathcal{E}(u,u) = \frac{1}{2}\mathbf{D}^{\mathbb{R}}(u,u) \quad \text{for } u \in \mathcal{F}.$$

This implies that X is a standard Brownian motion on $\mathbb{R}$. □ .

Next let us take positive numbers $\{p_n\}_{n\geq 0}$ such that

$$\alpha \leq p_n \leq \beta, \qquad n = 0, 1, 2, \cdots$$

for some positive constants α, β, and we let

$$\widetilde{m}(dx) := \sum_{n=0}^\infty p_n \mathbf{1}_{I_n}(x)dx.$$

The absorbed Brownian motion X^0 is symmetric with respect to the Lebesgue measure m but it can also be viewed as an $\widetilde{m}$-symmetric diffusion on E_0 as has been observed in [3] already. Let $(\widetilde{\mathcal{E}}^0, \widetilde{\mathcal{F}})$ and $(\widetilde{\mathcal{E}}^{\rm ref}, (\widetilde{\mathcal{F}}^0)^{\rm ref}_a)$ be the Dirichlet form of X^0 on $L^2(E_0; \widetilde{m})$ and its active reflected Dirichlet space, respectively. They are given by

$$
\begin{aligned}
\widetilde{\mathcal{F}}^0 &= \left\{ f \in L^2(\mathbb{R};\widetilde{m}) : \ f_n \in H^1_0(I_n) \text{ for every } n \geq 0 \right\}, \\
(\widetilde{\mathcal{F}}^0)^{\rm ref}_a &= \left\{ f \in L^2(\mathbb{R};\widetilde{m}) : \ f_n \in H^1(I_n) \text{ for every } n \geq 0 \right\}, \\
\widetilde{\mathcal{E}}^{\rm ref}(f,f) &= \sum_{n=0}^{\infty} \frac{p_n}{2} \mathbf{D}^{I_n}(f_n,f_n) \qquad \text{for } f \in (\widetilde{\mathcal{F}}^0)^{\rm ref}_a.
\end{aligned} \tag{6.4}
$$

Proposition 6.4 *Let $\widetilde{X}$ be an $\widetilde{m}$-symmetric diffusion process on $\mathbb{R}$ extending X^0 that has no killings at F. Assume that the Dirichlet form $(\widetilde{\mathcal{E}},\widetilde{\mathcal{F}})$ of $\widetilde{X}$ on $L^2(\mathbb{R};\widetilde{m})$ has $C^1_c(\mathbb{R})$ as its core. We have then the following:*

(i) $\widetilde{\mathcal{F}} = H^1(\mathbb{R})$ *and* $\widetilde{\mathcal{E}}(f,f) = \sum_{n=0}^{\infty} \frac{p_n}{2} \mathbf{D}^{I_n}(f_n,f_n)$ *for* $f \in \widetilde{\mathcal{F}}$.

(ii) *Let $\widetilde{\mathcal{A}}$ be the L^2-infinitesimal generator of $\widetilde{X}$ on $L^2(\mathbb{R};\widetilde{m})$. Then $f \in \mathcal{D}(\widetilde{\mathcal{A}})$ if and only if the following holds:*

$$
\begin{aligned}
& f \in H^1(\mathbb{R}), \quad f' \ \text{ is absolutely continuous on } I_n \text{ for every } n \geq 0, \quad f'' \in L^2(\mathbb{R};\widetilde{m}), \\
& p_{n+1} f'(a_n-) = p_n f'(a_n+) \text{ for every } n \geq 1 \text{ and } \lim_{n\to\infty} p_n f'(a_n+) = p_0 f'(0-).
\end{aligned} \tag{6.5}
$$

Further

$$
\widetilde{\mathcal{A}} f = \frac{1}{2} f'' \qquad \text{for } f \in \mathcal{D}(\widetilde{\mathcal{A}}).
$$

Proof. By making use of (6.4), (i) can be proved exactly in the same way as the proof of the preceding Proposition. We now apply Theorem 4.4 to the proof of (ii). Condition (4.3) is obviously satisfied in view of exact description of $(\widetilde{\mathcal{F}}^0)^{\rm ref}_a$. Descriptions in (6.4) also imply that $f \in \mathcal{D}(\mathcal{L})$ if and only if $f \in L^2(\mathbb{R};\widetilde{m}) (= L^2(\mathbb{R};m))$, $f_n \in H^1(I_n)$ and f_n' is absolutely continuous for every $n \geq 0$, and $f'' \in L^2(\mathbb{R};m)$. In this case, $\mathcal{L} f = \frac{1}{2} f''$. We further see from (i) that $\mathcal{G}'_F = H^1(\mathbb{R})\big|_F$.

The flux functional (4.7) now reads for $f \in H^1(\mathbb{R})$ and $\psi \in C^1_c(\mathbb{R})$ as follows:

$$
\begin{aligned}
\mathcal{N}(f)(\psi) &= \frac{1}{2} \sum_{n=0}^{\infty} \left(p_n \int_{I_n} f' \cdot (\mathbf{H}_\alpha \psi)' dx + p_n \int_{I_n} f'' \cdot \mathbf{H}_\alpha \psi dx \right) \\
&= -\frac{1}{2} p_1 f'(a_1+)\psi(a_1) + \frac{1}{2} \sum_{n=2}^{\infty} p_n \left(f'(a_{n-1}-)\psi(a_{n-1}) - f'(a_n+)\psi(a_n) \right) + \frac{1}{2} p_0 f'(0-)\psi(0).
\end{aligned}
$$

The lateral condition (4.18) holding for any $\psi \in C^1_c(\mathbb{R})$ with $J\big|_{F\times F} = 0$ and $\kappa\big|_F = 0$ is therefore equivalent to the equation (6.5), completing the proof of (ii) on account of Theorem 4.4. □

The process $\widetilde{X}$ characterized in the above Proposition is nothing but a diffusion process on $\mathbb{R}$ with Feller's generator $\widetilde{\mathcal{A}} = \frac{d}{d\widetilde{m}} \frac{d}{d\widetilde{s}}$, where

$$
\widetilde{s}(dx) := 2 \sum_{n=0}^{\infty} p_n^{-1} \mathbf{1}_{I_n}(x) dx, \qquad \text{and} \qquad \widetilde{m}(dx) := \sum_{n=0}^{\infty} p_n \mathbf{1}_{I_n}(x) dx.
$$

By repeating the one-point skew extensions formulated in [3, Theorem 4.10], X can be constructed from the Brownian motion as follows. Let B^- and B^+ be the absorbed Brownian motions on $\mathbb{R}_- = (-\infty, 0)$ and $\mathbb{R}_+ = (0, \infty)$, respectively. Let X^{01} be the subprocess of B^+ on $\mathbb{R}_+ \setminus \{a_1\}$ killed upon hitting a_1. The process X^{01} is symmetric with respect to the measure

$$m_1(dx) = p_2 \mathbf{1}_{(0,a_1)}(x)dx + p_1 \mathbf{1}_{(a_1,\infty)}(x)dx,$$

and we can apply [3, Theorem 4.10] to construct a unique m_1-symmetric diffusion X^1 on $\mathbb{R}_+$ extending X^{01} by darning the hole a_1 with entrance law μ_t^1 determined by $\int_0^\infty \mu_t^1 dt = m_1$.

We next consider the subprocess X^{02} of X^1 on $\mathbb{R}_+ \setminus \{a_2\}$ being killed upon hitting the point a_2. X^{02} is symmetric with respect to the measure

$$m_2(dx) = \frac{p_3}{p_2}\mathbf{1}_{(0,a_2)} m_1(dx) + \mathbf{1}_{(a_2,\infty)} m_1(dx) = p_3 \mathbf{1}_{(0,a_2)}dx + p_2 \mathbf{1}_{(a_2,1)}dx + p_1 \mathbf{1}_{(a_1,\infty)},$$

and we can construct a unique m_2-symmetric diffusion X^2 on $\mathbb{R}_+$ extending X^{02} just as above.

Repeating this procedure and taking the limit as in [4, §3], we get a diffusion X^+ on $\mathbb{R}_+$ satisfying the following: X^+ is symmetric with respect to the measure

$$m_+(dx) = \mathbf{1}_{\mathbb{R}_+}(x)\widetilde{m}(dx) = \sum_{n=1}^{\infty} p_n \mathbf{1}_{I_n}(x)dx$$

and it is actually an m_+-symmetric extension of the subprocess $X^{0,+}$ of X^0 on $E_0 \cap (0,\infty) = \cup_{n=1}^{\infty} I_n$. The process X^+ has a finite life time and approaches to 0 almost surely.

We finally piece X^+ together with B^- at 0 via [3, Theorem 4.10] to get a desired diffusion X on $\mathbb{R}$ which is symmetric with respect to

$$\widetilde{m}(dx) = \mathbf{1}_{\mathbb{R}_-}(x)dx + m_+(dx)$$

and actually an $\widetilde{m}$-symmetric extension of X^0.

An analogous method works in constructing skew Borwnian motions on a Sierpinski gasket.

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