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Logarithmic and Linear Potentials of Signed Measures and Markov Property of Associated Gaussian Fields

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Abstract We consider the family of finite signed measures on the complex plane $\mathbb{C}$ with compact support, of finite logarithmic energy and with zero total mass. We show directly that the logarithmic potential of such a measure sits in the Beppo Levi space, namely, the extended Dirichlet space of the Sobolev space of order 1 over $\mathbb{C}$, and that the half of its Dirichlet integral equals the logarithmic energy of the measure. We then derive the (local) Markov property of the Gaussian field $\mathbf{G}(\mathbb{C})$ indexed by this family of measures. Exactly analogous considerations will be made for the Beppo Levi space over the upper half plane $\mathbb{H}$ and the Cameron-Martin space over the real line $\mathbb{R}$. Some Gaussian fields appearing in recent literatures related to mathematical physics will be interpreted in terms of the present field $\mathbf{G}(\mathbb{C})$.

Keywords Logarithmic potential · Logarithmic energy · Beppo Levi space · Gaussian field · Markov property

Mathematics Subject Classification (2010) Primary 31A15 · Secondary 60G60, 31C25

1 Introduction

A basic relationship between a general transient Dirichlet form $\mathcal{E}$ and a Gaussian field $\mathbf{G}$ indexed by the family $\mathcal{M}_0$ of signed Radon measures of finite 0-order energy was established by Michael Röckner [18] in 1985. It was shown in [18] that the field $\mathbf{G}$ enjoys the global Markov property if and only if the form $\mathcal{E}$ has the local property by using the *balayage* operation on measures in $\mathcal{M}_0$ formulated in [8] by means of the transient

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extended Dirichlet space $(\mathcal{F}_e, \mathcal{E})$. See also [14]. We will be concerned about extending such a relationship to a general recurrent Dirichlet form.

In recent literatures related to mathematical physics, some investigations are being made about Gaussian fields indexed by measures on the complex plane $\mathbb{C}$ with finite logarithmic energy ([2, 7, 19]). A primary purpose of this paper is to clarify the role of such measures from the above mentioned general view point.

To be more precise, let $\mathcal{M}_{00}(\mathbb{C})$ be the linear space consisting of compactly supported finite signed measures on $\mathbb{C}$ of finite logarithmic energy and with vanishing total mass. In Section 2, we prove directly that the logarithmic potential $U\mu$ of any measure $\mu \in \mathcal{M}_{00}(\mathbb{C})$ sits in the Beppo Levi space BL($\mathbb{C}$) (cf. [4, 6]) which is just the extended Dirichlet space of the Sobolev space $H^1(\mathbb{C})$ over the plane $\mathbb{C}$, and that the half of the Dirichlet integral of $U\mu$ equals the logarithmic energy $I(\mu)$ of μ.

In particular the linear space $\mathcal{M}_{00}(\mathbb{C})$ equipped with the mutual logarithmic energy $I(\mu, \nu)$ is pre-Hilbertian so that the Gaussian field $\mathbf{G}(\mathbb{C})$ indexed by $\mathcal{M}_{00}(\mathbb{C})$ can be associated. We then derive the local Markov property of $\mathbf{G}(\mathbb{C})$ by invoking the *balayage* theorem for the logarithmic potentials well presented in S.C. Port and C.J. Stone [16, §6.7].

Actually the pre-Hilbertian structure of the space $\mathcal{M}_{00}(\mathbb{C})$ appeared already in the book by Ch.-J. de La Vallèe Poussin [13, II.§1] and its shortest direct proof was given by N.S. Landkof [12, I.§4] using the composition rule of Riesz kernels. On the other hand, by making use of Schwartz distributions and their Fourier transforms, Jacques Deny [5, III.5] introduced the distribution T of finite logarithmic energy $||T||$ together with the logarithmic potential U^T of T, and showed that, if T is compactly supported, then U^T is in BL($\mathbb{C}$) and its Dirichlet integral coincides with $||T||^2$ up to a constant factor.

Our result in Section 2 gives a first direct proof of this relation for the subfamily $\mathcal{M}_{00}(\mathbb{C})$ of such general distributions without using the distribution theory. A direct proof of the corresponding relation for the Newtonian potentials of measures in $\mathcal{M}_0$ was supplied by [12, I.§4] using the Gauss-Green formula. In Section 2, we shall instead employ an approximation by the Brownian semigroup.

Exactly the analogous consideration to Section 2 will be made in Section 3 for the Beppo Levi space BL($\mathbb{H}$) over the upper half plane $\mathbb{H}$ and the Gaussian field indexed by the space $\mathcal{M}_{00}(\overline{\mathbb{H}})$ of compactly supported finite signed measures on $\overline{\mathbb{H}}$ with vanishing total mass and of finite energy relative to the logarithmic kernel for the reflecting Brownian motion on $\overline{\mathbb{H}}$.

In Section 4, we continue to make an analogous consideration for the Cameron-Martin space $H^1_e(\mathbb{R})$ over the real line $\mathbb{R}$ and linear potentials of measures on $\mathbb{R}$.

In Section 5, Gaussian fields and intrinsically associated positive random measures appearing in [2, 19] will be interpreted in terms of the present field $\mathbf{G}(\mathbb{C})$.

Gaussian fields and their Markov property for more general recurrent Dirichlet forms will be investigated in [9].

2 Logarithmic Potentials and Gaussian Field Indexed by $\mathcal{M}_{00}(\mathbb{C})$

2.1 Logarithmic Potentials and Beppo Levi Space Over $\mathbb{C}$

For the complex plane $\mathbb{C}$. define

$$p_t(\mathbf{x}) = \frac{1}{2\pi t} \exp\left(-\frac{|\mathbf{x}|^2}{2t}\right), \quad t > 0, \ \mathbf{x} \in \mathbb{C}, \qquad k(\mathbf{x}) = \frac{1}{\pi} \log \frac{1}{|\mathbf{x}|}, \quad \mathbf{x} \in \mathbb{C}. \tag{2.1}$$

$p_t(\mathbf{x}-\mathbf{y})$ and $k(\mathbf{x}-\mathbf{y})$ are the transition density of the planar Brownian motion and the logarithmic kernel, respectively.

We fix a point $\mathbf{x}_0 \in \mathbb{C}$ with $|\mathbf{x}_0| = 1$ and let

$$k_T(\mathbf{x}) = \int_0^T (p_t(\mathbf{x}) - p_t(\mathbf{x}_0))dt, \quad T > 0, \quad \mathbf{x} \in \mathbb{C}$$

We then have the following. See Port-Stone [16, p 70].

Lemma 2.1

$$|k_T(\boldsymbol{x})| = \int_0^T |p_t(\boldsymbol{x}) - p_t(\boldsymbol{x}_0)|dt < |k(\boldsymbol{x})|, \quad T > 0, \ \boldsymbol{x} \in \mathbb{C},$$

and

$$\lim_{T\to\infty} k_T(\boldsymbol{x}) = k(\boldsymbol{x}), \quad x \in \mathbb{C}.$$

We consider the *Sobolev space of order* 1 over $\mathbb{C}$ and the *Beppo Levi space* over $\mathbb{C}$ defined respectively by

$$H^1(\mathbb{C}) = \{u \in L^2(\mathbb{C}) : |\nabla u| \in Ł^2(\mathbb{C})\},$$
$$\mathrm{BL}(\mathbb{C}) = \{u \in L^2_{\mathrm{loc}}(\mathbb{C}) : |\nabla u| \in Ł^2(\mathbb{C})\}.$$

Denote the Dirichlet integral $\int_{\mathbb{C}} \nabla f(\mathbf{x}) \cdot \nabla g(\mathbf{x}) d\mathbf{x}$ of functions f, g on $\mathbb{C}$ by $\mathbf{D}(f, g)$.

$\left(\frac{1}{2}\mathbf{D}, H^1(\mathbb{C})\right)$ is the Dirichlet form on $L^2(\mathbb{C})$ associated with the planar Brownian motion. Its extended Dirichlet space is known to be identical with the space $(\mathrm{BL}(\mathbb{C}), \frac{1}{2}\mathbf{D})$. In other words, the space $\mathrm{BL}(\mathbb{C})$ is the collection of those functions f on $\mathbb{C}$ for which there exist functions $f_n \in H^1(\mathbb{C}),\ n \geq 1,$such that $\{f_n\}$ is $\mathbf{D}$-Cauchy and $f_n \to f$ a.e. on $\mathbb{C}$ as $n \to \infty$. Such $\{f_n\}$ is called an *approximating sequence* for f. See Theorem 2.2.13 and the first part of Theorem 2.2.12 in [4] for a proof. The Beppo Levi space over $\mathbb{C}$ enjoys the following basic properties (cf. [4, p69]).

(BL.a) Denote by $\mathcal{N}$ the subspace of $\mathrm{BL}(\mathbb{C})$ consisting of constant functions on $\mathbb{C}$. Then the quotient space $\dot{\mathrm{BL}}(\mathbb{C}) = \mathrm{BL}(\mathbb{C})/\mathcal{N}$ is a real Hilbert space with inner product $\frac{1}{2}\mathbf{D}$.

(BL.b) If $u_n \in \mathrm{BL}(\mathbb{C})$ is $\mathbf{D}$-convergent to $u \in \mathrm{BL}(\mathbb{C})$, then there exist constants c_n such that $u_n + c_n$ converges to u in $L^2_{\mathrm{loc}}(\mathbb{C})$.

$\mathcal{L}(\mathbb{C})$ will denote the collcetion of bounded Borel functions on $\mathbb{C}$ vanishing outside some bounded sets. For $f \in \mathcal{L}(\mathbb{C})$, put

$$Uf(\mathbf{x}) = \int_{\mathbb{C}} k(\mathbf{x}-\mathbf{y}) f(\mathbf{y}) d\mathbf{y}, \ \ P_t f(\mathbf{x}) = \int_{\mathbb{C}} p_t(\mathbf{x}-\mathbf{y}) f(\mathbf{y}) d\mathbf{y}, \ \ S_T f(\mathbf{x}) = \int_0^T P_t f(\mathbf{x}) dt, \ \ x \in \mathbb{C}.$$

(f, g) will designate the integral $\int_{\mathbb{C}} f(\mathbf{x}) g(\mathbf{x}) d\mathbf{x}$ for functions f, g on $\mathbb{C}$.

Proposition 2.2 *If $f \in \mathcal{L}(\mathbb{C})$ satisfies*

$$\int_{\mathbb{C}} f(\boldsymbol{x}) d\boldsymbol{x} = 0, \tag{2.2}$$

then $Uf \in BL(\mathbb{C})$ and

$$\frac{1}{2}\mathbf{D}(Uf, u) = (f, u) \quad \text{for any } u \in \mathbf{BL}(\mathbb{C}). \tag{2.3}$$

In particular,

$$\frac{1}{2}\mathbf{D}(Uf, Uf) = (f, Uf). \tag{2.4}$$

Proof Since $\int_{\mathbb{C}} |k(\mathbf{x}-\mathbf{y})||f(\mathbf{y})|d\mathbf{y} < \infty,\ \mathbf{x} \in \mathbb{C}$, Lemma 2.1 guarantees the use of Fubini's theorem and the dominated convergence theorem to obtain

$$k_T f(\mathbf{x}) = \int_{\mathbb{C}} k_T(\mathbf{x}-\mathbf{y}) f(\mathbf{y}) d\mathbf{y} = S_T f(\mathbf{x}), \ \lim_{T\to\infty} S_T f(\mathbf{x}) = Uf(\mathbf{x}), \ \ \mathbf{x} \in \mathbb{C}.$$

Further we get $(f, S_T f) = (f, k_T f) \to (f, Uf),\ T \to \infty$,by noting that $\int_{\mathbb{C}\times\mathbb{C}} |k(\mathbf{x} - \mathbf{y})||f(\mathbf{x})||f(\mathbf{y})|d\mathbf{x}d\mathbf{y} < \infty$.

As $f \in L^2(\mathbb{C})$,we see by [10, Lemma 1.5.3] that $S_T \in H^1(\mathbb{C})$ and

$$\frac{1}{2}\mathbf{D}(S_T f, u) = (f - P_T f, u) \quad \text{for any } \ u \in H^1(\mathbb{C}). \tag{2.5}$$

Accordingly

$$\frac{1}{2}\mathbf{D}(S_T f - S_{T'} f, S_T f - S_{T'} f) = (f, 2S_{T+T'} f - S_{2T} f - S_{2T'} f) \to 0, \ \ T, T' \to \infty.$$

Therefore $Uf \in \mathrm{BL}(\mathbb{C})$. Since $(f, S_T f) = \int_0^T (P_{t/2} f.P_{t/2} f)dt$ increases to a finite limit (f, Uf) as $T \to \infty$, $|(P_t f, u)| \le \sqrt{(P_t f, P_t f)}\sqrt{(u, u)}$ tends to zero as $t \to \infty$ for any $u \in H^1(\mathbb{C})$. Consequently, we get Eq. 2.3 holding for any $u \in H^1(\mathbb{C})$ from Eq. 2.5.

For any $u \in \mathrm{BL}(\mathbb{C})$, there exist $u_n \in H^1(\mathbb{C})$ such that $\{u_n\}$ is $\mathbf{D}$-convergent to u. According to **(BL.b)**, there are some constants c_n such that $\{u_n + c_n\}$ is L^2_{loc}-convergent to u. By letting $n \to \infty$ in $\frac{1}{2}\mathbf{D}(Uf, u_n) = (f, u_n) = (f, u_n + c_n)$, we arrive at Eq. 2.3 for $u \in \mathrm{BL}(\mathbb{C})$. □

Denote by $\mathcal{M}^+(\mathbb{C})$ the collection of positive finite measures on $\mathbb{C}$ with compact support. The *logarithmic potential* $U\mu$ of $\mu \in \mathcal{M}^+(\mathbb{C})$ is defined by

$$U\mu(x) = \int_{\mathbb{C}} k(\mathbf{x}-\mathbf{y})\mu(d\mathbf{y}), \quad \mathbf{x} \in \mathbb{C}.$$

$U\mu$ is *superharmonic*, namely, it is lower semicontinuous and supermean valued. It is locally integrable and locally bounded below on $\mathbb{C}$.

For $r > 0$,consider the function $\psi_r(\mathbf{x}) = \frac{1}{\pi r^2} I_{B_r}(\mathbf{x}),\ \mathbf{x} \in \mathbb{C}$,where $B_r = \{\mathbf{y} \in \mathbb{C} : |\mathbf{y}| < r\}$. For $\mu \in \mathcal{M}^+$, define

$$\mu_r(\mathbf{x}) = \int_{\mathbb{C}} \psi_r(\mathbf{x}-\mathbf{y})\mu(d\mathbf{y}), \quad \mathbf{x} \in \mathbb{C}, \tag{2.6}$$

which is a continuous function on $\mathbb{C}$ belonging to $\mathcal{L}$. Furthermore

$$[\psi_r * (U\mu)](\mathbf{x}) = U\mu_r(\mathbf{x}) \uparrow U\mu(\mathbf{x}), \quad \mathbf{x} \in \mathbb{C}, \quad r \downarrow 0. \tag{2.7}$$

Here the notation $*$ designates the convolution of functions.

The integral $\int_{\mathbb{C}} u d\nu$ of a function u by a measure ν is denoted by $\langle u, \nu\rangle$ or $\langle \nu, u\rangle$. For $\mu, \nu \in \mathcal{M}^+(\mathbb{C})$, $\langle \mu, U\nu\rangle$ takes value in $(-\infty, +\infty]$. We define the *energy* $I(\mu)$ of $\mu \in \mathcal{M}^+$ by $I(\mu) = \langle \mu, U\mu\rangle$.

For $R > 0$, we let $k^R(\mathbf{x}) = \frac{1}{\pi}\log\frac{R}{|\mathbf{x}|}, \quad U^R\mu(\mathbf{x}) = \int_{\mathbb{C}} k^R(\mathbf{x}-\mathbf{y})\mu(d\mathbf{y})$. For $\mu, \nu \in \mathcal{M}^+(\mathbb{C})$, $\langle \mu, U\nu\rangle$ is finite if and only if so is $\langle \mu, U^R\nu\rangle$ for some $R > 0$, because

$$\langle \mu, U\nu\rangle = \langle \mu, U^R\nu\rangle - \frac{\log R}{\pi}\mu(\mathbb{C})\nu(\mathbb{C}). \tag{2.8}$$

Suppose $I(\mu+\nu)$ is finite for $\mu, \nu \in \mathcal{M}^+(\mathbb{C})$. Then $I(\mu)$, $I(\nu)$ and $\langle \mu, U\nu\rangle$ are all finite. To see this, it is enough to take $R > 0$ such that the supports of μ and ν are both contained in $B_{R/2}$ and to notice that $k^R(\mathbf{x}-\mathbf{y}) > 0,\ \mathbf{x}, \mathbf{y} \in B_{R/2}$.

Let us now introduce several classes of measures on $\mathbb{C}$.

$$\mathcal{M}_0^+(\mathbb{C}) = \{\mu \in \mathcal{M}^+(\mathbb{C}) : I(\mu) < \infty\},$$

$$\mathcal{M}_0(\mathbb{C}) = \{\mu : \text{ finite signed measure on } \mathbb{C},\ |\mu| \in \mathcal{M}_0^+(\mathbb{C})\}.$$

For $\mu \in \mathcal{M}_0(\mathbb{C})$,let $|\mu| = \mu^+ + \mu^-,\ \mu = \mu^+ - \mu^-,\ \mu^\pm \in \mathcal{M}_0^+(\mathbb{C})$,be its Jordan decomposition. Due to the observation made just above, the *energy* $I(\mu)$ of μ is well defined by

$$I(\mu) = I(\mu^+) + I(\mu^-) - 2\langle \mu^+, U\mu^- \rangle \ \in \mathbb{R}. \tag{2.9}$$

Finally we define the class

$$\mathcal{M}_{00}(\mathbb{C}) = \{\mu \in \mathcal{M}_0(\mathbb{C}) : \mu(\mathbb{C}) = 0\}.$$

Proposition 2.3 *For any $\mu \in \mathcal{M}_{00}(\mathbb{C})$, $U\mu \in BL(\mathbb{C})$ and*

$$\frac{1}{2}\mathbf{D}(U\mu, U\mu) = I(\mu), \tag{2.10}$$

$$\frac{1}{2}\mathbf{D}(U\mu, u) = \langle \mu, u \rangle \quad \text{for any } u \in C_c^1(\mathbb{C}). \tag{2.11}$$

Proof Let $\mu = \mu^+ - \mu^-,\ \mu^\pm \in \mathcal{M}_0^+(\mathbb{C})$,be the Jordan decomposition of $\mu \in \mathcal{M}_{00}(\mathbb{C})$. Taking a sequence $r_n \downarrow 0$ with $r_n < \frac{1}{2}$,define the functions $\mu_n^\pm = \mu_{r_n}^\pm$ according to Eq. 2.6 and let $\mu_n = \mu_n^+ - \mu_n^- (= \int_{\mathbb{C}} \psi_{r_n}(\mathbf{x} - \mathbf{y})\mu(d\mathbf{y}))$. Since each function μ_n belongs to $\mathcal{L}$ and satisfies (2.2), we obtain from Proposition 1.2 that $U\mu_n \in \mathrm{BL}(\mathbb{C})$ and $\frac{1}{2}\mathbf{D}(U\mu_n, u) = (\mu_n, u)$ for any $u \in \mathrm{BL}(\mathbb{C})$. In particular,

$$\frac{1}{2}\mathbf{D}(U\mu_n, U\mu_m) = (\mu_n, U\mu_m), \quad n,\ m \in \mathbb{N}. \tag{2.12}$$

We first use Eq. 2.12 to get

$$\frac{1}{2}\mathbf{D}(U\mu_n, U\mu_n) = (\mu_n^+, U\mu_n^+) + (\mu_n^-, U\mu_n^-) - 2(\mu_n^+, U\mu_n^-).$$

By virtue of Eq. 2.7, $(\mu_n^\pm, U\mu_n^\pm) \le (\mu_n^\pm, U\mu^\pm) = \langle U\mu_n^\pm, \mu^\pm \rangle \le \langle U\mu^\pm, \mu^\pm \rangle < \infty$. On the other hand, if we take a disk B_R large enough to contain the supports of $\mu^\pm$,then the supports of the measures $\mu_n^\pm(\mathbf{x})d\mathbf{x}$ are contained in B_{R+1} so that $(\mu_n^+, U^{R+1}\mu_n^-) \ge 0$. Further $\int_{\mathbb{C}} \mu_n^\pm(\mathbf{x})d\mathbf{x} = \mu^\pm(\mathbb{C}) < \infty$. Hence it follows from Eq. 2.8 that

$$-(\mu_n^+, U\mu_n^-) \le \frac{1}{\pi}\log(R+1)\mu^+(\mathbb{C})\mu^-(\mathbb{C}) < \infty.$$

We thus obtain the boundedness $\sup_{n\in\mathbb{N}} \frac{1}{2}\mathbf{D}(U\mu_n, U\mu_n) < \infty$.

By the Banach-Saks theorem (cf. [4, Theorem A.4.1]), there exists a subsequence $\{\mu_{n_k}\}$ of $\{\mu_n\}$ such that, for its Cesàro mean denoted by ν_k, $U\nu_k$ is $\mathbf{D}$-convergent to some $v \in \mathrm{BL}(\mathbb{C})$ as $k \to \infty$. Then, by **(BL.b)** again, there exist constants c_k such that $U\nu_k + c_k$ converges to v in $L^2_{\mathrm{loc}}(\mathbb{C})$ as $k \to \infty$.

$U\nu_k = U\nu_k^+ - U\nu_k^-$ for the Cesàro mean $\nu_k^\pm$ of $\{\mu_{n_k}^\pm\}$ and $U\nu_k^\pm(\mathbf{x}) \uparrow U\mu^\pm(\mathbf{x}),\ \mathbf{x} \in \mathbb{C}$,as $k \to \infty$ by Eq. 2.7 so that $\lim_{k\to\infty} U\nu_k(\mathbf{x}) = U\mu^+(\mathbf{x}) - U\mu^-(\mathbf{x}) = U\mu(\mathbf{x})$ for any $\mathbf{x} \in \mathbb{C} \setminus N$, where $N = \{\mathbf{x} \in \mathbb{C} : U\mu^-(\mathbf{x}) = +\infty\}$. But the Lebesgue measure of N is zero because $U\mu^-$ is locally integrable. Hence the limit $c = \lim_{k\to\infty} c_k$ exists and $v = U\mu + c$, and consequently $U\mu \in \mathrm{BL}(\mathbb{C})$ and $U\nu_k \to U\mu,\ k \to \infty$, $\mathbf{D}$-strongly in $(\mathrm{BL}(\mathbb{C}), \frac{1}{2}\mathbf{D})$.

We next use Eq. 2.12 to get

$$\frac{1}{2}\mathbf{D}(U\nu_k, U\nu_\ell) = (U\nu_k, \nu_\ell) = (U\nu_k^+, \nu_\ell^+) + (U\nu_k^-, \nu_\ell^-) - (U\nu_k^+, \nu_\ell^-) - (U\nu_k^-, \nu_\ell^+).$$

As $U\nu_k^{\pm}$ are locally bounded below and $U\mu^{\pm}$ are locally integrable, we let $k \to \infty$ using the monotone convergence theorem to obtain

$$\frac{1}{2}\mathbf{D}(U\mu, U\nu_\ell) = (U\mu^+, \nu_\ell^+) + (U\mu^-, \nu_\ell^-) - (U\mu^+, \nu_\ell^-) - (U\mu^-, \nu_\ell^+)$$
$$= \langle \mu^+, U\nu_\ell^+ \rangle + \langle \mu^-, U\nu_\ell^- \rangle - \langle \mu^+, U\nu_\ell^- \rangle - \langle \mu^-, U\nu_\ell^+ \rangle.$$

By noting (2.9), we finally let $\ell \to \infty$ using the monotone convergence theorem to arrive at Eq. 2.10.

To prove (2.11), take any $u \in C_c^1(\mathbb{C})$. Denote by $\widetilde{\psi}_k$ the Cesàro mean of $\{\psi_{n_k}\}$. Since $\psi_n * u$ converges to u locally uniformly on $\mathbb{C}$ as $n \to \infty$, so does $\widetilde{\psi}_k * u_0$ as $k \to \infty$. We get from Eq. 2.3

$$\frac{1}{2}\mathbf{D}(U\nu_k, u) = (\nu_k, u) = \langle \mu, \widetilde{\psi}_k * u \rangle.$$

By letting $k \to \infty$, we obtain (2.11). □

The rest of this section will be concerned about an extension of the Eq. 2.11 to $u \in \mathrm{BL}(\mathbb{C})$ and balayage (sweeping out) of measures. We need some preparations.

Let $\mathbf{X} = (X_t, \{\mathbb{P}_\mathbf{x}\}_{\mathbf{x}\in\mathbb{C}})$ be the planar Brownian motion. A subset N of $\mathbb{C}$ is called a *polar set* (relative to $\mathbf{X}$) if N is contained in a Borel set B such that

$$\mathbb{P}_\mathbf{x}(\sigma_B < \infty) = 0 \quad \text{for any } \mathbf{x} \in \mathbb{C}, \quad \text{where } \sigma_B = \inf\{t > 0 : X_t \in B\}.$$

For $u \in H^1(\mathbb{C})$, we put $\mathbf{D}_1(u,u) = \frac{1}{2}\mathbf{D}(u,u) + (u,u)$ and define the $\mathbf{D}_1$-capacity of an open set $G \subset \mathbb{C}$ by

$$\mathrm{Cap}(G) = \inf\{\mathbf{D}_1(u,u) : u \in H^1(\mathbb{C}),\ u \geq 1 \text{ a.e. on } G\}.$$

The $\mathbf{D}_1$-capacity of an arbitrary set $B \subset \mathbb{C}$ is defined by $\mathrm{Cap}(B) = \inf\{\mathrm{Cap}(G) : G \text{ open},\ G \supset B\}$.

It is known that a subset N of $\mathbb{C}$ is polar if and only if $\mathrm{Cap}(N) = 0$ (cf. [10, Theorems 4.1.2, 4.2.1]). '*quasi-everywhere*' or 'q.e.' means 'except for a polar set'. An extended real valued function u defined q.e. on $\mathbb{C}$ is said to be $\mathbf{D}_1$-*quasi continuous* if, for any $\epsilon > 0$, there exists an open set $G \subset \mathbb{C}$ with $\mathrm{Cap}(G) < \epsilon$ such that $u\big|_{\mathbb{C}\setminus G}$ is finite and continuous.

Lemma 2.4 *(i) Any function in $BL(\mathbb{C})$ admits a $\boldsymbol{D}_1$-quasi continuous version. If $f_n \in H^1(\mathbb{C})$, $n \geq 1$, constitute an approximating sequence of $f \in BL(\mathbb{C})$ and each f_n is $\boldsymbol{D}_1$-quasi continuous, then there exists an subsequence $\{n_k\}$ such that f_{n_k} converges to a $\boldsymbol{D}_1$-quasi continuous version $\widetilde{f}$ of f q.e. on $\mathbb{C}$.*

(ii) Any $u \in BL(\mathbb{C})$ admits an approximating sequence of functions in $C_c^1(\mathbb{C})$.

(iii) Take any $\mu \in \mathcal{M}_0^+(\mathbb{C})$. Then μ charges no polar set. Further $U\mu(\boldsymbol{x}) < \infty$ for q.e. $\boldsymbol{x} \in \mathbb{C}$.

Proof (i) follows from [4, Theirem 2.3.4].

(ii). As in the proof of Theorem 2.2.13 in [4], it suffices to prove this assertion for bounded $u \in \mathrm{BL}(\mathbb{C})$. Further, for such u, it is shown there that $\sup_n \mathbf{D}(u_n, u_n) < \infty$ for $u_n(\mathbf{x}) = u(\mathbf{x})\eta_n(|\mathbf{x}|)$, $\mathbf{x} \in \mathbb{C}$, where η_n is a non-negative smooth function on $[0, \infty)$ satisfying

$$\begin{cases} \eta_n(x) = 1 \quad \text{for } 0 \leq x < n, \quad \eta_n(x) = 0 \quad \text{for } x > 2n+1, \\ |\eta_n'(x)| \leq 1/n \quad \text{for } n \leq x \leq 2n+1, \quad 0 \leq \eta_n(x) \leq 1 \quad \text{for } x \in [0,\infty). \end{cases}$$

Consider a non-negative smooth function φ on $\mathbb{C}$ with $\mathrm{supp}(\varphi) \subset B_1$ and $\int_{B_1} \varphi(\mathbf{x})d\mathbf{x} = 1$. We set $\varphi_n(\mathbf{x}) = n^2\varphi(n\mathbf{x})$, $x \in \mathbb{C}$, and let $v_n = \varphi_n * u_n$. Then $v_n \in C_c^1(\mathbb{C})$ and

$\lim_{n\to\infty} v_n(\mathbf{x}) = u(\mathbf{x})$ for a.e. $\mathbf{x} \in \mathbb{C}$. Moreover $\mathbf{D}(v_n, v_n) \le \mathbf{D}(u_n, u_n)$ so that $\sup_n \mathbf{D}(v_n, v_n) < \infty$. This implies that the Cesàro mean sequence of a suitable subsequence of $\{v_n\}$ is a desired approximating sequence of u.

(iii). We use the following *fundamental identity for the logarithmic potential* due to [16, Theorem 3.4.2]: for any non-polar compact set $K \subset \mathbb{C}$,

$$k(\mathbf{x}, \mathbf{y}) = g^{\mathbb{C}\setminus K}(\mathbf{x}, \mathbf{y}) + \int_K h_K(\mathbf{x}, d\mathbf{z})k(\mathbf{z}, \mathbf{y}) - W_K(\mathbf{x}), \quad \mathbf{x}, \mathbf{y} \in \mathbb{C}, \tag{2.13}$$

where $k(\mathbf{x}, \mathbf{y}) = k(\mathbf{x} - \mathbf{y})$, $g^{\mathbb{C}\setminus K}$ is the 0-order resolvent density of the absorbing Brownian motion on $\mathbb{C} \setminus K$, h_K is the hitting distribution for K of the planar Brownian motion $\mathbf{X}$ defined by $h_K(\mathbf{x}, B) = \mathbb{P}_\mathbf{x}(\sigma_K < \infty,\ X_{\sigma_K} \in B)$ for any Borel set B and W_K is a certain non-negative locally bounded Borel function on $\mathbb{C}$ vanishing q.e. on K.

Take an open disk B_R containing the support of μ and write $D = B_R$ and $S = \partial B_R$. As $W_S(\mathbf{x}) = 0,\ \mathbf{x} \in D$, by [16, §3, Prop.4.7], we have from Eq. 2.13

$$U\mu(\mathbf{x}) = \int_D g^D(\mathbf{x}, \mathbf{y})\mu(d\mathbf{y}) + \mathbb{E}_\mathbf{x}\left[U\mu(X_{\sigma_S}); \sigma_S < \infty\right], \quad \mathbf{x} \in D, \tag{2.14}$$

where $g^D(\mathbf{x}, \mathbf{y})$ is the 0-order resolvent density of the absorbing Brownian motion $\mathbf{X}_D$ on D. Denote by $g_1^D(\mathbf{x}, \mathbf{y})$ the 1-order resolvent density of $\mathbf{X}_D$.

Since $I(\mu) < \infty$ and the last term of the right hand side of the above identity are bounded in $\mathbf{x} \in D$, we have $\int_{D\times D} g^D(\mathbf{x}, \mathbf{y})\mu(d\mathbf{x})\mu(d\mathbf{y}) < \infty$, and so $\int_{D\times D} g_1^D(\mathbf{x}, \mathbf{y})\mu(d\mathbf{x})\mu(d\mathbf{y}) < \infty$. This means that the measure μ is of finite energy integral relative to $\mathbf{X}_D$ ([10, Exercise 4.2.2]). Accordingly μ charges no polar set relative to $\mathbf{X}_D$, and equivalently, relative to $\mathbf{X}$ (cf. [10, Lemma 2.2.3, Theorem 4.4.3]).

Since $U\mu(\mathbf{x}) < \infty$ for a.e. $\mathbf{x} \in D$, so is the function $G^D\mu(\mathbf{x}) = \int_D g^D(\mathbf{x}, \mathbf{y})\mu(d\mathbf{y})$. As $G^D\mu$ is $\mathbf{X}_D$-excessive, we can conclude from [4, Theorem A.2.13 (v)] that it is finite q.e. on D. We then get the last statement by the above identity because we can take an arbitrarily large $R > 0$. □

For $u \in \mathrm{BL}(\mathbb{C})$, we set

$$L(u) = \int_{B_1} \psi_1(\mathbf{x})u(\mathbf{x})d\mathbf{x},$$

which is well defined because $u \in L^2_{\mathrm{loc}}(\mathbb{C})$.

Proposition 2.5 *It holds for any $\nu \in \mathcal{M}_0^+(\mathbb{C})$ that*

$$\langle \nu, |\widetilde{u} - L(u)|\rangle \le C(\nu)\sqrt{\boldsymbol{D}(u, u)} \quad \text{for any } u \in BL(\mathbb{C}), \tag{2.15}$$

where $\widetilde{u}$ is a $\boldsymbol{D}_1$-quasi continuous version of u and $C(\nu)$ is a positive constant independent of $u \in BL(\mathbb{C})$. In particular, $\widetilde{u}$ for $u \in BL(\mathbb{C})$ is ν-integrable.

Proof We proceed as in [10, p 61]. For fixed $\nu \in \mathcal{M}_0^+(\mathbb{C})$ and $u \in C_c^1(\mathbb{C})$, define the measures $\widehat{\nu}$ and μ by

$$\widehat{\nu} = \mathrm{sgn}(u - L(u)) \cdot \nu, \quad \mu(d\mathbf{x}) = \widehat{\nu}(d\mathbf{x}) - \widehat{\nu}(\mathbb{C})\psi_1(\mathbf{x})d\mathbf{x}.$$

As $\mu \in \mathcal{M}_{00}(\mathbb{C})$, we have from Eq. 2.11 that

$$\frac{1}{2}\mathbf{D}(U\mu, u) = \langle \mu, u\rangle = \langle \widehat{\nu}, u\rangle - \widehat{\nu}(\mathbb{C})L(u) = \langle \widehat{\nu}, u - L(u)\rangle = \langle \nu, |u - L(u)|\rangle.$$

Consequently, it follows from Eq. 2.10 that

$$\langle \nu, |u - L(u)|\rangle \leq \frac{1}{2}\sqrt{I(\mu)} \cdot \sqrt{\mathbf{D}(u,u)}.$$

Let us show that $\frac{1}{2}\sqrt{I(\mu)}$ is dominated by a constant $C(\nu)$ that is independent of $u \in C_c^1(\mathbb{C})$. Take $R > 2$ such that the support of ν is contained in $B_{R/2}$. We have

$$I(\mu) = I(\widehat{\nu}) + \widehat{\nu}(\mathbb{C})^2(\psi_1, U\psi_1) - 2\widehat{\nu}(\mathbb{C})\langle\widehat{\nu}, U\psi_1\rangle.$$

From Eq. 2.8, we get

$$\begin{aligned} I(\widehat{\nu}) &\leq \langle\widehat{\nu^+}, U^R\widehat{\nu^+}\rangle + \langle\widehat{\nu^-}, U^R\widehat{\nu^-}\rangle + \frac{2\log R}{\pi}\widehat{\nu^+}(\mathbb{C})\widehat{\nu^-}(\mathbb{C}) \\ &\leq 2\langle\nu, U^R\nu\rangle + \frac{2\log R}{\pi}\nu(\mathbb{C})^2, \end{aligned}$$

and

$$|\langle\widehat{\nu}, U\psi_1\rangle| = |\langle\widehat{\nu^+}, U\psi_1\rangle - \langle\widehat{\nu^-}, U\psi_1\rangle| \leq 2\langle\nu, U^R\psi_1\rangle + \frac{2\log R}{\pi}\nu(\mathbb{C}),$$

so that, if we put

$$c(\nu) = 2\langle\nu, U^R\nu\rangle + 4\nu(\mathbb{C})\langle\nu, U^R\psi_1\rangle + \nu(\mathbb{C})^2\left[(\psi_1, U\psi_1) + \frac{6\log R}{\pi}\right],$$

then $\frac{1}{2}\sqrt{I(\mu)}$ is dominated by $C(\nu) = \frac{1}{2}\sqrt{c(\nu)}$, which is independent of $u \in C_c^1(\mathbb{C})$.

We now have Eq. 2.15 holding for $\nu \in \mathcal{M}_0^+(\mathbb{C})$ and for any $u \in C_c^1(\mathbb{C})$ with this constant $C(\nu)$. For any $u \in \mathrm{BL}(\mathbb{C})$,one can choose its approximating sequence $u_n \in C_c^1(\mathbb{C})$ by Lemma 2.4 (ii). In view of Lemma 2.4 (i), u_n converges to a $\mathbf{D}_1$-quasi continuous version $\widetilde{u}$ of u q.e. on $\mathbb{C}$ by selecting a suitable subsequence if necessary. On the other hand, according to **(BL.b)**, there exist constants c_n such that $u_n + c_n$ converges to u in $L^2_{\mathrm{loc}}(\mathbb{C})$ and consequently a.e. on $\mathbb{C}$ by choosing a subsequence if necessary. Therefore $\lim_{n\to\infty} c_n = 0$. Thus

$$u_n - L(u_n) \;\to\; \widetilde{u} - L(u), \quad n \to \infty, \quad \text{q.e. on } \mathbb{C},$$

and

$$\langle\nu, |u_n - L(u_n)|\rangle \leq C(\nu)\sqrt{\mathbf{D}(u_n, u_n)}. \quad n \geq 1.$$

We let $n \to \infty$. Since ν charges no polar set by Lemma 2.4 (iii), we can use Fatou's lemma to get the desired inequality (2.15).

$\widetilde{u}$ for $u \in \mathrm{BL}(\mathbb{C})$ is ν-integrable as $\langle\nu, |\widetilde{u}|\rangle \leq C(\nu)\sqrt{\mathbf{D}(u,u)} + \nu(\mathbb{C})|L(u)| < \infty$. □

Theorem 2.6 *For any $\mu \in \mathcal{M}_{00}(\mathbb{C})$, $U\mu \in BL(\mathbb{C})$ and $U\mu$ is $\boldsymbol{D}_1$-quasi continuous. Furthermore*

$$\frac{1}{2}\boldsymbol{D}(U\mu, u) = \langle\widetilde{u}, \mu\rangle, \quad \text{for any } u \in BL(\mathbb{C}), \tag{2.16}$$

where $\widetilde{u}$ is any $\boldsymbol{D}_1$-quasi continuous version of u.

Proof For any $u \in \mathrm{BL}(\mathbb{C})$, choose its approximating sequence $\{u_n\}$ from $C_c^1(\mathbb{C})$ according to Lemma 4.1 (ii). For $\mu \in \mathcal{M}_{00}(\mathbb{C})$,we have from Proposition 2.5

$$\begin{aligned} &|\langle\mu, \widetilde{u}\rangle - \langle\mu, u_n\rangle| \leq |\langle\mu, \widetilde{u} - u_n - L(u - u_n)\rangle| \\ &\leq \langle\mu^+, |\widetilde{u} - u_n - L(u - u_n)|\rangle + \langle\mu^-, |\widetilde{u} - u_n - L(u - u_n)|\rangle \\ &\leq (C(\mu^+) + C(\mu^-))\sqrt{\mathbf{D}(u - u_n, u - u_n)} \to 0, \quad n \to \infty. \end{aligned}$$

So, by letting $n \to \infty$ in Eq. 2.11 for $u = u_n$,we arrive at Eq. 2.16.

We have seen in the proof of Proposition 2.3 that, for $\mu \in \mathcal{M}_{00}(\mathbb{C})$, $U\mu$ can be $\mathbf{D}$-approximated by a series $\{U\nu_k\}$ of continuous functions and furthermore, $\lim_{k\to\infty} U\nu_k(\mathbf{x}) = U\mu(\mathbf{x})$ for $\mathbf{x} \in \mathbb{C} \setminus N$, where $N = \{\mathbf{x} \in \mathbb{C} : U\mu^-(\mathbf{x}) = \infty\}$. As $\mu^- \in \mathcal{M}_0^+$,the set N is polar by Lemma 2.4 (iii). Therefore $U\mu$ is $\mathbf{D}_1$-quasi continuous in view of Lemma 2.4 (i). □

Proposition 2.7 *(i)* $\{U\mu : \mu \in \mathcal{M}_{00}(\mathbb{C})\}$ *is dense in* $(\dot{\mathrm{BL}}(\mathbb{C}), \frac{1}{2}\mathbf{D})$.

(ii) The linear space $\mathcal{M}_{00}(\mathbb{C})$ *is pre-Hilbertian with inner product* $I(\mu, \nu) = \langle \mu, U\nu \rangle,\ \ \mu, \nu \in \mathcal{M}_{00}(\mathbb{C})$.

Proof (i). Suppose $u \in \mathrm{BL}(\mathbb{C})$ is $\mathbf{D}$-orthogonal to $\{U\mu : \mu \in \mathcal{M}_{00}(\mathbb{C})\}$. In view of **(BL.a)**, it suffices to show that u is constant a.e.

For $\mathbf{x} \in \mathbb{C}$ and $r > 0$, consider the measure $\mu^{\mathbf{x},r}(d\mathbf{y}) = \psi_r(\mathbf{x}-\mathbf{y})d\mathbf{y} - \psi_1(\mathbf{y})d\mathbf{y} \in \mathcal{M}_{00}(\mathbb{C})$. Then we have from Eq. 2.16 that

$$\frac{1}{2}\mathbf{D}(U\mu^{\mathbf{x},r}, u) = (\psi_r * u)(\mathbf{x}) - (\psi_1 * u)(\mathbf{0}) = 0, \quad \mathbf{x} \in \mathbb{C},$$

and so, $(\psi_r * u)(\mathbf{x})$ is a constant for all $\mathbf{x} \in \mathbb{C}$ and $r > 0$. As $u \in L^2_{\mathrm{loc}}(\mathbb{C})$, $\lim_{r\downarrow 0}(\psi_r * u)(\mathbf{x}) = u(\mathbf{x})$ for a.e. $\mathbf{x} \in \mathbb{C}$. Thus u is constant a.e. on $\mathbb{C}$.

(ii). By Eq. 2.10, $\langle \mu, U\mu \rangle = \frac{1}{2}\mathbf{D}(U\mu, U\mu) \geq 0$ for any $\mu \in \mathcal{M}_{00}(\mathbb{C})$. Suppose $\langle \mu, U\mu \rangle = 0$ for $\mu \in \mathcal{M}_{00}(\mathbb{C})$. Then, for any $u \in C_c^1(\mathbb{C})$, $\langle \mu, u \rangle = \frac{1}{2}\mathbf{D}(U\mu, u) = 0$ by Eqs. 2.10 and 2.11, yielding $\mu = 0$. □

This proposition means that the abstract completion of the pre-Hilbert space $(\mathcal{M}_{00}(\mathbb{C}), I)$ is isometrically isomorphic with $(\dot{\mathrm{BL}}(\mathbb{C}), \frac{1}{2}\mathbf{D})$ by the map $\mu \in \mathcal{M}_{00}(\mathbb{C}) \mapsto U\mu \in \dot{\mathrm{BL}}(\mathbb{C})$. The space $(\dot{\mathrm{BL}}(\mathbb{C}), \frac{1}{2}\mathbf{D})$ can be actually viewed as a reproducing kernel Hilbert space for $(\mathcal{M}_{00}(\mathbb{C}), I)$.

Let K be a non-polar compact set in $\mathbb{C}$. For $\mu \in \mathcal{M}_0(\mathbb{C})$,define

$$\mu_K(B) = \int_{\mathbb{C}} \mu(d\mathbf{y}) h_K(\mathbf{y}, B), \quad \text{for any Borel set } B \subset \mathbb{C}. \tag{2.17}$$

The measure μ_K is called the *balayage* of μ to K.

Proposition 2.8 *Let* $\mu \in \mathcal{M}_{00}(\mathbb{C})$. *Then* $\mu_K \in \mathcal{M}_{00}(\mathbb{C})$ *and, for any* $\nu \in \mathcal{M}_{00}(\mathbb{C})$ *with supp*$[|\nu|] \subset K$,

$$\langle U\mu, \nu \rangle = \langle U\mu_K, \nu \rangle. \tag{2.18}$$

Proof Since K is a non-polar compact set, $h_K(\mathbf{y}, K) = 1$ for any $\mathbf{y} \in \mathbb{C}$ so that μ_K vanishes on $\mathbb{C} \setminus K$ and $\mu_K(\mathbb{C}) = \mu(\mathbb{C})$ for any $\mu \in \mathcal{M}_0(\mathbb{C})$. For $\mu \in \mathcal{M}^+(\mathbb{C})$,we get the followings from the fundamental identity (2.13) (cf. [16, Theorem 6.7.17]): there exists a constant $c(\mu)$ and

$$U\mu_K(\mathbf{x}) \leq U\mu(\mathbf{x}) + c(\mu), \quad \text{for any } \mathbf{x} \in \mathbb{C}, \tag{2.19}$$

$$U\mu_K(\mathbf{x}) = U\mu(\mathbf{x}) + c(\mu), \quad \text{for q.e. } \mathbf{x} \in K. \tag{2.20}$$

If $\mu \in \mathcal{M}_0^+(\mathbb{C})$,then Eq. 2.19 implies that

$$\begin{aligned}\langle \mu_K, U\mu_K \rangle &\leq \langle \mu_K, U\mu \rangle + c(\mu)\mu_K(\mathbb{C}) \\ &= \langle U\mu_K, \mu \rangle + c(\mu)\mu(\mathbb{C}) \leq \langle U\mu, \mu \rangle + 2c(\mu)\mu(\mathbb{C}) < \infty,\end{aligned}$$

and consequently, $\mu_K \in \mathcal{M}_0^+(\mathbb{C})$. One can thus see that, if μ belongs to $\mathcal{M}_{00}(\mathbb{C})$,then so does μ_K.

Clearly the identity (2.20) remains valid for any $\mu \in \mathcal{M}_{00}(\mathbb{C})$. Integrating the both hand sides of this identity by $\nu \in \mathcal{M}_{00}(\mathbb{C})$ with $\mathrm{supp}[|\nu|] \subset K$ and noting that $|\nu|$ charges no polar set (Lemma 2.4 (iii)), we arrive at Eq. 2.18). □

2.2 Markov property of the Gaussian field index by $\mathcal{M}_{00}(\mathbb{C})$

In view of Proposition 2.7 (ii), there exist a system of centered Gaussian random variables $\mathbf{G}(\mathbb{C}) = \{X_\mu : \mu \in \mathcal{M}_{00}(\mathbb{C})\}$ on a certain probability space $(\Omega, \mathcal{B}, \mathbb{P})$ with covariance

$$\mathbb{E}[X_\mu X_\nu] = \langle \mu, U\nu \rangle, \quad \mu, \nu \in \mathcal{M}_{00}(\mathbb{C}). \tag{2.21}$$

For a subset A of $\mathbb{C}$, define the sub-σ-field of $\mathcal{B}$ by

$$\sigma(A) = \sigma\{X_\mu : \mu \in \mathcal{M}_{00}(\mathbb{C}),\ \mathrm{supp}[|\mu|] \subset A\}.$$

$\mathbf{G}(\mathbb{C})$ is said to have the *local Markov property* if, for any bounded open subset G of $\mathbb{C}$, the identity

$$\mathbb{E}[YZ \big| \sigma(\partial G)] = \mathbb{E}[Y \big| \sigma(\partial G)]\ \mathbb{E}[Z \big| \sigma(\partial G)] \tag{2.22}$$

holds for any bounded $\sigma(\overline{G})$-measurable function Y on Ω and any bounded $\sigma(\mathbb{C} \setminus G)$-measurable function Z on Ω. The following is known to be a necessary and sufficient condition for the local Markov property of $\mathbf{G}(\mathbb{C})$ (cf. [18, §6]): for any bounded open set $G \subset \mathbb{C}$

$$\sigma\{\mathbb{E}[Y \big| \sigma(\overline{G})] : Y \text{ is bounded and } \sigma(\mathbb{C} \setminus G)\text{-measurable}\} \subset \sigma(\partial G). \tag{2.23}$$

Theorem 2.9 *The Gaussian field* $\mathbf{G}(\mathbb{C})$ *indexed by* $\mathcal{M}_{00}(\mathbb{C})$ *enjoys the local Markov property.*

Proof Let G be a bounded open subset of $\mathbb{C}$ and take any $\mu \in \mathcal{M}_{00}(\mathbb{C})$ with $\mathrm{supp}[|\mu|] \subset \mathbb{C} \setminus G$. Due to the continuity of the path X_t of the planar Brownian motion $X = (X_t, \mathbb{P}_\mathbf{x})$,

$$\mu_{\overline{G}}(B) = \int_{\mathbb{C}\setminus G} \mu(d\mathbf{y})\mathbb{P}_\mathbf{y}(X_{\sigma_{\overline{G}}} \in B) = \int_{\mathbb{C}\setminus G} \mu(d\mathbf{y})\mathbb{P}_\mathbf{y}(X_{\sigma_{\partial G}} \in B) = \mu_{\partial G}(B),$$

for any Borel set $B \subset \mathbb{C}$.

By virtue of Proposition 2.8, we have for any $\nu \in \mathcal{M}_{00}(\mathbb{C})$ with $\mathrm{supp}[|\nu|] \subset \overline{G}$

$$\langle U\mu, \nu \rangle = \langle U\mu_{\overline{G}}, \nu \rangle,$$

which means $\mathbb{E}[X_\mu X_\nu] = \mathbb{E}[X_{\mu_{\overline{G}}} X_\nu]$. Hence $X_\mu - X_{\mu_{\overline{G}}}$ is orthogonal to X_ν, and consequently independent of $\sigma(\overline{G})$ because all random varibles involved are Gaussian. Accordingly

$$\mathbb{E}[X_\mu - X_{\mu_{\overline{G}}} \big| \sigma(\overline{G})] = \mathbb{E}[X_\mu - X_{\mu_{\overline{G}}}] = 0.$$

Thus we obtain

$$\mathbb{E}[X_\mu \big| \sigma(\overline{G})] = E[X_{\mu_{\overline{G}}} \big| \sigma(\overline{G})] = X_{\mu_{\overline{G}}} = X_{\mu_{\partial G}} \in \sigma(\partial G).$$

□

3 RBM on $\overline{\mathbb{H}}$ and Gaussian field indexed by $\mathcal{M}_{00}(\overline{\mathbb{H}})$

We consider the upper half plane $\mathbb{H} = \{\mathbf{x} = (x, y) \in \mathbb{C} : y > 0\}$. $\mathbb{H}$ will be also denoted by $\mathbb{H}_+$, while $\mathbb{H}_-$ denotes the lower half plane $\{\mathbf{x} = (x, y) \in \mathbb{C} : y < 0\}$. For $\mathbf{x} = (x, y) \in \mathbb{C}$, $\mathbf{x}^* = (x, -y)$ denotes its reflection relative to $\partial\mathbb{H}$.

Let $\widehat{\mathbf{X}} = (\widehat{X}_t, \{\widehat{\mathbb{P}}_{\mathbf{x}}\}_{\mathbf{x}\in\overline{\mathbb{H}}})$ be the *reflecting Brownian motion* (RBM in abbreviation) on $\overline{\mathbb{H}}$. $\widehat{\mathbf{X}}$ is obtained from the planar Brownian motion $\mathbf{X} = (X_t = (X_t^{(1)}, X_t^{(2)}), \{\mathbb{P}_{\mathbf{x}}\}_{\mathbf{x}\in\mathbb{C}})$ by

$$\widehat{X}_t = (X_t^{(1)}, |X_t^{(2)}|), \qquad \widehat{\mathbb{P}}_{\mathbf{x}} = \mathbb{P}_{\mathbf{x}}, \quad \mathbf{x} \in \overline{\mathbb{H}}. \tag{3.1}$$

Consider a closed subset F of $\overline{\mathbb{H}}$ and denote by F^r the set of all regular points for F relative to $\widehat{\mathbf{X}}$; $F^r = \{\mathbf{x} \in F; \widehat{\mathbb{P}}_{\mathbf{x}}(\sigma_F = 0) = 1\}$. $F \setminus F^r$ is then *polar relative to* $\widehat{\mathbf{X}}$, namely, $\widehat{\mathbb{P}}_{\mathbf{x}}(\sigma_{F\setminus F^r} < \infty) = 0$ for any $\mathbf{x} \in \overline{\mathbb{H}}$ (cf. [10, Theorems A.2.6, 4.1.2, 4.1.3]). We also consider the part $\widehat{\mathbf{X}}_{\overline{\mathbb{H}}\setminus F}$ on $\overline{\mathbb{H}} \setminus F$ of $\widehat{\mathbf{X}}$, that is to say, the process obtained from $\widehat{\mathbf{X}}$ by killing upon hitting the set F. Define for any Borel set $B \subset \overline{\mathbb{H}}$

$$\widehat{g}^{\overline{\mathbb{H}}\setminus F}(\mathbf{x}, B) = \widehat{\mathbb{E}}_{\mathbf{x}}\left[\int_0^{\sigma_F} I_B(\widehat{X}_s)ds\right], \quad \mathbf{x} \in \overline{\mathbb{H}}.$$

For a set $B \subset \overline{\mathbb{H}}_\pm$, $B^* = \{\mathbf{x}^*; \mathbf{x} \in B\} \subset \overline{\mathbb{H}}_\mp$ denotes its reflection relative to $\partial\mathbb{H}$.

Lemma 3.1 *If F be a closed subset of $\overline{\mathbb{H}}$ that is non-polar for $\widehat{\boldsymbol{X}}$,then, for any bounded Borel set $B \subset \overline{\mathbb{H}}$,*

$$\sup_{\boldsymbol{x}\in\overline{\mathbb{H}}} \widehat{g}^{\overline{\mathbb{H}}\setminus F}(\boldsymbol{x}, B) < \infty. \tag{3.2}$$

Proof We compare $\widehat{g}^{\overline{\mathbb{H}}\setminus F}(\mathbf{x}, B)$ with its counterpart for the planar Brownian motion $\mathbf{X}$:

$$g^{\mathbb{C}\setminus F^*}(\mathbf{x}, B) = \mathbb{E}_{\mathbf{x}}\left[\int_0^{\sigma_{F^*}} I_B(X_s)ds\right], \quad \mathbf{x} \in \mathbb{C}.$$

In view of [16, Proposition 2.2.7], it holds that

$$\sup_{\mathbf{x}\in\mathbb{C}} g^{\mathbb{C}\setminus F^*}(\mathbf{x}, \ B) < \infty, \tag{3.3}$$

whenever $F^* \subset \mathbb{C}$ is a non-polar closed set for $\mathbf{X}$ and $B \subset \mathbb{C}$ is a bounded Borel set.

Suppose F and B satisfy the stated conditions. We then have $\widehat{g}^{\overline{\mathbb{H}}\setminus F}(\mathbf{x}, B) = \text{I+II+III}$, $\mathbf{x} \in \overline{\mathbb{H}}$, where

$$\text{I} = \widehat{\mathbb{E}}_{\mathbf{x}}\left[\int_0^{\sigma_F} I_B(\widehat{X}_s)ds \ : \ \sigma_F < \sigma_{\partial\mathbb{H}}\right], \quad \text{II} = \widehat{\mathbb{E}}_{\mathbf{x}}\left[\int_0^{\sigma_{\partial\mathbb{H}}} I_B(\widehat{X}_s)ds; \sigma_{\partial\mathbb{H}} < \sigma_F\right],$$

and $\quad \text{III} = \widehat{\mathbb{E}}_{\mathbf{x}}\left[\widehat{\mathbb{E}}_{\widehat{X}_{\sigma_{\partial\mathbb{H}}}}[\int_0^{\sigma_F} I_B(\widehat{X}_s)ds]; \sigma_{\partial\mathbb{H}} < \sigma_F\right].$

Accordingly

$$\text{I} \le g^{F\cup\overline{\mathbb{H}}_-}(\mathbf{x}, B), \quad \text{II} \le g^{\overline{\mathbb{H}}_-}(\mathbf{x}, B), \quad \mathbf{x} \in \overline{\mathbb{H}}.$$

The definition (3.1) implies that, for $\mathbf{y} \in \partial\mathbb{H}$,

$$\widehat{\mathbb{E}}_{\mathbf{y}}\left[\int_0^{\sigma_F} I_B(\widehat{X}_s)ds\right] = \mathbb{E}_{\mathbf{y}}\left[\int_0^{\sigma_{F\cup F^*}} I_{B\cup B^*}(X_s)ds\right],$$

and consequently,

$$\text{III} \le \sup_{\mathbf{y}\in\mathbb{C}} g^{F\cup F^*}(\mathbf{y}, B \cup B^*).$$

Thus Eq. 3.2 follows from Eq. 3.3. □

Recall $p_t(\mathbf{x})$, $t > 0$, $\mathbf{x} \in \mathbb{C}$,and $k(\mathbf{x})$, $\mathbf{x} \in \mathbb{C}$,defined by Eq. 2.1. We let

$$\widehat{p}_t(\mathbf{x}, \mathbf{y}) = p_t(\mathbf{x}-\mathbf{y}) + p_t(\mathbf{x}-\mathbf{y}^*), \ \ t > 0, \quad \widehat{k}(\mathbf{x}, \mathbf{y}) = k(\mathbf{x}-\mathbf{y}) + k(\mathbf{x}-\mathbf{y}^*), \quad \mathbf{x}, \ \mathbf{y} \in \overline{\mathbb{H}}. \tag{3.4}$$

$\widehat{p_t}(\mathbf{x},\mathbf{y})$, $\mathbf{x},\mathbf{y} \in \overline{\mathbb{H}}$, is the transition density of the RBM on $\overline{\mathbb{H}}$. Let us call $\widehat{k}(\mathbf{x},\mathbf{y})$, $\mathbf{x},\mathbf{y} \in \overline{\mathbb{H}}$, the *logarithmic kernel for* the RBM on $\overline{\mathbb{H}}$.

The following proposition is a counterpart of Eq. 2.13 for the RBM on $\overline{\mathbb{H}}$.

Proposition 3.2 *For any compact subset K of $\overline{\mathbb{H}}$ that is non-polar for $\widehat{\boldsymbol{X}}$, we have*

$$\widehat{k}(\boldsymbol{x},\boldsymbol{y}) = \widehat{g}^{\overline{\mathbb{H}}\setminus K}(\boldsymbol{x},\boldsymbol{y}) + \int_K \widehat{h}_K(\boldsymbol{x},dz)\widehat{k}(z,\boldsymbol{y}) - \hat{W}_K(\boldsymbol{x}), \quad \boldsymbol{x},\boldsymbol{y} \in \overline{\mathbb{H}}. \tag{3.5}$$

Here $\widehat{g}^{\overline{\mathbb{H}}\setminus K}(\boldsymbol{x},\boldsymbol{y})$ is the 0-order resolvent density of $\widehat{\boldsymbol{X}}_{\overline{\mathbb{H}}\setminus K}$, $\widehat{h}_K$ is the hitting distribution for K of $\widehat{\boldsymbol{X}}$ defined by $\widehat{h}_K(\boldsymbol{x},B) = \widehat{\mathbb{P}}_{\boldsymbol{x}}(\sigma_K < \infty, \widehat{X}_{\sigma_K} \in B)$ for any Borel set $B \subset \overline{\mathbb{H}}$, and $\hat{W}_K$ is a certain non-negative locally bounded function on $\overline{\mathbb{H}}$ vanishing on K^r. For $\boldsymbol{x} \neq \boldsymbol{y}$, both $\widehat{g}^{\overline{\mathbb{H}}\setminus K}(\boldsymbol{x},\boldsymbol{y})$ and $\int_K \widehat{h}_K(\boldsymbol{x},dz)\widehat{k}(z,\boldsymbol{y})$ are finite.

Proof For $\alpha > 0$, we set

$$g_\alpha(\mathbf{x}) = \int_0^\infty e^{-\alpha t} p_t(\mathbf{x})dt, \quad k_\alpha(\mathbf{x}) = g_\alpha(\mathbf{x}) - g_\alpha(\mathbf{x}_0), \quad \mathbf{x} \in \mathbb{C},$$

where $\mathbf{x}_0$ is a fixed point in $\overline{\mathbb{H}}$ with $|\mathbf{x}_0| = 1$. As Lemma 2.1, we can then see that

$$0 \le k_\alpha(\mathbf{x}) \uparrow k(\mathbf{x}), \text{ for } |\mathbf{x}| \le 1; \quad 0 \le -k_\alpha(\mathbf{x}) \uparrow -k(\mathbf{x}) < \infty, \text{ for } |\mathbf{x}| > 1, \quad \alpha \downarrow 0. \tag{3.6}$$

We next set, for $\mathbf{x},\ \mathbf{y} \in \overline{\mathbb{H}}$,

$$\widehat{g_\alpha}(\mathbf{x},\mathbf{y}) = g_\alpha(\mathbf{x}-\mathbf{y}) + g_\alpha(\mathbf{x}-\mathbf{y}^*), \quad \widehat{k_\alpha}(\mathbf{x},\mathbf{y}) = k_\alpha(\mathbf{x}-\mathbf{y}) + k_\alpha(\mathbf{x}-\mathbf{y}^*).$$

Since $\widehat{g_\alpha}(\mathbf{x},\mathbf{y})$ is the α-order resolvent density of $\widehat{\mathbf{X}}$, the strong Markov property of $\widehat{\mathbf{X}}$ yields the identity

$$\widehat{g_\alpha}(\mathbf{x},\mathbf{y}) = \widehat{g_\alpha}^{\overline{\mathbb{H}}\setminus K}(\mathbf{x},\mathbf{y}) + \int_K \widehat{h}_K^\alpha(\mathbf{x},d\mathbf{z})\widehat{g_\alpha}(\mathbf{z},\mathbf{y}), \quad \mathbf{x},\ \mathbf{y} \in \overline{\mathbb{H}},$$

for the α-order resolvent density $\widehat{g_\alpha}^{\overline{\mathbb{H}}\setminus K}$ of $\widehat{\mathbf{X}}_{\overline{\mathbb{H}}\setminus K}$ and for the α-order hitting distribution $\widehat{h}_K^\alpha$ of $\widehat{\mathbf{X}}$ for K. By substituting $\widehat{g_\alpha}(\mathbf{x},\mathbf{y}) = \widehat{k_\alpha}(\mathbf{x},\mathbf{y}) - 2g_\alpha(\mathbf{x}_0)$ into the above identity, we obtain

$$\widehat{k_\alpha}(\mathbf{x},\mathbf{y}) = \widehat{g_\alpha}^{\overline{\mathbb{H}}\setminus K}(\mathbf{x},\mathbf{y}) + \int_K \widehat{h}_K^\alpha(\mathbf{x},d\mathbf{z})\widehat{k_\alpha}(\mathbf{z},\mathbf{y}) - \hat{W}_K^\alpha(\mathbf{x}), \quad \mathbf{x},\ \mathbf{y} \in \overline{\mathbb{H}}, \tag{3.7}$$

where

$$\hat{W}_K^\alpha(\mathbf{x}) = 2g_\alpha(\mathbf{x}_0)\ (1 - \widehat{\mathbb{E}}_{\mathbf{x}}[e^{-\alpha\sigma_K}]), \quad \mathbf{x} \in \overline{\mathbb{H}}. \tag{3.8}$$

Integrating the both hand sides of Eq. 3.7 by $d\mathbf{y}$ over a bounded Borel set $B \subset \mathbb{H}$ with a positive Lebesgue measure $|B|$, we get

$$\widehat{k_\alpha}(\mathbf{x},B) = \widehat{g_\alpha}^{\overline{\mathbb{H}}\setminus K}(\mathbf{x},B) + \int_K \widehat{h}_K^\alpha(\mathbf{x},d\mathbf{z})\widehat{k_\alpha}(\mathbf{z},B) - |B|\hat{W}_K^\alpha(\mathbf{x}), \quad \mathbf{x} \in \overline{\mathbb{H}}.$$

By Lemma 3.1, $g^{\overline{\mathbb{H}}\setminus K}(\mathbf{x},B)$ is bounded in $\mathbf{x} \in \overline{\mathbb{H}}$. Moreover $\int_B \widehat{k}(\mathbf{x},\mathbf{y})d\mathbf{y}$ is locally bounded on $\overline{\mathbb{H}}$. Hence we can see by letting $\alpha \to 0$ in the above identity that the limit $\hat{W}_K(\mathbf{x}) = \lim_{\alpha\downarrow 0} \hat{W}_K^\alpha(\mathbf{x})$ exists and further that the limit is locally bounded on $\overline{\mathbb{H}}$ and vanishing on K^r. We finally let $\alpha \downarrow 0$ in the Eq. 3.7 by noticing (3.6) to arrive at the desired conclusions. □

Denote by $\mathcal{M}^+(\overline{\mathbb{H}})$ the collection of positive finite measures on $\overline{\mathbb{H}}$ with compact support. The logarithmic potential $\widehat{U}\mu$ of $\mu \in \mathcal{M}^+(\overline{\mathbb{H}})$ for RBM is defined by

$$\widehat{U}\mu(\mathbf{x}) = \int_{\overline{\mathbb{H}}} \widehat{k}(\mathbf{x},\mathbf{y})\mu(d\mathbf{y}), \quad \mathbf{x} \in \overline{\mathbb{H}}. \tag{3.9}$$

For $\mu \in \mathcal{M}^+(\overline{\mathbb{H}})$ and a compact set $K \subset \overline{\mathbb{H}}$, define

$$\widehat{\mu}_K(B) = \int_{\overline{\mathbb{H}}} \mu(d\mathbf{y})\widehat{h}_K(\mathbf{y}, B), \quad \text{for any Borel } B \subset \overline{\mathbb{H}}. \tag{3.10}$$

Then $\widehat{\mu}_K \in \mathcal{M}^+(\overline{\mathbb{H}})$ and $\mathrm{supp}[\widehat{\mu}_K] \subset K$. $\widehat{\mu}_K$ is called the *balayage* of μ to K relative to $\widehat{\mathbf{X}}$.

$\langle \nu, u\rangle_{\overline{\mathbb{H}}}$ or $\langle u, \nu\rangle_{\overline{\mathbb{H}}}$ will designate the integral $\int_{\overline{\mathbb{H}}} u(\mathbf{x})\nu(d\mathbf{x})$ for a function u and a measure ν on $\overline{\mathbb{H}}$.

Proposition 3.3 *Let K be a compact subset of $\overline{\mathbb{H}}$ that is non-polar for $\widehat{\mathbf{X}}$. It holds then for any $\mu \in \mathcal{M}^+(\overline{\mathbb{H}})$*

$$\widehat{U}\widehat{\mu}_K(\boldsymbol{x}) = \widehat{U}\mu(\boldsymbol{x}) + \langle \hat{W}_K, \mu\rangle_{\overline{\mathbb{H}}}, \quad \textit{for any } \boldsymbol{x} \in K^r, \tag{3.11}$$

$$\widehat{U}\widehat{\mu}_K(\boldsymbol{x}) \le \widehat{U}\mu(\boldsymbol{x}) + \langle \hat{W}_K, \mu\rangle_{\overline{\mathbb{H}}}, \quad \textit{for any } \boldsymbol{x} \in \overline{\mathbb{H}}. \tag{3.12}$$

Proof Both $\widehat{k}(\mathbf{x}, \mathbf{y})$ and $\widehat{g}^{\overline{\mathbb{H}}\setminus K}(\mathbf{x}, \mathbf{y})$ being symmetric, Eq. 3.5 yields

$$\int_K \widehat{h}_K(\mathbf{x}, d\mathbf{z})\widehat{k}(\mathbf{z}, \mathbf{y}) - \hat{W}_K(\mathbf{x}) = \int_K \widehat{h}_K(\mathbf{y}, d\mathbf{z})\widehat{k}(\mathbf{z}, \mathbf{x}) - \hat{W}_K(\mathbf{y}),$$

and consequently,

$$\widehat{\mathbb{E}}_{\mathbf{x}}\left[\widehat{U}\mu(\widehat{X}_{\sigma_K}); \sigma_K < \infty\right] = \widehat{U}\widehat{\mu}_K(\mathbf{x}) - \langle \hat{W}_K, \mu\rangle_{\overline{\mathbb{H}}} + \mu(\overline{\mathbb{H}})\hat{W}(\mathbf{x}), \quad \mathbf{x} \in \overline{\mathbb{H}}. \tag{3.13}$$

We obtain Eq. 3.11 from Eq. 3.13 for $\mathbf{x} \in K^r$. Further Eqs. 3.5 and 3.13 lead us to

$$\begin{aligned}\widehat{U}\mu(\mathbf{x}) &= \int_{\overline{\mathbb{H}}} \widehat{g}^{\overline{\mathbb{H}}\setminus K}(\mathbf{x}, \mathbf{y})\mu(d\mathbf{y}) + \widehat{\mathbb{E}}_{\mathbf{x}}\left[\widehat{U}\mu(\widehat{X}_{\sigma_K}); \sigma_K < \infty\right] - \mu(\overline{\mathbb{H}})\hat{W}(\mathbf{x}) \\ &\ge \widehat{U}\widehat{\mu}_K(\mathbf{x}) - \langle \hat{W}_K, \mu\rangle_{\overline{\mathbb{H}}}, \quad \mathbf{x} \in \overline{\mathbb{H}},\end{aligned}$$

which yields (3.12). □

Let us introduce classes of measures on $\overline{\mathbb{H}}$ by

$$\mathcal{M}_0^+(\overline{\mathbb{H}}) = \{\mu \in \mathcal{M}^+(\overline{\mathbb{H}}) : \langle \mu, \widehat{U}\mu\rangle_{\overline{\mathbb{H}}} < \infty\},$$

$$\mathcal{M}_0(\overline{\mathbb{H}}) = \{\mu : \text{ finite signed measure on } \overline{\mathbb{H}},\ |\mu| \in \mathcal{M}_0^+(\overline{\mathbb{H}})\},$$

$$\mathcal{M}_{00}(\overline{\mathbb{H}}) = \{\mu \in \mathcal{M}_0(\overline{\mathbb{H}}) : \mu(\overline{\mathbb{H}}) = 0\}.$$

Given $\mu \in \mathcal{M}_0(\overline{\mathbb{H}})$, its extension μ^* to a measure on $\mathbb{C}$ by reflection relative to $\partial\mathbb{H}$ is defined as follows: for a Borel set B, $\mu^*(B) = \mu(B)$ if $B \subset \overline{\mathbb{H}}_+$, and $\mu^*(B) = \mu(B^*)$ if $B \subset \mathbb{H}_-$. Further, given a function f on $\overline{\mathbb{H}}$, its extension f^* to a function on $\mathbb{C}$ by reflection relative to $\partial\mathbb{H}$ is defined as follows: $f^*(\mathbf{x}) = f(\mathbf{x})$ if $\mathbf{x} \in \mathbb{H}_+$, and $f^*(\mathbf{x}) = f(\mathbf{x}^*)$ if $x \in \mathbb{H}_-$.

We readily obtain the following.

Lemma 3.4 *It holds for $\mu \in \mathcal{M}^+(\overline{\mathbb{H}})$ that*

$$U\mu^*(\boldsymbol{x}) = (\widehat{U}\mu)^*(\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{C}, \tag{3.14}$$

and

$$\langle \mu, \widehat{U}\mu\rangle_{\overline{\mathbb{H}}} = \frac{1}{2}\langle \mu^*, U\mu^*\rangle. \tag{3.15}$$

Equation 3.15 implies that $\mu \in \mathcal{M}_0(\overline{\mathbb{H}})$ (resp. $\mu \in \mathcal{M}_{00}(\overline{\mathbb{H}})$) if and only if $\mu^* \in \mathcal{M}_0(\mathbb{C})$ (resp. $\mu^* \in \mathcal{M}_{00}(\mathbb{C})$).

Analogously to the case of the whole plane $\mathbb{C}$, we consider the *Sobolev space of order* 1 and the *Beppo Levi space* over $\mathbb{H}$ defined respectively by

$$H^1(\mathbb{H}) = \{u \in L^2(\mathbb{H}) : |\nabla u| \in Ł^2(\mathbb{H})\}, \quad \mathrm{BL}(\mathbb{H}) = \{u \in L^2_{\mathrm{loc}}(\mathbb{H}) : |\nabla u| \in Ł^2(\mathbb{H})\}.$$

Denote the Dirichlet integral $\int_{\mathbb{H}} \nabla f(\mathbf{x}) \cdot \nabla g(\mathbf{x}) d\mathbf{x}$ of functions f, g on $\mathbb{H}$ by $\mathbf{D}_{\mathbb{H}}(f, g)$. $(\frac{1}{2}\mathbf{D}_{\mathbb{H}}, H^1(\mathbb{H}))$ is the regular Dirichlet form on $L^2(\overline{\mathbb{H}})$ associated with the RBM on $\overline{\mathbb{H}}$. Its extended Dirichlet space is known to be identical with the space $(\mathrm{BL}(\mathbb{H}), \frac{1}{2}\mathbf{D}_{\mathbb{H}})$ just as the case of $\mathbb{C}$ in place of $\mathbb{H}$. Moreover the quotient space $\dot{\mathrm{BL}}(\mathbb{H})$ of $\mathrm{BL}(\mathbb{H})$ by its subspace of constant functions on $\mathbb{H}$ is a real Hilbert space with inner product $\frac{1}{2}\mathbf{D}_{\mathbb{H}}$.

Lemma 3.5 *$f \in BL(\mathbb{H})$ if and only if $f^* \in BL(\mathbb{C})$. It holds for any $f \in BL(\mathbb{H})$ that*

$$\boldsymbol{D}_{\mathbb{H}}(f, f) = \frac{1}{2}\boldsymbol{D}(f^*, f^*). \tag{3.16}$$

Proof For $f \in L^2(\mathbb{H})$,we let

$$\widehat{P}_t f(\mathbf{x}) = \int_{\mathbb{H}} \widehat{p}_t(\mathbf{x}, \mathbf{y}) f(\mathbf{y}) d\mathbf{y}, \ \mathbf{x} \in \mathbb{H}, \quad P_t f^*(\mathbf{x}) = \int_{\mathbb{C}} p_t(\mathbf{x} - \mathbf{y}) f^*(\mathbf{y}) d\mathbf{y}, \ \mathbf{x} \in \mathbb{C}.$$

We then easily see that $P_t f^*(\mathbf{x}) = (\widehat{P}_t f)^*(\mathbf{x}), \ \mathbf{x} \in \mathbb{C}$,so that

$$\begin{aligned}(f^*, f^* - P_t f^*)_{L^2(\mathbb{C})} &= \int_{\mathbb{H}_+} f(\mathbf{x})(f(\mathbf{x}) - \widehat{P}_t f(\mathbf{x}))d\mathbf{x} \\ &\quad + \int_{\mathbb{H}_-} f(\mathbf{x}^*)(f(\mathbf{x}^*) - \widehat{P}_t f(\mathbf{x}^*))d\mathbf{x} = 2(f, f - \widehat{P}_t f)_{L^2(\mathbb{H})}.\end{aligned}$$

Dividing the above identity by t and letting $t \downarrow 0$,we get the stated relations between the Sobolev spaces H^1 of order 1, that can be readily extended to the ones between the Bepp Levi spaces. □

Let us define $\mathbf{D}_{\mathbb{H},1}$-capacity $\mathrm{Cap}^{\mathbb{H}}$ of a subset $B \subset \overline{\mathbb{H}}$ and $\mathbf{D}_{\mathbb{H},1}$-quasi continuity of a function on $\overline{\mathbb{H}}$ in the same way as in Section 2 but with $\mathbf{D}_{\mathbb{H}}$ in place of $\mathbf{D}$. Then a set $N \subset \overline{\mathbb{H}}$ is polar relative to $\widehat{\mathbf{X}}$ if and only if $\mathrm{Cap}^{\mathbb{H}}(N) = 0$.

Lemma 3.6 *(i) Any function in $BL(\mathbb{H})$ admits its $\boldsymbol{D}_{\mathbb{H},1}$-quasi continuous version.*
(ii) Any $\mu \in \mathcal{M}_0^+(\overline{\mathbb{H}})$ charges no polar set relative to $\widehat{\mathbf{X}}$.
(iii) If f is a $\boldsymbol{D}_1$-quasi continuous function on $\mathbb{C}$, then $f\big|_{\overline{\mathbb{H}}}$ is $\boldsymbol{D}_{\mathbb{H},1}$-quasi continuous.
(iv) If f is a $\boldsymbol{D}_{\mathbb{H},1}$-quasi continuous function on $\overline{\mathbb{H}}$, then f^* is $\boldsymbol{D}_1$-quasi continuous on $\mathbb{C}$.

Proof (i). See [10, Theorem 2.1.7].

(ii). This can be shown using the identity (3.5) that any measure $\nu \in \mathcal{M}_{00}(\overline{\mathbb{H}})$ charges no polar set for $\widehat{\mathbf{X}}$, just as the corresponding statement is shown in the proof of Lemma 2.4 (iii) using the identity (2.13).

(iii). For an open set $G \subset \mathbb{C}, \quad \mathrm{Cap}^{\mathbb{H}}(G \cap \overline{\mathbb{H}}) \le \mathrm{Cap}(G)$.

(iv). For an open set $G \subset \overline{\mathbb{H}}, \quad \mathrm{Cap}(G \cup G^*) \le \mathrm{Cap}^{\mathbb{H}}(G)$. □

Theorem 3.7 *For any $\mu \in \mathcal{M}_{00}(\overline{\mathbb{H}})$, $\widehat{U}\mu \in BL(\mathbb{H})$ and $\widehat{U}\mu$ is $\boldsymbol{D}_{\mathbb{H},1}$-quasi continuous. Furthermore*

$$\frac{1}{2}\boldsymbol{D}_{\mathbb{H}}(\widehat{U}\mu, u) = \langle \widetilde{u}, \mu \rangle_{\overline{\mathbb{H}}}, \quad \text{for any } u \in BL(\mathbb{H}), \tag{3.17}$$

where $\widetilde{u}$ is any $\boldsymbol{D}_{\mathbb{H},1}$-quasi continuous version of u.

Proof $\mu^* \in \mathcal{M}_{00}(\mathbb{C})$ by Eq. 3.15 so that $U\mu^*$ is a $\mathbf{D}_1$-quasi continuous function in $\mathrm{BL}(\mathbb{C})$ by Theorem 2.6 and $\widehat{U}\mu$ is $\mathbf{D}_{\mathbb{H},1}$-quasi continuous by Eq. 3.14 and Lemma 3.6 (iii). For any $\mathbf{D}_{\mathbb{H},1}$-quasi continuous version $\widetilde{u}$ of $u \in \mathrm{BL}(\mathbb{H})$, $\widetilde{u}^*$ is $\mathbf{D}_1$-quasi continuous by Lemma 3.6 (iv). Therefore it follows from Eq. 3.15, Lemma 3.5 and Theorem 2.6 that

$$\frac{1}{2}\mathbf{D}_{\mathbb{H}}(\widehat{U}\mu, u) = \frac{1}{4}\mathbf{D}(U\mu^*, u^*) = \frac{1}{2}\langle \mu^*, \widetilde{u}^* \rangle = \langle \mu, \widetilde{u} \rangle_{\overline{\mathbb{H}}}.$$

□

Proposition 3.8 *(i)* $\{\widehat{U}\mu : \mu \in \mathcal{M}_{00}(\overline{\mathbb{H}})\}$ *is dense in* $(\dot{\mathbf{B}}\mathbf{L}(\mathbb{H}), \frac{1}{2}\mathbf{D}_{\mathbb{H}})$.

(ii) The linear space $\mathcal{M}_{00}(\overline{\mathbb{H}})$ *is pre-Hilbertian with inner product* $I_{\mathbb{H}}(\mu, \nu) = \langle \mu, U\nu \rangle_{\overline{\mathbb{H}}} : \mu, \nu \in \mathcal{M}_{00}(\overline{\mathbb{H}})$.

Proof Using Eq. 3.17, (i) can be shown as the proof of Proposition 2.7 (i). (ii) follows from Eq. 3.15 and Proposition 2.7 (ii). □

This proposition implies that the abstract completion of the pre-Hilbert space $(\mathcal{M}_{00}(\overline{\mathbb{H}}), I_{\mathbb{H}})$ is isometrically isomorphic with $(\dot{\mathrm{B}}\mathrm{L}(\mathbb{H}), \frac{1}{2}\mathbf{D}_{\mathbb{H}})$ by the map $\mu \in \mathcal{M}_{00}(\overline{\mathbb{H}}) \mapsto U\mu \in .\dot{\mathrm{B}}\mathrm{L}(\mathbb{H})$.

We now state a counterpart of Proposition 2.8 for the upper-half plane. For $\mu \in \mathcal{M}_0(\overline{\mathbb{H}})$ and a compact set $K \subset \overline{\mathbb{H}}$,define the balayage $\widehat{\mu}_K$ of μ on K by Eq. 3.10.

Proposition 3.9 *Let* $\mu \in \mathcal{M}_{00}(\overline{\mathbb{H}})$ *and* $K \subset \overline{\mathbb{H}}$ *be a compact set that is non-polar relative to the RBM* $\widehat{\mathbf{X}}$. *Then* $\widehat{\mu}_K \in \mathcal{M}_{00}(\overline{\mathbb{H}})$ *and, for any* $\nu \in \mathcal{M}_{00}(\overline{\mathbb{H}})$ *with supp*$[|\nu|] \subset K$,

$$\langle \widehat{U}\mu, \nu \rangle_{\overline{\mathbb{H}}} = \langle \widehat{U}\widehat{\mu}_K, \nu \rangle_{\overline{\mathbb{H}}}. \tag{3.18}$$

Proof One can proceed along the same line as the proof of Proposition 2.9. Since the compact set K is non-polar for the RBM $\widehat{\mathbf{X}}$ which is irreducible recurrent, $\widehat{h}_K(\mathbf{y}, K) = 1$ for any $\mathbf{y} \in \overline{\mathbb{H}}$ (cf. [10, Exercise 4.7.1]) so that $\widehat{\mu}_K(\overline{\mathbb{H}} \setminus K) = 0$ and $\widehat{\mu}_K(\overline{\mathbb{H}}) = \mu(\overline{\mathbb{H}})$ for any $\mu \in \mathcal{M}_0(\overline{\mathbb{H}})$. By making use of the inequality (3.12), we can show that, if μ belongs to $\mathcal{M}_{00}(\mathbb{H})$, so does $\widehat{\mu}_K$, just as the corresponding statment is derived in the proof of Proposition 2.8 from the inequality (2.19).

The equality (3.11) holding for $\mathbf{x} \in K^r$ remains valid for $\mu \in \mathcal{M}_{00}(\overline{\mathbb{H}})$ in place of $\mu \in \mathcal{M}^+(\overline{\mathbb{H}})$. Integrating the both hand sides of this identity with respect to $\nu \in \mathcal{M}_{00}(\overline{\mathbb{H}})$ by taking Lemma 3.6 (ii) into account, we arrive at Eq. 3.18. □

By virtue of Proposition 3.8 (ii), there exist a system of centered Gaussian random variables $\mathbf{G}(\overline{\mathbb{H}}) = \{X_\mu : \mu \in \mathcal{M}_{00}(\overline{\mathbb{H}})\}$ on a certain probability space $(\Omega, \mathcal{B}, \mathbb{P})$ with covariance

$$\mathbb{E}[X_\mu X_\nu] = \langle \mu, \widehat{U}\nu \rangle, \quad \mu, \nu \in \mathcal{M}_{00}(\overline{\mathbb{H}}). \tag{3.19}$$

Exactly in the same way as Theorem 2.9 is derived from Proposition 2.8, we can obtain the following from Proposition 3.9:

Theorem 3.10 *The Gaussian field* $\mathbf{G}(\overline{\mathbb{H}})$ *indexed by* $\mathcal{M}_{00}(\overline{\mathbb{H}})$ *enjoys the local Markov property.*

4 Linear potentials and Gaussian field indexed by $\mathcal{M}_{00}(\mathbb{R})$

For the real line $\mathbb{R}$, let

$$p_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}, \quad t > 0,\ x \in \mathbb{R}, \qquad k(x) = -|x|, \quad x \in \mathbb{R}. \tag{4.1}$$

$p_t(x-y)$ is the transition density of the Brownian motion on $\mathbb{R}$ and $k(x-y)$ is called the *linear potential kernel*. If we set

$$k_T(x) = \int_0^T (p_t(x) - p_t(0))dt, \qquad x \in \mathbb{R}, \quad T > 0,$$

then (cf. [16, p. 82]),

$$0 > k_T(x) \ \downarrow \ k(x), \qquad T \uparrow \infty, \quad x \in \mathbb{R}. \tag{4.2}$$

For functions f, g on $\mathbb{R}$, the integrals $\int_{\mathbb{R}} f(x)g(x)dx$ and $\int_{\mathbb{R}} f'(x)g'(x)dx$ are denoted by (f, g) and $\mathbf{D}(f, g)$, respectively. The *Cameron-Martin space* on $\mathbb{R}$ is defined by

$$H_e^1(\mathbb{R}) = \{u : \text{ absolutely continuous on } \mathbb{R}, \quad \mathbf{D}(u,u) < \infty\}. \tag{4.3}$$

Put $H^1(\mathbb{R}) = H_e^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then $(\frac{1}{2}\mathbf{D}, H^1(\mathbb{R}))$ is the Dirichlet form on $L^2(\mathbb{R})$ associated with the Brownian motion on $\mathbb{R}$ and $(H_e^1(\mathbb{R}), \frac{1}{2}\mathbf{D})$ is its extended Dirichlet space (cf. [10, Exercise 1.6.2]).

Lemma 4.1 *The quotient space $\dot{H}_e^1(\mathbb{R})$ of $H_e^1(\mathbb{R})$ by constant functions on $\mathbb{R}$ is a Hilbert space with inner product $\frac{1}{2}\boldsymbol{D}$.*

If $u_n \in H_e^1(\mathbb{R})$ is $\boldsymbol{D}$-convergent to $u \in H_e^1(\mathbb{R})$ as $n \to \infty$,then there are constants c_n such that $u_n - c_n$ converges to u locally uniformly on $\mathbb{R}$ as $n \to \infty$.

Proof From $f(b) - f(a) = \int_a^b f'(\xi)d\mathbf{x}$, we have $(f(a) - f(b))^2 \le |a-b|\mathbf{D}(f,f)$. So $\{u_n - u_n(0)\}$ is locally uniformly convergent to $u + c$ for some constant c, □

The linear potential of a Borel function f on $\mathbb{R}$ is defined by
$Uf(x) = \int_{\mathbb{R}} k(x-y)f(y)dy, \quad x \in \mathbb{R}$,whenever the integral makes sense.

Proposition 4.2 *Let f be a bounded Borel function on $\mathbb{R}$ vanishing outside some bounded set and satisfying $\int_{\mathbb{R}} f(x)dx = 0$. Then $Uf \in H_e^1(\mathbb{R})$ and*

$$\frac{1}{2}\boldsymbol{D}(Uf, u) = (f, u) \quad \textit{for any } u \in \boldsymbol{H}_e^1(\mathbb{R}). \tag{4.4}$$

Proof For f with the stated properties, we get by noting (4.2) that

$$Uf(x) = \lim_{T\to\infty} \int_{\mathbb{R}} k_T(x-y)f(y)dy = \lim_{T\to\infty} \int_0^T p_t(x-y)f(y)dy \quad x \in \mathbb{R}.$$

Hence we can verify as in the proof of Proposition 2.2 that $Uf \in H_e^1(\mathbb{R})$ and the Eq. 4.4 holds for any $u \in H^1(\mathbb{R})$. For any $u \in H_e^1(\mathbb{R})$, choose $u_n \in H^1(\mathbb{R}),\ n \ge 1$,that are $\mathbf{D}$-convergent to u. Then ,taking constants $c_n,\ n \ge 1$, as in Lemma 4.1, we can obtain (4.4) for u from those for $u_n,\ n \ge 1$. □

Denote by $\mathcal{M}^+(\mathbb{R})$ the collection of positive finite measures on $\mathbb{R}$ with compact support and define

$$\mathcal{M}_0(\mathbb{R}) = \{\mu;\ \text{finite signed measure}, |\mu| \in \mathcal{M}^+(\mathbb{R})\}, \quad \mathcal{M}_{00}(\mathbb{R}) = \{\mu \in \mathcal{M}_0(\mathbb{R}) : \mu(\mathbb{R}) = 0\}.$$

The linear potential of $\mu \in \mathcal{M}_0(\mathbb{R})$ is defined by $U\mu(x) = \int_{\mathbb{R}} k(x-y)\mu(dy),\ x \in \mathbb{R}$. For a function f and a measure ν on $\mathbb{R}$, the integral $\int_{\mathbb{R}} f(x)\nu(dx)$ will be disignated as $\langle \nu, f\rangle$ or $\langle f, \nu\rangle$.

Theorem 4.3 *For any $\mu \in \mathcal{M}_{00}(\mathbb{R})$, $U\mu \in H_e^1(\mathbb{R})$ and*

$$\frac{1}{2}\boldsymbol{D}(U\mu, u) = \langle \mu, u\rangle, \quad \text{for any } u \in H_e^1(\mathbb{R}). \tag{4.5}$$

In particular,

$$\frac{1}{2}\boldsymbol{D}(U\mu, U\mu) = \langle \mu, U\mu\rangle. \tag{4.6}$$

Proof Let $\psi_n(x) = \frac{n}{2} I_{(-1/n.1/n)}(x),\ x \in \mathbb{R}$,and $J_R = [-R, R],\ R > 0$. Consider $\mu \in \mathcal{M}^+(\mathbb{R})$ with $\text{supp}[\mu] \subset J_R$ for some $R > 0$ and define

$$\mu_n(x) = \int_{\mathbb{R}} \psi_n(x-y)\mu(dy), \quad x \in \mathbb{R}, \quad n \geq 1. \tag{4.7}$$

μ_n is a continuous function whose support is contained in J_{R+1} and $\int_{\mathbb{R}} \mu_n(x)dx = \mu(\mathbb{R})$. Hence

$$|(\mu_n, U\mu_n)| \leq M\mu(\mathbb{R})^2 \quad \text{where } M = \sup_{x,y \in J_{R+1}} |x-y|.$$

We further see that

$$\lim_{n\to\infty} U\mu_n(x) = U\mu(x), \quad x \in \mathbb{R}; \quad \lim_{n\to\infty} (u, \mu_n) = \langle \mu, u\rangle, \tag{4.8}$$

where u is any continuous function on $\mathbb{R}$.

We now take any $\mu \in \mathcal{M}_{00}$ and define μ_n by Eq. 4.7. Then μ_n is a continous function on $\mathbb{R}$ with compact support and $\int_{\mathbb{R}} \mu_n(x)dx = 0$. Therefore, by Proposition 4.2, $U\mu_n \in H_e^1(\mathbb{R})$ and

$$\frac{1}{2}\mathbf{D}(U\mu_n, u) = (u, \mu_n), \quad \text{for any } u \in H_e^1(\mathbb{R}). \tag{4.9}$$

Since $\frac{1}{2}\mathbf{D}(U\mu_n, U\mu_n) = (\mu_n, U\mu_n)$ is uniformly bounded in n by the preceding observation, for a Cesàro mean sequence (denoted by $\{\nu_n\}$) of a suitable subsequence of $\{\mu_n\}$, $U\nu_n$ is $\mathbf{D}$-convergent to some $v \in H_e^1(\mathbb{R})$. According to Lemma 4.1, there are constants c_n such that $U\nu_n + c_n$ is locally uniformly convergent to v. As Eq. 4.8 remains valid for ν_n in place of μ_n, the limit $\lim_{n\to\infty} c_n = c$ exists and $v = U\mu + c$. By letting $n \to \infty$ in the Eq. 4.9 for ν_n in place of μ_n, we get to Eq. 4.5. □

Proposition 4.4 *(i) $\{U\mu : \mu \in \mathcal{M}_{00}(\mathbb{R})\}$ is dense in $(\dot{H}_e^1(\mathbb{R}), \frac{1}{2}\boldsymbol{D})$.*

(ii) The linear space $\mathcal{M}_{00}(\mathbb{R})$ is pre-Hilbertian with inner product $I_{\mathbb{R}}(\mu, \nu) = \langle \mu, U\nu\rangle : \mu, \nu \in \mathcal{M}_{00}(\mathbb{R})$.

Proof (i). Suppose $u \in H_e^1(\mathbb{R})$ is $\mathbf{D}$-orthogonal to $\{U\mu : \mu \in \mathcal{M}_{00}(\mathbb{R})\}$. Then $\langle \mu, u\rangle = 0$ for any $\mu \in \mathcal{M}_{00}(\mathbb{R})$. Taking $\mu = \delta_x - \delta_0 \in \mathcal{M}_{00}(\mathbb{R})$,where δ_x denotes the delta-measure on $\mathbb{R}$ concentrated on $\{x\}$, we get $u(x) = u(0)$ for any $x \in \mathbb{R}$, so that u is a constant function.

(ii). By Eq. 4.6, $I_{\mathbb{R}}(\mu, \mu) \geq 0$ for any $\mu \in \mathcal{M}_{00}(\mathbb{R})$. Suppose $I_{\mathbb{R}}(\mu, \mu) = 0$ for $\mu \in \mathcal{M}_{00}(\mathbb{R})$. Then $\langle \mu, u\rangle = \frac{1}{2}\mathbf{D}(U\mu, u) = 0$ for any $u \in C_c(\mathbb{R})$ by Eqs. 4.6 and 4.5, yielding $\mu = 0$. □

This proposition implies that the abstract completion of the pre-Hilbert space $(\mathcal{M}_{00}(\mathbb{R}), I_{\mathbb{R}})$ is isometrically isomorphic with $(\dot{H}_e^1(\mathbb{R}), \frac{1}{2}\mathbf{D})$ by the map $\mu \in \mathcal{M}_{00}(\mathbb{R}) \mapsto U\mu \in \dot{H}_e^1(\mathbb{R})$.

We finally state a counterpart of Proposition 2.8 and Proposition 3.9 for the present linear case. The *balayage* of $\mu \in \mathcal{M}_{00}(\mathbb{R})$ to a non-empty compact set $K \subset \mathbb{R}$ is defined by $\mu_K(\cdot) = \int_{\mathbb{R}} \mu(dy)h_k(y, \cdot)$ using the hitting distribution $h_K(x, \cdot) = \mathbb{P}_x(X_{\sigma_K} \in \cdot)$ of the one-dimensional Brownian motion $(X_t, \{\mathbb{P}_x\}_{x\in\mathbb{R}})$.

Proposition 4.5 *Let $\mu \in \mathcal{M}_{00}(\mathbb{R})$ and K be a non-empty compact set of $\mathbb{R}$. Then $\mu_K \in \mathcal{M}_{00}(\mathbb{R})$ and, for any $\nu \in \mathcal{M}_{00}(\mathbb{R})$ with supp$[|\nu|] \subset K$, $\langle U\mu, \nu\rangle = \langle U\mu_K, \nu\rangle$.*

Proof Due to the recurrence of the one-dimensional Brownian motion, $h_K(y, K) = 1$ for every $y \in \mathbb{R}$ so that $\mu_K \in \mathcal{M}_{00}(\mathbb{R})$ for $\mu \in \mathcal{M}_{00}(\mathbb{R})$. The fundamental identity for linear potentials presented by formula (6) in page 83 of [16] reads

$$k(x-y) = g_{\mathbb{R}\setminus K}(x, y) + \int_K h_K(x, dz)k(z-y) - W_K(x), \quad x,\ y \in \mathbb{R}, \tag{4.10}$$

where $g_{\mathbb{R}\setminus K}(x, y)$ is the 0-order resolvent density of the absorbing Brownian motion on $\mathbb{R} \setminus K$ and $W_K(x)$ is a certain non-negative locally bounded function on $\mathbb{R}$ vanishing on K. Just as in the proof of Proposition 3.3, one can deduce from Eq. 4.10 the identity

$$U\mu_K(x) = U\mu(x) + \langle W_K, \mu\rangle$$

holding for every $x \in K$,which yields $\langle U\mu, \nu\rangle = \langle U\mu_K, \nu\rangle$ immediately. □

In view of Proposition 4.4 (ii), there exist a system of centered Gaussian random variables $\mathbf{G}(\mathbb{R}) = \{X_\mu : \mu \in \mathcal{M}_{00}(\mathbb{R})\}$ on a certain probability space $(\Omega, \mathcal{B}, \mathbb{P})$ with covariance

$$\mathbb{E}[X_\mu X_\nu] = \langle \mu, U\nu\rangle, \quad \mu, \nu \in \mathcal{M}_{00}(\mathbb{R}). \tag{4.11}$$

Exactly in the same way as Theorem 2.9 is derived from Proposition 2.8, we can obtain the following from Proposition 4.5:

Theorem 4.6 *The Gaussian field* $\mathbf{G}(\mathbb{R})$ *indexed by* $\mathcal{M}_{00}(\mathbb{R})$ *enjoys the local Markov property.*

5 Gaussian fields and processes induced by G(ℂ) and G(ℝ)

We exhibit several examples of Gaussian fields and processes that can be obtained as subfields of $\mathbf{G}(\mathbb{C})$ and $\mathbf{G}(\mathbb{R})$. A special attention will be paid on positive random measures intrinsically associated with the fields.

We first recall the equilibrium measure in the logarithmic potential theory (cf.[16, §3.4]). For any non-polar bounded Borel set $B \subset \mathbb{C}$, there exists a unique probability measure μ_B concentrated on B^r whose logarithmic potential $U\mu_B(\cdot) = \int_{\mathbb{C}} k(\cdot - \mathbf{y})\mu_B(d\mathbf{y})$ is constant on B^r. Here B^r denotes the set of all regular points of B relative to the planar Brownian motion $\mathbf{X} = (X_t, \{\mathbb{P}_\mathbf{x}\}_{\mathbf{x}\in\mathbb{C}})$. μ_B is called the *equilibrium measure* of B and it is concentrated on ∂B. Actually μ_B equals $\lim_{|\mathbf{x}|\to\infty} \mathbb{P}_\mathbf{x}(X_{\sigma_B} \in \cdot)$ the hitting distribution on B of $\mathbf{X}$ from ∞. For $r > 0$, let $B_r = \{\mathbf{x} \in \mathbb{C} : |\mathbf{x}| < r\}$ and σ_r be the uniform probability measure on ∂B_r. Then $\mu_{B_r} = \sigma_r$ and

$$U\sigma_r(\mathbf{x}) = \frac{1}{\pi} \log \frac{1}{|\mathbf{x}| \vee r}, \qquad \mathbf{x} \in \mathbb{C}. \tag{5.1}$$

5.1 Restriction G(∂B₁) of G(ℂ) to ∂B₁

We put

$$\mathcal{L}(\partial B_1) = \{\psi : \text{ bounded Borel function on } \partial B_1\},$$

and define for $\psi \in \mathcal{L}(\partial B_1)$

$$\mu_\psi(d\mathbf{y}) = \psi(\mathbf{y})\sigma_1(d\mathbf{y}) \ - \ \langle \sigma_1, \psi\rangle \cdot \sigma_1(d\mathbf{y}). \tag{5.2}$$

μ_ψ is a member of $\mathcal{M}_{00}(\mathbb{C})$. We denote the Gaussian random variable $X_{\mu_\psi} \in \mathbf{G}(\mathbb{C})$ by Y_ψ and consider the Gaussian field

$$\mathbf{G}(\partial B_1) = \{Y_\psi : \psi \in \mathcal{L}(\partial B_1)\} \tag{5.3}$$

indexed by $\mathcal{L}(\partial B_1)$. We then have for $\psi_1, \psi_2 \in \mathcal{L}(\partial B_1)$

$$\mathbb{E}[Y_{\psi_1} Y_{\psi_2}] = \int_{\partial B_1 \times \partial B_1} k(\mathbf{x} - \mathbf{y})\psi_1(\mathbf{x})\psi_2(\mathbf{y})\sigma_1(d\mathbf{x})\sigma_1(d\mathbf{y}), \tag{5.4}$$

because of Eq. 2.21 and $U\sigma_1(\mathbf{x}) = 0,\ \mathbf{x} \in \partial B_1$.

We identify ∂B_1 with the torus $\mathbb{T} = [0, 1)$ and denote by $\mathcal{D}(\mathbb{T})$ the collection of C^∞-functions on $\mathbb{T}$. With this identification

$$k(\mathbf{x} - \mathbf{x}') = \frac{1}{\pi} \log \frac{1}{2 \sin \pi |t - s|}, \quad \mathbf{x} = e^{2\pi ti}, \ \mathbf{x}' = e^{2\pi si}, \ \ t, s \in \mathbb{T}. \tag{5.5}$$

Equation 5.4 implies that, if $\psi_n \in \mathcal{D}(\mathbb{T})$ converges to 0 uniformly on ∂B_1,then $\mathbb{E}[Y_{\psi_n}^2] \to 0$ and consequently $Y_{\psi_n} \to 0$ a.s. as $n \to \infty$. Thus the map $\psi \in \mathcal{D}(\mathbb{T}) \mapsto Y_\psi$ is in the space of distributions $\mathcal{D}'(\mathbb{T})$ a.s., namely, we may view $\mathbf{G}(\partial B_1)\big|_{\mathcal{D}(\mathbb{T})}$ as a Gaussian random distribution.

Recently this Gaussian random distribution was introduced and studied in [2] via an informal definition of a Gaussian field $\{Y_t : t \in \mathbb{T}\}$ indexed by $\mathbb{T}$ whose covariance $\mathbb{E}[Y_t Y_s]$ is identical with Eq. 5.5 in the following manner. Using independent random variables $\{A_n, B_n, \ n \geq 1\}$ with common distribution $N(0, 1)$, consider a random Fourier series

$$Y_t^N = \frac{1}{\sqrt{\pi}} \sum_{n=1}^{N} \frac{1}{\sqrt{n}} (A_n \cos 2\pi nt + B_n \sin 2\pi nt), \quad t \in \mathbb{T}, \tag{5.6}$$

and let $Y_t = \lim_{N\to\infty} Y_t^N$. Since $\lim_{N\to\infty} \mathbb{E}[(Y_t^N)^2]$ diverges, Y_t is not well defined.
But it can be verified that

$$\lim_{N\to\infty} \mathbb{E}[Y_t^N Y_s^N] = \frac{1}{\pi} \log \frac{1}{2 \sin \pi |t - s|}, \quad t \neq s,$$

both hand sides being equal to a convergent cosine series $\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \cos 2\pi (t - s)n$. As is indicated by this identity, one can give a representation $\widetilde{Y}_\psi$ of our Gaussian random variable Y_ψ for $\psi \in \mathcal{L}(\mathbb{T})$ by

$$\widetilde{Y}_\psi = \lim_{N\to\infty} \int_{\mathbb{T}} Y_t^N \psi(t) dt. \tag{5.7}$$

Indeed one can verify that

$$\mathbb{E}[\widetilde{Y}_{\psi_1} \widetilde{Y}_{\psi_2}] = \int_{\mathbb{T}\times\mathbb{T}} k(e^{2\pi ti} - e^{2\pi si}) \psi_1(e^{2\pi ti}) \psi_2(e^{2\pi si}) \, dtds,$$

both hand sides being identical with $\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n} (a_n^1 a_n^2 + b_n^1 b_n^2)$ for the Fourier coeffiesients $a_n^i, \ b_n^i$ of $\psi_i \in \mathcal{L}(\mathbb{T}), \ i = 1, 2$.

A positive random measure $\mu_{\mathbf{G}(\mathbb{T})}$ concentrated on the circle $\mathbb{T}$ was constructed in [2] by approximating the field $\mathbf{G}(\partial B_1)$ via a certain white noise expansion. An alternative conceivable approximation would be

$$\langle \varphi, \mu_{\mathbf{G}(\mathbb{T})} \rangle = \lim_{\varepsilon \downarrow 0} \int_{\mathbb{T}} \exp \left(\gamma \widetilde{Y}_{\psi^{t,\varepsilon}} - \frac{\gamma^2}{2} \mathbb{E}(\widetilde{Y}^2_{\psi^{t,\varepsilon}}) \right) \varphi(t) dt, \quad \varphi \in C(\mathbb{T}),$$

where $\psi^{t,\varepsilon}(e^{2\pi si})$ is defined to be $1/2\varepsilon$ if $s \in (t - \varepsilon, t + \varepsilon)$ and 0 otherwise.

More generally we may consider the restriction of $\mathbf{G}(\mathbb{C})$ to ∂B for any (not necessarily connected) non-polar bounded Borel set $B \subset \mathbb{C}$. Let μ_B be the equilibrium measure for B. By choosing $R > 0$ with $B \subset B_R$,define for $\psi \in \mathcal{L}(\partial B)$

$$\mu_\psi(d\mathbf{y}) = \psi(\mathbf{y}) \mu_B(d\mathbf{y}) - \langle \mu_B, \psi \rangle \cdot \sigma_R(d\mathbf{y}) \ \in \mathcal{M}_{00}(\mathbb{C}), \tag{5.8}$$

and denote X_{μ_ψ} by Y_ψ. Then $\mathbf{G}(\partial B) = \{Y_\psi : \psi \in \mathcal{L}(\partial B)\}$ is a Gaussian field with covariance

$$\mathbb{E}[Y_{\psi_1} Y_{\psi_2}] = \int_{\partial B \times \partial B} k(\mathbf{x} - \mathbf{y}) \psi_1(\mathbf{x}) \psi_2(\mathbf{y}) \mu_B(d\mathbf{x}) \mu_B(d\mathbf{y}) + \frac{1}{\pi} \log R, \ \ \psi_1, \ \psi_2 \in \mathcal{L}(\partial B). \tag{5.9}$$

In connection with this subsection, we mention that the logarithmic potential of a variant of the measure (5.2) was considered in [5, III.5] already.

5.2 Gaussian field $\{Y^{\mathbf{x},\varepsilon}\}$ indexed by $\{\mu^{\mathbf{x},\varepsilon}\}$ and Liouville measure $\mu_{\mathbf{G}(\mathbb{C})}$

For $\mathbf{x} \in \mathbb{C}$, $\varepsilon > 0$,let $B_\varepsilon(\mathbf{x}) = \{\mathbf{y} \in \mathbb{C} : |\mathbf{x} - \mathbf{y}| < \varepsilon\}$ and $\mu^{\mathbf{x},\varepsilon}$ be the uniform probability measure on $\partial B_\varepsilon(\mathbf{x})$. $\mu^{\mathbf{x},\varepsilon}$ is the equilibrium measure for the set $B_\varepsilon(\mathbf{x})$ with the equilibrium potential

$$U\mu^{\mathbf{x},\varepsilon}(\mathbf{y}) = \frac{1}{\pi} \log \frac{1}{|\mathbf{x} - \mathbf{y}| \vee \varepsilon}, \quad \mathbf{y} \in \mathbb{C}.$$

Fix $R > 0$. For any $\mathbf{x} \in \mathbb{C}$ and $\varepsilon > 0$ with $B_\varepsilon(\mathbf{x}) \subset B_R$,define

$$\widetilde{\mu}^{\mathbf{x},\varepsilon} = \mu^{\mathbf{x},\varepsilon} - \sigma_R \ (\in \mathcal{M}_{00}(\mathbb{C})). \tag{5.10}$$

The associated Gaussian random variable $X_{\widetilde{\mu}^{\mathbf{x},\varepsilon}} \in \mathbf{G}(\mathbb{C})$ is denoted by $Y^{\mathbf{x},\varepsilon}$. In view of Eq. 5.1, the Gaussian field $\{Y^{\mathbf{x},\varepsilon}\}$ indexed by $(\mathbf{x}, \varepsilon)$ has the convariance

$$\mathbb{E}[Y^{\mathbf{x},\varepsilon} Y^{\mathbf{y},\varepsilon}] = \langle \mu^{\mathbf{x},\varepsilon}, U\mu^{\mathbf{y},\varepsilon} \rangle + \frac{1}{\pi} \log R. \tag{5.11}$$

In particular, $\mathbb{E}[(Y^{\mathbf{x},\varepsilon})^2] = \frac{1}{\pi}(\log R - \log \varepsilon)$ so that $\mathbb{E}[e^{\gamma Y^{\mathbf{x},\varepsilon}}] = (R/\varepsilon)^{\gamma^2/(2\pi)}$ for a constant $\gamma > 0$. Denote by m the Lebesgue measure on $\mathbb{C}$. It is plausible that the almost sure limit

$$\lim_{\varepsilon \downarrow 0} \left(\frac{\varepsilon}{R} \right)^{\frac{\gamma^2}{2\pi}} \exp(\gamma\, Y^{\mathbf{x},\varepsilon}) \cdot m(d\mathbf{x}) = \mu_{\mathbf{G}(\mathbb{C})}(d\mathbf{x}) \tag{5.12}$$

exists in the sense of weak convergence of measures on B_R and the limit random measure $\mu_{\mathbf{G}(\mathbb{C})} = \mu^{\gamma,R}_{\mathbf{G}(\mathbb{C})}$ is non-degenerate for small $\gamma > 0$.

A similar assertion is being made in [19] without proof but by quoting [7]. In [7], the existence of a positive random measure analogous to $\mu_{\mathbf{G}(\mathbb{C})}$ called a *Liouville* (*quantum gravity*) *measure* is studied for the Gaussian field $\mathbf{G}(D)$ associated with the transient Dirichlet form $(\frac{1}{2}\mathbf{D}_D, H^1_0(D))$ on $L^2(D)$ of the absorbing Brownian motion on a planar domain $D \subset \mathbb{C}$. $\mathbf{G}(D)$ can be formulated as the Gaussian field indexed by signed Radon measures on D of finite 0-order energy. [7] is just treating its special subfield indexed by $\{\mu^{\mathbf{x},\epsilon}; \mathbf{x} \in D, \epsilon > 0\}$. One can then well use the Markov property of $\mathbf{G}(D)$ due to [18] or its weak version. D is assumed to be bounded so that the extra term $W_{\partial D}$ in the fundamental identity for the logarithmic potentials vanishes on D (cf. (2.14)). See the proof of [9, Proposition 2.5 (ii)] for a justification of the formulation mentioned above.

In this connection, we mention the work [1] that studies a time changed planar Brownian motion with the symmetrizing measure being the Liouville random measure $\mu_{\mathbf{G}}$ for the *massive free field* $\mathbf{G}$, namely, the Gaussian field associated with the transient Dirichlet form $(\frac{1}{2}\mathbf{D}(f, g) + \alpha(f, g), H^1(\mathbb{C}))$ on $L^2(\mathbb{C})$ for a fixed $\alpha > 0$ (cf. [15, §4]). This Liouville measure $\mu_{\mathbf{G}}$ has been rigorously constructed (cf. [17]) by an approximation of the 0-order resolvent kernel $r(\mathbf{x}, \mathbf{y}) = \int_0^\infty q_t(\mathbf{x}, \mathbf{y})dt$ for $q_t(\mathbf{x}, \mathbf{y}) = (2\pi t)^{-1} e^{-\alpha t} e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{2t}}$ as a sum of non-singular kernels $\int_{a_n}^{a_{n+1}} q_t(\mathbf{x}, \mathbf{y})dt$ and by applying Kahane's multiplicative chaos [11] to Gaussian fields $\{X_n(\mathbf{x}),\ \mathbf{x} \in \mathbb{C}\}$ indexed by $\{\delta_{\mathbf{x}}; \mathbf{x} \in \mathbb{C}\}$. See [3] and references therein for some more general considerations on Liouville random measures.

5.3 Brownian motions produced by $\mathbf{G}(\mathbb{C})$ and $\mathbf{G}(\mathbb{R})$

For $r \geq 1$, define $\mu_r = \sqrt{\pi}(\sigma_r - \sigma_1)$ $(\in \mathcal{M}_{00}(\mathbb{C}))$. It follows from Eq. 5.1 that $\langle \mu_{r_1}, U\mu_{r_2} \rangle = (\log r_1) \wedge (\log r_2)$, $r_1, r_2 \geq 1$. Therefore, if we denote by W_t the Gaussian random variable $X_{\mu_{e^t}}$ for $t \in [0, \infty)$, then

$$\mathbb{E}[W_t\, W_s] = t \wedge s, \qquad t,\ s \geq 0, \tag{5.13}$$

namely, $\{W_t\}_{t\in[0,\infty)}$ is a Brownian motion with time parameter $[0, \infty)$.

Finally we take the Gaussian field $\mathbf{G}(\mathbb{R})$ indexed by $\mathcal{M}_{00}(\mathbb{R})$ considered in Section 4. For $x \in \mathbb{R}$, δ_x denotes the δ-measure concentrated at $\{x\}$. We let

$$\mu_x = \frac{1}{2}(\delta_x - \delta_0)\ (\in \mathcal{M}_{00}(\mathbb{R})), \quad x \in \mathbb{R},$$

and denote by B_x the assoicated Gaussian random variable X_{μ_x}. We then have from Eq. 4.11

$$\mathbb{E}[B_x B_y] = |x| + |y| - |x - y|, \quad x, y \in \mathbb{R}. \tag{5.14}$$

The right hand side equals $x \wedge y$; 0; and $-(x \vee y)$, in accordance with $x, y \geq 0$; $x > 0$, $y < 0$; and $x, y < 0$, respectively. This means that $\{B_x; x \geq 0\}$ and $\{B_{-x} : x \leq 0\}$ are independent Brownian motion so that $\{B_x : x \in \mathbb{R}\}$ is a Brownian motion with time parameter $\mathbb{R}$.

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