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Reflections at infinity of time changed RBMs on a domain with Liouville branches

By Zhen-Qing Chen and Masatoshi Fukushima

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Abstract. Let Z be the transient reflecting Brownian motion on the closure of an unbounded domain $D \subset \mathbb{R}^d$ with N number of Liouville branches. We consider a diffuion X on $\overline{D}$ having finite lifetime obtained from Z by a time change. We show that X admits only a finite number of possible symmetric conservative diffusion extensions Y beyond its lifetime characterized by possible partitions of the collection of N ends and we identify the family of the extended Dirichlet spaces of all Y (which are independent of time change used) as subspaces of the space $\mathrm{BL}(D)$ spanned by the extended Sobolev space $H^1_e(D)$ and the approaching probabilities of Z to the ends of Liouville branches.

1. Introduction.

The boundary problem of a Markov process X concerns all possible Markovian prolongations Y of X beyond its life time ζ whenever ζ is finite. For a conservative but transient Markov process, we can still consider its extension, after a time change to speed up the original process. Let $Z = (Z_t, \mathbf{Q}_x)$ be a conservative right process on a locally compact separable metric space E and ∂ be the point at infinity of E. Suppose Z is transient relative to an excessive measure m: for the 0-order resolvent R of Z, $Rf(z) < \infty$, m-a.e. for some strictly positive function (or equivalently, for any non-negative function) $f \in L^1(E; m)$. Then

$$\mathbf{Q}_x \left(\lim_{t\to\infty} Z_t = \partial \right) = 1 \quad \text{for q.e. } x \in E,$$

if Rf is lower semicontinuous for any non-negative Borel function f ([**FTa**]). The last condition is not needed when X is m-symmetric ([**CF2**]). Here, 'q.e.' means 'except for an m-polar set'.

Take any strictly positive bounded function $f \in L^1(E; m)$. Then $A_t = \int_0^t f(Z_s)ds$, $t \geq 0$ is a strictly increasing PCAF of Z with $\mathbf{E}^{\mathbf{Q}}_x[A_\infty] = Rf(x) < \infty$ for q.e. $x \in E$. The time changed process $X = (X_t, \zeta, \mathbf{P}_x)$ of Z by means of A is defined by

$$X_t = Z_{\tau_t},\ t \geq 0, \quad \tau = A^{-1}, \quad \zeta = A_\infty, \quad \mathbf{P}_x = \mathbf{Q}_x,\ x \in E.$$

Since $\mathbf{P}_x(\zeta < \infty,\ \lim_{t\to\zeta} X_t = \partial) = \mathbf{P}_x(\zeta < \infty) = 1$ for q.e. $x \in E$, the boundary problem for X at ∂ makes perfect sense. We denote X also by X^f to indicate its dependence on the function f. For different choices of f, X^f have a common geometric structure related

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each other only by time changes. Making a closer look at the geometric behaviors of a conservative transient process Z around ∂ is a right way toward the study of the boundary problem for $X = X^f$. A strong Markov process $\widehat{X}$ on a topological space $\widehat{E}$ is said to be an extension of X on E if (i) E can be embedded homomorphically as a dense open subset of $\widehat{E}$, (ii) the part process of $\widehat{X}$ killed upon leaving E has the same distribution as X, and (iii) $\widehat{X}$ has no sojourn on $\widehat{E} \setminus E$; that is, $\widehat{X}$ spends zero Lebesgue amount of time on $\widehat{E} \setminus E$.

In this paper, Z is the transient reflecting Brownian motion on the closure of an unbounded domain $D \subset \mathbb{R}^d$ with N number of Liouville branches. Our main aim is to prove in Section 5 that a time changed process X^f of Z admits essentially only a finite number of possible symmetric conservative diffusion extensions Y beyond its lifetime. They are characterized by the partition of the collection of N ends. Moreover, all the corresponding extended Dirichlet spaces $(\mathcal{E}^Y, \mathcal{F}_e^Y)$ are identified in terms of the extended Dirichlet space of Z and the approaching probabilities of Z to the ends of Liouville branches in an extremely simple manner. These extended Dirichlet spaces are independent of the choice of f. The L^2-generator of each extension Y is also characterized in Section 6 by means of zero flux conditions at the ends of branches. Each extension Y may be called a *many point reflection at infinity of* X^f generalizing the notion of the one point reflection in [**CF3**] in the present specific context. The characterization of possible extensions also uses quasi-homeomorphism and equivalence between Dirichlet forms. See the Appendix, Section 8, of this paper for details.

In fact, our results are valid for a time changed process X^μ of Z by means of a more general finite smooth measure μ on $\overline{D}$ than $f(x)dx$. This is demonstrated in Section 7.

Although we formulate our results for the reflecting Brownian motion on an unbounded domain in $\mathbb{R}^d$ with several Liouville branches, all of them except for Theorem 6.1 remain valid without any essential change for the reflecting diffusion process associated with the uniformly elliptic second order self-adjoint partial differential operator with measurable coefficients that was constructed in [**C**] and [**FTo**]. Since we need strong Feller property of the reflecting diffusion process, we assume the underlying unbounded domain is Lipschitz in the sense of [**FTo**]; see Remark 5.3. Thus we are effectively investigating common path behaviors at infinity holding for such a general family of diffusion processes.

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2. Preliminaries.

For a domain $D \subset \mathbb{R}^d$, let us consider the spaces

$$\mathrm{BL}(D) = \{u \in L^2_{\mathrm{loc}}(D) : \ |\nabla u| \in L^2(D)\}, \quad H^1(D) = \mathrm{BL}(D) \cap L^2(D). \tag{2.1}$$

The space $\mathrm{BL}(D)$ called the *Beppo Levi space* was introduced by Deny and Lions [**DL**] as the space of Schwartz distributions whose first order derivatives are in $L^2(D)$, which can be identified with the function space described above. The quotient space $\dot{\mathrm{BL}}(D)$ of $\mathrm{BL}(D)$ by the space of all constant functions on D is a real Hilbert space with inner product

$$\mathbf{D}(u,v) = \int_D \nabla u(x) \cdot \nabla v(x) dx.$$

See Section 1.1 of Maz'ja [**M**] for proofs of the above stated facts, where the space $\mathrm{BL}(D)$ is denoted by $L_2^1(D)$ and studied in a more general context of the spaces $L_p^\ell(D),\ \ell \geq 1,\ p \geq 1$.

Define

$$(\mathcal{E}, \mathcal{F}) = \left(\frac{1}{2}\mathbf{D}, H^1(D) \right), \tag{2.2}$$

which is a Dirichlet form on $L^2(D)$. The collection of those domains $D \subset \mathbb{R}^d$ for which (2.2) is regular on $L^2(\overline{D})$ will be denoted by $\mathcal{D}$. It is known that $D \in \mathcal{D}$ if D is either a domain of continuous boundary or an extendable domain relative to $H^1(D)$ (cf. [**CF1**, p.866]). For $D \in \mathcal{D}$, the diffusion process Z on $\overline{D}$ associated with (2.2) is by definition the *reflecting Brownian motion* (RBM in abbreviation) which is known to be conservative. Furthermore, the space $\mathrm{BL}(D)$ is nothing but the *reflected Dirichlet space* of the form (2.2) ([**CF2**, Section 6.5]). The Dirichlet form (2.2) is either recurrent or transient and the latter case occurs only when $d \geq 3$ and D is unbounded. For $D_1,\ D_2 \in \mathcal{D}$ with $D_1 \subset D_2$, (2.2) is transient for D_2 whenever so it is for the smaller domain D_1. If (2.2) is recurrent, then we have the identity

$$\mathrm{BL}(D) = H_e^1(D)$$

where $H_e^1(D)$ denotes the *extended Dirichlet space* of the form (2.2) or of the RBM Z ([**CF2**]) that may be called the *extended Sobolev space of order* 1.

Suppose $D \in \mathcal{D}$ and (2.2) is transient. Then $H_e^1(D)$ is a Hilbert space with inner product $\mathbf{D}/2$ possessing the space $C_c^\infty(\overline{D})$ as its core. $H_e^1(D)$ can be regarded as a proper closed subspace of the quotient space $\dot{\mathrm{BL}}(D)$. Define

$$\mathcal{H}^*(D) = \{u \in \mathrm{BL}(D) :\ \mathbf{D}(u,v) = 0 \text{ for every } v \in H_e^1(D)\}. \tag{2.3}$$

Any function $u \in \mathrm{BL}(D)$ admits a unique decomposition

$$u = u_0 + h, \quad u_0 \in H_e^1(D), \quad h \in \mathcal{H}^*(D). \tag{2.4}$$

Any function $h \in \mathcal{H}^*(D)$ is of finite Dirichlet integral and harmonic on D. Furthermore, the quasi-continuous version of h is harmonic on $\overline{D}$ with respect to the RBM Z.

In what follows, we restrict our attention to the case where the form (2.2) is transient and so we assume that $d \geq 3$ and $D \in \mathcal{D}$ is unbounded.

Definition 2.1. A domain $D \in \mathcal{D}$ is called a *Liouville domain* if the Dirichlet form (2.2) is transient and $\dim \mathcal{H}^*(D) = 1$.

A domain $D \in \mathcal{D}$ is a Liouville domain if and only if the form (2.2) is transient and any function $u \in \mathrm{BL}(D)$ admits a unique decomposition

$$u = u_0 + c, \qquad \text{where } u_0 \in H^1_e(D) \text{ and } c \in \mathbb{R}. \tag{2.5}$$

We shall denote by $c(u)$ the constant c in (2.5) uniquely associated with $u \in \mathrm{BL}(D)$ for a Liouville domain D.

A trivial but important example of a Liouville domain is $\mathbb{R}^d$ with $d \geq 3$, see Brelot [**B**]. Another important example of a Liouville domain is provided by an unbounded uniform domain that has been shown by Jones [**J**] (see also [**HK**]) to be an extendable domain relative to the space $\mathrm{BL}(D)$.

A domain $D \subset \mathbb{R}^d$ is called a *uniform domain* if there exists $C > 0$ such that for every $x, y \in D$, there is a rectifiable curve γ in D connecting x and y with $\mathrm{length}(\gamma) \leq C|x-y|$, and moreover

$$\min\{|x-z|,\ |z-y|\} \leq C\mathrm{dist}(z, D^c) \quad \text{for every } z \in \gamma.$$

It was proved in Theorem 3.5 of [**CF1**] that any unbounded uniform domain is a Liouville domain in the sense of Definition 2.1. An unbounded uniform domain is such a domain that is broaden toward the infinity. The truncated infinite cone $C_{A,a} = \{(r, \omega) : r > a,\ \omega \in A\} \subset \mathbb{R}^d$ for any connected open set $A \subset S^{d-1}$ with Lipschitz boundary is an unbounded uniform domain. To the contrary, (2.2) is recurrent for the cylinder $D = \{(x, x') \in \mathbb{R}^d : x \in \mathbb{R},\ |x'| < 1\}$. See Pinsky [**P**] for transience criteria for other types of domains. On the other hand, it has been shown in [**CF2**, Proposition 7.8.5] that (2.2) is transient but $\dim(\mathcal{H}^*(D)) = 2$ for a special domain

$$D = B_1(\mathbf{0}) \cup \left\{(x, x') \in \mathbb{R}^d : x \in \mathbb{R},\ |x| > |x'|\right\}, \quad d \geq 3 \tag{2.6}$$

with two symmetric cone branches. Here $B_r(\mathbf{0}),\ r > 0$, denotes an open ball with radius r centered at the origin. This domain is not uniform because of a presence of a bottleneck. We shall consider much more general domains than this. But before proceeding to the main setting of the present paper, we state a simple property of Liouville domains:

PROPOSITION 2.2. *For $D_1, D_2 \in \mathcal{D}$ with $D_1 \subset D_2$, suppose D_1 is a Liouville domain and $D_2 \setminus D_1$ is bounded. Then D_2 is a Liouville domain. Furthermore, for any $u \in \mathrm{BL}(D_2)$, it holds that $c(u) = c(u\big|_{D_1})$.*

PROOF. The proof is similar to that of [**CF1**, Proposition 3.6]. Note that (2.2) is transient for D_2. We show that any $u \in \mathrm{BL}(D_2)$ admits a decomposition (2.5) with $u_0 \in H^1_e(D_2)$ and $c = c(u\big|_{D_1})$. Due to the normal contraction property of $\mathrm{BL}(D_2)$ and the transience of $(\mathbf{D}/2, H^1(D))$, we may assume that u is bounded on D_2. By noting that $u\big|_{D_1} \in \mathrm{BL}(D_1)$ and D_1 is a Liouville domain, we let $c = c(u\big|_{D_1})$ and $u_0(x) = u(x) - c,\ x \in D_2$. Then $u_0\big|_{D_1} \in H^1_e(D_1)$. To prove that $u_0 \in H^1_e(D_2)$, choose an open ball $B_r(\mathbf{0}) \supset \overline{D_2 \setminus D_1}$ and a function $w \in C_c^\infty(\mathbb{R}^d)$ with $w(x) = 1,\ x \in B_r(\mathbf{0})$. Clearly $wu_0 \in H^1_e(D_2)$.

It remains to show $(1-w)u_0 \in H^1_e(D_2)$. Take $g_n \in H^1(D_1)$ converging to u_0 a.e. on

D_1 and in the Dirichlet norm on D_1. By truncation, we may assume that g_n is uniformly bounded on D_1. Then

$$\begin{aligned}&\int_{D_2} |\nabla[(1-w(x))g_n(x)]|^2 dx \\ &\le 2\sup_{x\in\mathbb{R}^d}(1-w(x))^2\int_{D_1}|\nabla g_n(x)|^2dx + 2\sup_{x\in D_1}|g_n(x)|^2\int_{\mathbb{R}^d}|\nabla w(x)|^2dx,\end{aligned}$$

which is uniformly bounded in n, yielding by the Banach–Saks theorem that $(1-w)u_0 \in H^1_e(D_2)$. $\square$

We shall work under the regularity condition
(A.1) D is of a Lipschitz boundary ∂D,

which means the following: there are constants $M > 0$, $\delta > 0$ and a locally finite covering $\{U_j\}_{j\in J}$ of ∂D such that, for each $j \in J$, $D \cap U_j$ is a upper part of a graph of a Lipschitz continuous function under an appropriate coodinate system with the Lipschitz constant bounded by M and $\partial D \subset \bigcup_{j\in J}\{x \in U_j : \mathrm{dist}(x, \partial U_j) > \delta\}$. According to [**FTo**], there exists then a conservative diffusion process $Z = (Z_t, \mathbf{Q}_x)$ on $\overline{D}$ associated with the regular Dirichlet form (2.2) on $L^2(\overline{D})$ whose resolvent $\{G^Z_\alpha; \alpha > 0\}$ has the strong Feller property in the sense that

$$G^Z_\alpha(bL^1(D)) \subset bC(\overline{D}). \tag{2.7}$$

Z is a precise version of the RBM on $\overline{D}$. In particular, the transition probability of Z is absolutely continuous with respect to the Lebesgue measure.

Under the condition (A.1) and the transience assumption on (2.2), the RBM $Z = (Z_t, \mathbf{Q}_x)$ on $\overline{D}$ enjoys the properties that

$$\mathbf{Q}_x\left(\lim_{t\to\infty} Z_t = \partial\right) = 1 \quad \text{for every } x \in \overline{D}, \tag{2.8}$$

where ∂ denotes the point at infinity of $\overline{D}$, and

$$\mathbf{Q}_x\left(\lim_{t\to\infty} u(Z_t) = 0\right) = 1 \quad \text{for every } x \in \overline{D}, \tag{2.9}$$

for any $u \in H^1_e(D)$, u being taken to be quasi-continuous. See [**CF2**, Section 7.8, (4^o)].

In the rest of this paper, we fix a domain D of $\mathbb{R}^d$, $d \ge 3$, satisfying (A.1) and

(A.2) $$D \setminus \overline{B_r(\mathbf{0})} = \bigcup_{j=1}^N C_j$$

for some $r > 0$ and an integer N, where $C_1, \dots, C_N$ are Liouville domains with Lipschitz boundaries such that $\overline{C}_1, \dots, \overline{C}_N$ are mutually disjoint. D may be called a *Lipschitz domain with N number of Liouville branches.*

Let ∂_j be the point at infinity of the unbounded closed set $\overline{C}_j$ for each $1 \le j \le N$. Denote the N-points set $\{\partial_1, \dots, \partial_N\}$ by F and put $\overline{D}^* = \overline{D} \cup F$. $\overline{D}^*$ can be made to be a compact Hausdorff space if we employ as a local base of neighborhoods of each point $\partial_j \in F$ the neighborhoods of ∂_j in $\overline{C}_j \cup \{\partial_j\}$. $\overline{D}^*$ may be called the *N-points compactification of* $\overline{D}$.

Obviously the Dirichlet form (2.2) is transient for D. We shall verify in Section 4 that $\dim(\mathcal{H}^*(D)) = N$. Here we note the following implication of Proposition 2.2; if a domain D is of the type (A.2) for different $0 < r_1 < r_2$, and if D is a domain with N number of Liouville branches relative to r_2, then so it is relative to r_1.

3. Approaching probabilities of RBM Z and limits of BL-functions along Z_t.

For each $1 \leq j \leq N$, define the approaching probability of the RBM $Z = (Z_t, \mathbf{Q}_x)$ to ∂_j by

$$\varphi_j(x) = \mathbf{Q}_x \left(\lim_{t \to \infty} Z_t = \partial_j \right), \quad x \in \overline{D}. \tag{3.1}$$

PROPOSITION 3.1. *It holds that*

$$\sum_{j=1}^{N} \varphi_j(x) = 1 \quad \text{for every} \quad x \in \overline{D}, \tag{3.2}$$

and, for each $1 \leq j \leq N$,

$$\varphi_j(x) > 0 \quad \text{for every} \quad x \in \overline{D}. \tag{3.3}$$

PROOF. (3.2) is a consequence of (2.8). As φ_j is a non-negative harmonic function on the domain D, it is either identically zero on D or strictly positive on D. Since $\varphi_j(x) = Q_t \varphi_j(x),\ x \in \overline{D}$, where Q_t is the transition semigroup of the RBM Z, which has a strictly positive transition density kernel, the above dichotomy extends from D to $\overline{D}$.

Suppose $\varphi_j(x) \equiv 0$ on $\overline{D}$. Then by (2.8)

$$\mathbf{Q}_x \left(\sigma_{\partial B_r(\mathbf{0})} < \infty \right) = 1, \quad \text{for any } x \in \overline{C}_j \setminus B_{r+1}(\mathbf{0}). \tag{3.4}$$

Let $Z^j = (Z_t^j, \mathbf{Q}_x^j),\ x \in \overline{C}_j$, be the RBM on $\overline{C}_j$, which is transient as C_j is a Liouville domain. Since Z and Z^j share the common part process on $\overline{C}_j \setminus \partial B_r(\mathbf{0})$, (3.4) remains valid if $\mathbf{Q}_x$ is replaced by $\mathbf{Q}_x^j$. By the Markov property of Z^j and the conservativeness of Z^j, we have

$$\mathbf{Q}_x^j \left(\sigma_{\partial B_r(\mathbf{0})} \circ \theta_\ell < \infty \text{ for every integer } \ell \right) = 1,$$

for any $x \in \overline{C}_j \setminus B_{r+1}(\mathbf{0})$. This however contradicts to the transience property (2.8) of Z^j. $\square$

PROPOSITION 3.2. *For any $u \in \mathrm{BL}(D)$, let $c_j(u) = c(u|_{C_j})$ for $1 \leq j \leq N$. Then*

$$\mathbf{Q}_x \left(Z_{\infty-} = \partial_j,\ \lim_{t \to \infty} u(Z_t) = c_j(u) \right) = \mathbf{Q}_x \left(Z_{\infty-} = \partial_j \right),\ x \in \overline{D}, \quad 1 \leq j \leq N. \tag{3.5}$$

If $c_j(u) = 0$ for every $1 \leq j \leq N$, then $u \in H_e^1(D)$.

Proof. We prove (3.5) for $j = 1$. Let $r > 0$ be the radius in (A.2) and $Z^1 = (Z^1_t, \mathbf{Q}^1_x)$ be the RBM on $\overline{C_1}$. The hitting times of $B_r(\mathbf{0})$ and $B_R(\mathbf{0})$ for $R > r$ will be denoted by σ_r and σ_R, respectively. Observe that Z and Z^1 share in common the part process on $\overline{C}_1 \setminus \partial B_r(\mathbf{0})$. Since C_1 is a Liouville domain, we see from (2.5) and (2.9) that

$$\mathbf{Q}^1_x \left(\lim_{t\to\infty} u(Z^1_t) = c_1(u) \right) = 1 \quad \text{for every } x \in \overline{C}_1.$$

For $R > r$, we consider the event

$$\Gamma_R = \{Z_{\sigma_R} \in \overline{C}_1,\ \sigma_r \circ \theta_{\sigma_R} = \infty\}.$$

Then $\Gamma_R \cap \{Z_{\infty-} = \partial\}$ increases as R increases and $\{Z_{\infty-} = \partial_1\} = \bigcup_{R>r}[\Gamma_R \cap \{Z_{\infty-} = \partial\}]$. In view of (2.8), we have for $x \in \overline{D}$,

$$\begin{aligned}
\mathbf{Q}_x(Z_{\infty-} = \partial_1) &= \lim_{R\to\infty} \mathbf{Q}_x(\Gamma_R \cap \{Z_{\infty-} = \partial\}) = \lim_{R\to\infty} \mathbf{Q}_x(\Gamma_R) \\
&= \lim_{R\to\infty} \mathbf{E}^{\mathbf{Q}_x} \left[\mathbf{Q}_{Z_{\sigma_R}}(\sigma_r = \infty); Z_{\sigma_R} \in \overline{C}_1 \right] \\
&= \lim_{R\to\infty} \mathbf{E}^{\mathbf{Q}_x} \left[\mathbf{Q}^1_{Z_{\sigma_R}}(\sigma_r = \infty); Z_{\sigma_R} \in \overline{C}_1 \right] \\
&= \lim_{R\to\infty} \mathbf{E}^{\mathbf{Q}_x} \left[\mathbf{Q}^1_{Z_{\sigma_R}}(\sigma_r = \infty,\ \lim_{t\to\infty} u(Z^1_t) = c_1(u)); Z_{\sigma_R} \in \overline{C}_1 \right].
\end{aligned}$$

In exactly the same way, we can see that $\mathbf{Q}_x(Z_{\infty-} = \partial_1,\ \lim_{t\to\infty} u(Z_t) = c_1(u))$ equals the last expression in the above display, proving (3.5) for $j = 1$

Suppose $u \in \mathrm{BL}(D)$ satisfies $c_j(u) = 0$ for every $1 \le j \le N$. Then $u\big|_{C_j} \in H^1_e(C_j)$ for every $1 \le j \le N$ and we can conclude as the proof of Proposition 2.2 that $u \in H^1_e(D)$. □

We remark that, in view of Proposition 2.2 the constants $c_j(u)$, $1 \le j \le N$, in the above proposition are independent of the choice of the radius r in (A.2).

4. Reflecting extension X^* of a time changed RBM X and dimension of $\mathcal{H}^*(D)$.

Fix a strictly positive bounded integrable function f on $\overline{D}$ and define

$$A_t = \int_0^t f(Z_s)ds, \quad t \ge 0. \tag{4.1}$$

A_t is a positive continuous additive functional (PCAF) of the RBM $Z = (Z_t, \mathbf{Q}_x)$ on $\overline{D}$ in the strict sense with full support. Notice that

$$\mathbf{Q}_x(A_\infty < \infty) = 1 \quad \text{for every } x \in \overline{D}, \tag{4.2}$$

because $\mathbf{E}^{Q_x}[A_\infty] = G^Z_{0+} f(x) < \infty$ for a.e. $x \in \overline{D}$ due to the transience of Z ([**CF2**, Proposition 2.1.3]) and hence

$$\mathbf{Q}_x(A_\infty = \infty) = \mathbf{Q}_x(A_\infty \circ \theta_t = \infty) = \mathbf{E}^{\mathbf{Q}_x}[\mathbf{Q}_{Z_t}(A_\infty = \infty)] = 0 \quad \text{for every } x \in \overline{D}, \tag{4.3}$$

on account of the stated absolute continuity of the transition function of Z.

Let $X = (X_t, \zeta, \mathbf{P}_x)$ be the time changed process of Z by means of A:

$$X_t = Z_{\tau_t}, \quad \tau = A^{-1}, \quad \zeta = A_\infty, \quad \mathbf{P}_x = \mathbf{Q}_x \text{ for } x \in \overline{D}.$$

The Markov process $X = X^f$ is a diffusion process on $\overline{D}$ symmetric with respect to the measure $m(dx) = f(x)dx$ and the Dirichlet form $(\mathcal{E}^X, \mathcal{F}^X)$ of X on $L^2(\overline{D}; m)$ is given by

$$\mathcal{E}^X = \frac{1}{2}\mathbf{D}, \qquad \mathcal{F}^X = H^1_e(D) \cap L^2(\overline{D}; m). \tag{4.4}$$

Since the extended Dirichlet space and the reflected Dirichlet space are invariant under a time change by a fully supported PCAF ([**CF2**, Corollary 5.2.12, Proposition 6.4.6]), these spaces for $\mathcal{E}^X$ are still given by $H^1_e(D)$ and $\mathrm{BL}(D)$, respectively. But the life time ζ of X is finite $\mathbf{P}_x$-a.s. for every $x \in \overline{D}$ in view of (4.2) so that we may consider the problem of extending X after ζ, particularly, from $\overline{D}$ to its N-points compactification $\overline{D}^* = \overline{D} \cup F$ with $F = \{\partial_1, \dots, \partial_N\}$.

We can rewrite the approaching probability φ_j of Z to ∂_j defined by (3.1) as

$$\varphi_j(x) = \mathbf{P}_x\left(\zeta < \infty, \quad X_{\zeta-} = \partial_j\right), \quad x \in \overline{D}, \quad 1 \le j \le N, \tag{4.5}$$

in terms of the time changed process X. The measure $m(dx) = f(x)dx$ is extended from $\overline{D}$ to $\overline{D}^*$ by setting $m(F) = 0$. An m-symmetric conservative diffusion process X^* on $\overline{D}^*$ will be called a *symmetric conservative diffusion extension* of X if its part process on $\overline{D}$ being killed upon hitting F is equivalent in law with X. The resolvent of X is denoted by $\{G^X_\alpha,\ \alpha > 0\}$.

Proposition 4.1. *There exists a unique symmetric conservative diffusion extension X^* of X from $\overline{D}$ to $\overline{D}^* = \overline{D} \cup F$. The process X^* is recurrent. Let $(\mathcal{E}^*, \mathcal{F}^*)$ and $\mathcal{F}^*_e$ be the Dirichlet form of X^* on $L^2(\overline{D}^*, m)$ $(= L^2(D; m))$ and its extended Dirichlet space, respectively. Then*

$$\mathcal{F}^*_e = H^1_e(D) \ \oplus \left\{ \sum_{j=1}^N c_j \varphi_j \ : \ c_j \in \mathbb{R} \right\} \ \subset \ \mathrm{BL}(D), \tag{4.6}$$

$$\mathcal{E}^*(u, v) = \frac{1}{2}\, \mathbf{D}(u, v), \qquad u, v \in \mathcal{F}^*_e. \tag{4.7}$$

Proof. We apply a general existence theorem of a many-point extension formulated in [**CF2**, Theorem 7.7.4] to the m-symmetric diffusion X on $\overline{D}$ and the N-points compactification $\overline{D}^* = \overline{D} \cup F$ of $\overline{D}$. We verify conditions (M.1), (M.2), (M.3) for X required in this theorem. $\psi_j(x) := \mathbf{P}_x(\zeta < \infty, X_{\zeta-} = \partial_j)$ is positive for every $x \in \overline{D}, 1 \le j \le N$, by (3.3) and (4.5), and so (M.1) is satisfied. Since $m(\overline{D}) = \int_{\overline{D}} f dx < \infty$, the m-integrability (M.2) of the function $u^{(j)}_\alpha(x) = \mathbf{E}_x\left[e^{-\alpha\zeta}; X_{\zeta-} = \partial_j\right],\ x \in \overline{D}$, is trivially fulfilled, $1 \le j \le N$. For any $1 \le j \le N$ and any compact set $V \subset \overline{D}$, $\inf_{x \in V} G^X_\alpha \psi_j(x)$ is positive because $G^X_\alpha \psi_j = G^X_{0+} u^{(j)}_\alpha = G^Z_{0+}(u^{(j)}_\alpha f)$ is lower semi-continuous on account of (2.7) and $u^{(j)}_\alpha$ is positive on $\overline{D}$. Accordingly, condition (M.3) is also satisfied.

Therefore there exists an m-symmetric diffusion extension X^* of X from $\overline{D}$ to $\overline{D}^*$ admitting no killing on F. We can then use a general characterization theorem [**CF2**, Theorem 7.7.3] to conclude that such an extension X^* of X is unique in law and its extended Dirichlet space $(\mathcal{F}_e^*, \mathcal{E}^*)$ is given by (4.6) and (4.7) as $\psi_j = \varphi_j,\ 1 \le j \le N$. In particular, (3.2) implies $1 \in \mathcal{F}_e^*,\ \mathcal{E}^*(1,1) = 0$, so that X^* is recurrent and consequently conservative. This also means the unique existence of an m-symmetric conservative diffusion extension X^* of X to $\overline{D}^*$. □

THEOREM 4.2. *$\dim(\mathcal{H}^*(D)) = N$ and*

$$\mathcal{H}^*(D) = \left\{ \sum_{j=1}^{N} c_j\, \varphi_j \ :\ c_j \in \mathbb{R} \right\}. \tag{4.8}$$

The m-symmetric conservative diffusion extension X^* of the time changed RBM X constructed in Proposition 4.1 *is a reflecting extension of X in the sense that the extended Dirichlet space $(\mathcal{F}_e^*, \mathcal{E}^*)$ of X^* equals* $(\mathrm{BL}(D), \mathbf{D}/2)$ *the reflected Dirichlet space of X.*

PROOF. By Proposition 4.1, $\{\varphi_j; 1 \le j \le N\} \subset \mathcal{H}^*(D) \subset \mathrm{BL}(D)$. For $1 \le j, k \le N$, let $c_k^{(j)} = c_k(\varphi_j)$. We claim that

$$c_k^{(j)} = \delta_{jk}, \qquad 1 \le k \le N. \tag{4.9}$$

Let τ_n be the exit time of Z from the set $\overline{D} \cap B_n(\mathbf{0}),\ n \ge 1$. Then $\{\varphi_j(Z_{\tau_n})\}_{n \ge 1}$ is a bounded $\mathbf{Q}_x$-martingale and possesses an a.s. limit Φ with $\varphi_j(x) = \mathbf{E}^{Q_x}[\Phi]$. By (3.5),

$$\Phi = \sum_{k=1}^{N} c_k^{(j)} \mathbf{1}_{\{Z_{\infty-} = \partial_k\}}. \tag{4.10}$$

For $k \ne j$, put $F_{k,n} = C_k \cap \{|x| = n\}$. Then by (3.5) again

$$\begin{aligned} c_k^{(j)} \varphi_k(x) &= \lim_{n\to\infty} \mathbf{E}^{\mathbf{Q}_x}\left[\varphi_j(Z_{\tau_n}) \mathbf{1}_{\{Z_{\infty-} = \partial_k\}}\right] \le \limsup_{n\to\infty} \mathbf{E}^{\mathbf{Q}_x}\left[\varphi_j(Z_{\tau_n}) \mathbf{1}_{\{Z_{\tau_n} \in C_k\}}\right] \\ &= \limsup_{n\to\infty} \mathbf{E}^{\mathbf{Q}_x}\left[\mathbf{Q}_x\left(Z_{\infty-} \circ \theta_{\tau_n} = \partial_j,\ Z_{\tau_n} \in C_k \middle| \mathcal{F}_{\tau_n}\right)\right] \\ &\le \lim_{n\to\infty} \mathbf{Q}_x\left(Z_{\infty-} = \partial_j,\ \sigma_{F_{k,n}} < \infty\right) = 0, \end{aligned}$$

yielding $c_k^{(j)} = 0,\ k \ne j$. Taking $\mathbf{Q}_x$-expectation in (4.10) proves the claim (4.9).

Next for any $u \in \mathrm{BL}(D)$, let $u_0 = u - \sum_{j=1}^N c_j(u)\varphi_j$. Then $u_0 \in \mathrm{BL}(D)$ with $c_j(u_0) = 0$ for every $1 \le j \le N$. So by Proposition 3.2, $u_0 \in H_e^1(D)$. This establishes (4.8). The linear independence of $\{\varphi_j; 1 \le j \le N\}$ follows from (4.9), while (4.6) and (4.8) yield the last assertion of the theorem. □

REMARK 4.3. This theorem for the special domain (2.6) was stated in [**CF2**, Proposition 7.8.5]. We take this opportunity to mention that the proof of the latter given in the book [**CF2**] contained a flaw (on the third line of page 386), that should be corrected in the above way.

5. Partitions Π of F and all possible symmetric diffusion extensions Y of a time changed RBM X.

We continue to consider the N-points compactification $\overline{D}^* = \overline{D} \cup F$ of $\overline{D}$ introduced at the end of Section 1. A map Π from the boundary set $F = \{\partial_1, \dots, \partial_N\}$ onto a finite set $\widehat{F} = \{\widehat{\partial}_1, \dots, \widehat{\partial}_\ell\}$ with $\ell \leq N$ is called a *partition* of F. We let $\overline{D}^{\Pi,*} = \overline{D} \cup \widehat{F}$. We extend the map Π from F to $\overline{D}^*$ by setting $\Pi x = x$, $x \in \overline{D}$, and introduce the quotient topology on $\overline{D}^{\Pi,*}$ by Π. In other words, we employ $\mathcal{U}_\Pi = \{U \subset \overline{D}^{\Pi,*} : \Pi^{-1}(U)$ is an open subset of $\overline{D}^*\}$ as the family of open subsets of $\overline{D}^{\Pi,*}$. Then $\overline{D}^{\Pi,*}$ is a compact Hausdorff space and may be called an *ℓ-points compactification of $\overline{D}$ obtained from $\overline{D}^*$ by identifying the points in the set $\Pi^{-1}\widehat{\partial}_i \subset F$ as a single point $\widehat{\partial}_i$ for each $1 \leq i \leq \ell$.*

Given a partition Π of F, the approaching probabilities $\widehat{\varphi}_i$ of the RBM $Z = (Z_t, \mathbf{Q}_x)$ to $\widehat{\partial}_i \in \widehat{F}$ are defined by

$$\widehat{\varphi}_i(x) = \sum_{j \in \Pi^{-1}\widehat{\partial}_i} \varphi_j(x), \quad x \in \overline{D}, \quad 1 \leq i \leq \ell. \tag{5.1}$$

As in the preceding section, we define the time changed process $X = (X_t, \zeta, \mathbf{P}_x)$ on $\overline{D}$ of Z by means of a strictly positive bounded integrable function f on $\overline{D}$. The measure $m(dx) = f(x)dx$ is extended from $\overline{D}$ to $\overline{D}^{\Pi,*}$ by setting $m(\widehat{F}) = 0$. Just as in Proposition 4.1, there exists then a unique m-symmetric conservative diffusion extension $X^{\Pi,*}$ of X from $\overline{D}$ to $\overline{D}^{\Pi,*}$ and the Dirichlet form $(\mathcal{E}^{\Pi,*}, \mathcal{F}^{\Pi,*})$ of $X^{\Pi,*}$ on $L^2(\overline{D}^{\Pi,*}; m)$ $(= L^2(D; m))$ admits the extended Dirichlet space $(\mathcal{F}_e^{\Pi,*}, \mathcal{E}^{\Pi,*})$ expressed as

$$\mathcal{F}_e^{\Pi,*} = H_e^1(D) \oplus \left\{ \sum_{i=1}^{\ell} c_i \widehat{\varphi}_i \ : \ c_i \in \mathbb{R} \right\} \subset \mathrm{BL}(D), \tag{5.2}$$

$$\mathcal{E}^{\Pi,*}(u, v) = \frac{1}{2}\, \mathbf{D}(u, v), \qquad u, v \in \mathcal{F}_e^{\Pi,*}. \tag{5.3}$$

$X^{\Pi,*}$ is recurrent. $\mathcal{E}^{\Pi,*}$ is a quasi-regular Dirichlet form on $L^2(\overline{D}^{\Pi,*}; m)$.

We now prove that the family $\{X^{\Pi,*} : \Pi$ is a partition of $F\}$ exhausts all possible m-symmetric conservative diffusion extensions of the time changed RBM X on $\overline{D}$.

Let E be a Lusin space into which $\overline{D}$ is homeomorpically embedded as an open subset. The measure $m(dx) = f(x)dx$ on $\overline{D}$ is extended to E by setting $m(E \backslash \overline{D}) = 0$. Let $Y = (Y_t, \mathbf{P}_x^Y)$ be an m-symmetric conservative diffusion process on E whose part process on $\overline{D}$ is identical in law with X. We denote by $(\mathcal{E}^Y, \mathcal{F}^Y)$ and $\mathcal{F}_e^Y$ the Dirichlet form of Y on $L^2(E; m)$ and its extended Dirichlet space. We call Y an *m-symmetric conservative diffusion extension* of X. The following theorem extends [**CF1**, Theorem 3.4]. See also [**F4**, Theorem 6.1] for analogous statements in a different context.

Theorem 5.1. *There exists a partition Π of F such that, as Dirichlet forms on $L^2(\overline{D}; m)$,*

$$(\mathcal{E}^Y, \mathcal{F}^Y) = (\mathcal{E}^{\Pi,*}, \mathcal{F}^{\Pi,*}). \tag{5.4}$$

Y under $\mathbf{P}_{g\cdot m}$ and $X^{\Pi,*}$ under $\mathbf{P}^{\Pi,*}_{g\cdot m}$ have the same finite dimensional distribution for any non-negative $g \in L^2(\overline{D}; m)$. Furthermore, a quasi-homeomorphic image of Y is identical with $X^{\Pi,*}$ in the sense of Theorem 8.2 *in Appendix.*

Proof. As has been noted in the preceding section, the extended Dirichlet space $(\mathcal{F}^X_e, \mathcal{E}^X)$ and the reflected Dirichlet space $((\mathcal{F}^X)^{\mathrm{ref}}, (\mathcal{E}^X)^{\mathrm{ref}})$ of the Dirichlet form (4.4) are given by

$$\mathcal{F}^X_e = H^1_e(D), \qquad \mathcal{E}^X = \frac{1}{2}\mathbf{D}, \tag{5.5}$$

$$(\mathcal{F}^X)^{\mathrm{ref}} = \mathrm{BL}(D) = H^1_e(D) \oplus \mathcal{H}^*(D), \qquad (\mathcal{E}^X)^{\mathrm{ref}} = \frac{1}{2}\mathbf{D}, \tag{5.6}$$

respectively.

$\mathcal{E}^Y$ is a quasi-regular Dirichlet form on $L^2(E; m)$ and Y is properly associated with it by virtue of Ma and Röckner [**MR**]. By Chen–Ma–Röckner [**CMR**], $\mathcal{E}^Y$ is therefore quasi homeomorphic with a regular Dirichlet form. In particular, via a quasi homeomorphism j in [**CF2**, Theorems 3.1.13]), we can assume that E is a locally compact separable metric space, $\mathcal{E}^Y$ is a regular Dirichlet form on $L^2(E; m)$, Y is an associated Hunt process on E, and $\widetilde{F} := E\backslash\overline{D}$ is quasi-closed. Since Y is a conservative extension of the non-conservative process X, $\widetilde{F}$ must be non $\mathcal{E}^Y$-polar. Y can be also shown to be irreducible as in the proof of [**CF2**, Lemma 7.2.7 (ii)]. Thus we are in the same setting as in Section 7.1 of [**CF2**] and Theorem 7.1.6 in it applies to Y and $\widetilde{F}$.

Every function in $\mathcal{F}^Y_e$ will be taken to be $\mathcal{E}^Y$-quasi continuous. As Y is a diffusion with no killing inside, the jumping measure J and the killing measure k in the Beurling-Deny decomposition of $\mathcal{E}^Y$ vanish so that we have by [**CF2**, Theorem 7.1.6]

$$H^1_e(D) \subset \mathcal{F}^Y_e \subset \mathrm{BL}(D), \quad \mathcal{H}^Y := \{\mathbf{H}u : u \in \mathcal{F}^Y_e\} \subset \mathcal{H}^*(D), \tag{5.7}$$

$$\mathcal{E}^Y(u,u) = \frac{1}{2}\mathbf{D}(u,u) + \frac{1}{2}\mu^c_{\langle \mathbf{H}u\rangle}(\widetilde{F}), \quad u \in \mathcal{F}^Y_e, \tag{5.8}$$

where $\mathbf{H}u(x) = \mathbf{E}^Y_x[u(Y_{\sigma_{\widetilde{F}}})],\ x \in E$.

Let us prove that

$$\mu^c_{\langle u\rangle}(\widetilde{F}) = 0, \qquad u \in \mathcal{H}^Y. \tag{5.9}$$

To this end, we consider a finite measure ν on E defined by

$$\nu(B) = \int_{\overline{D}} \mathbf{P}^Y_x \left(Y_{\sigma_{\widetilde{F}}} \in B,\ \sigma_{\widetilde{F}} < \infty\right) m(dx), \quad B \in \mathcal{B}(E).$$

ν vanishes off $\widetilde{F}$ and charges no $\mathcal{E}^Y$-polar set. In view of [**CF2**, Lemma 5.2.9 (i)], $\widetilde{F}$ is a quasi support of ν in the following sense: $\nu(E \setminus \widetilde{F}) = 0$ and $\widetilde{F} \subset \widehat{F}$ q.e. for any quasi closed set $\widehat{F}$ with $\nu(E \setminus \widehat{F}) = 0$.

Now, for $u \in \mathcal{H}^Y$, (4.8) and (5.7) imply that $u = \sum_{j=1}^N c_j\varphi_j$ for some constants c_j. Take $\widehat{F} = \{\xi \in E : u(\xi) \in \{c_1, \dots, c_N\}\}$. Since u is quasi continuous, $\widehat{F}$ is a quasi closed set. As u is continuous along the sample path of Y (cf. [**CF2**, Theorem 3.1.7]), we have

$\nu(E \setminus \widehat{F}) = \mathbf{P}_m(u(Y_{\sigma_{\widetilde{F}}}) \notin \{c_1, \ldots, c_N\}) = 0$ on account of Proposition 3.2 and (4.9). Accordingly $\widetilde{F} \subset \widehat{F}$ q.e., namely, u takes only finite values $\{c_1, \ldots, c_N\}$ q.e. on $\widetilde{F}$. By the *energy image density property* of $\mu^c_{\langle u \rangle}$ due to Bouleau and Hirsch [**BH**] (cf. [**CF2**, Theorem 4.3.8]), we thus get (5.9).

Relation (5.7) and Proposition 3.2(ii) imply that every function $u \in \mathcal{H}^Y (\subset \mathrm{BL}(D))$ admits a limit $u(\partial_j)$ at each boundary point $\partial_j \in F$ along the path of Z. Define an equivalence relation $\sim$ on F by $\partial_j \sim \partial_k$ if and only if $u(\partial_j) = u(\partial_k)$ for every $u \in \mathcal{H}^Y$. Notice that, for every $1 \leq j \leq N$, there exists $u \in \mathcal{H}^Y$ with $u(\partial_j) \neq 0$. Otherwise, for the resolvent $\{G^Y_\alpha : \alpha > 0\}$ of Y, $G^Y_\alpha 1 \in \mathcal{F}^Y_e (\subset \mathrm{BL}(D))$ approaches to zero at some ∂_j along the path of Z, contradiction to the conservativeness of Y. Let Π be the corresponding partition of F: Π maps F onto $\{\widehat{\partial}_1, \ldots, \widehat{\partial}_\ell\}$ the set of all equivalence classes with respect to $\sim$. Then $\mathcal{H}^Y = \left\{ \sum_{i=1}^{\ell} c_i \widehat{\varphi}_i : \ c_i \in \mathbb{R} \right\}$ for $\widehat{\varphi}_i$ defined by (5.1). Hence (5.2), (5.3), (5.7), (5.8) and (5.9) lead us to the desired identity (5.4).

Since the both Dirichlet forms share a common semigroup on $L^2(\overline{D}; m)$, we get the first conclusion of the theorem. Further the Dirichlet spaces

$$(E,\ m,\ \mathcal{E}^Y, \mathcal{F}^Y), \qquad (\overline{D}^{\Pi,*},\ m,\ \mathcal{E}^{\Pi,*}, \mathcal{F}^{\Pi,*})$$

are equivalent in the sense of Appendix (Section 8) by the identity map Φ from $\mathcal{F}^Y_b$ onto $\mathcal{F}^{\Pi,*}_b$ so that we get the second conclusion from Theorem 8.2. □

Remark 5.2. (i) For different choices of f, the family of all symmetric conservative extensions Y of X^f is invariant up to time changes because it shares a common family of extended Dirichlet spaces (5.2)–(5.3). The same can be said for more general time changed RBM X^μ, which will be formulated in Section 7.

(ii) We can replace the conservativeness assumption on Y by a weaker one that Y is a proper extension of X with no killing on $E \setminus \overline{D}$. Then the above theorem remains valid if $X^{\Pi,*}$ is allowed to be replaced by its subprocess being killed upon hitting some (but not all) $\widehat{\partial}_i$.

Remark 5.3 (Symmetric diffusion for a uniformly elliptic differential operator). Given measurable functions $a_{ij}(x)$, $1 \leq i, j \leq d$, on D such that

$$a_{ij}(x) = a_{ji}(x), \quad \Lambda^{-1}|\xi|^2 \leq \sum_{1 \leq i,j \leq d} a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2, \quad x \in D,\ \xi \in \mathbb{R}^d, \tag{5.10}$$

for some constant $\Lambda \geq 1$, we consider a Dirichlet form

$$(\mathcal{E}, \mathcal{F}) = (\boldsymbol{a}, H^1(D)) \tag{5.11}$$

on $L^2(D)$ where

$$\boldsymbol{a}(u, v) = \int_D \sum_{i,j=1}^d a_{ij}(x) \frac{\partial u}{\partial x_i}(x) \frac{\partial v}{\partial x_j}(x) dx, \quad u, v \in H^1(D).$$

If we replace the Dirichlet form (2.2) on $L^2(D)$ and the associated RBM Z on $\overline{D}$, respectively, by the Dirichlet form (5.11) on $L^2(D)$ and the associated reflecting diffusion process on $\overline{D}$ constructed in [**FTo**], all results from Section 3 to Section 5 still hold without any change as we shall see now.

By this replacement, the extended Dirichlet space and the reflected Dirichlet space are still $H_e^1(D)$ and $\mathrm{BL}(D)$, respectively, although the inner product $\mathbf{D}/2$ is replaced by $\boldsymbol{a}$. The transience of (5.11) is equivalent to that of (2.2). The space $\mathcal{H}^*(D)$ is now defined by (2.3) with $\boldsymbol{a}$ in place of $\mathbf{D}/2$. But, by noting that $\boldsymbol{a}(c,c)=0$ for any constant c and by taking the characterization of a Liouville domain stated below Definition 2.1 into account, we readily see that $D \in \mathcal{D}$ is a Liouville domain relative to (5.11) if and only if so it is relative to (2.2).

Remark 5.4 (All possible symmetric conservative diffusion extensions of a one-dimensional minimal diffusion). Consider a minimal diffusion X on a one-dimensional open interval $I=(r_1,r_2)$ with no killing inside for which both boundaries r_1,r_2 are regular. Let E be a Lusin space into which I is homeomorphically embedded as an open subset. The speed measure m of X is extended to E by setting $m(E\setminus I)=0$. Let Y be an m-symmetric conservative diffusion extension of X from I to E. Then, by removing some m-polar open set for Y from $\widetilde{F}=E\setminus I$, a homeomorphic image of Y is identical with either the two point extension of X to $[r_1,r_2]$ or its one-point extension to the one-point compactification of I. This fact was implicitly indicated in [**F2**, Section 5] and [**F3**, Section 5] without proof. This can be shown in a similar manner to the proof of Theorem 5.1 by establishing the counterpart of the identity (5.9) and by noting that, for the one-point and two-point extensions of X, every non-empty subset of the state space has a positive 1-capacity uniformly bounded away from zero due to the bound [**CF2**, (2.2.31)] and so a quasi-homeomorphism is reduced to a homeomorphism.

To put it another way, Theorem 5.1 reveals that the time changed RBM X on an unbounded domain with N-Liouville branches has a very similar structure to the one-dimensional diffusion only by changing two boundary points to N boundary points.

We note that the connected sum of non-parabolic manifolds being studied by Kuz'menko and Molchanov [**KM**], Grigor'yan and Saloff-Coste [**GS**] bears a strong similarity to the present paper in the setting although the main concern in these papers was the heat kernel estimates.

6. Characterization of L^2-generator of extension Y by zero flux condition at infnity.

For a strictly positive bounded integrable function f on D, we put $m(dx)=f(x)dx$ and denote by $(\cdot,\cdot)$ the inner product for $L^2(D;m)$. Let Y be any m-symmetric conservative diffusion extension of the time changed process $X=X^f=(X_t,\zeta,\mathbf{P}_x)$ of the RBM Z on $\overline{D}$. Let $\Pi: F\mapsto\{\widehat{\partial}_1,\dots,\widehat{\partial}_\ell\}$, $\ell\le N$, be the corresponding partition of the boundary $F=\{\partial_1,\dots,\partial_N\}$ appearing in Theorem 5.1. The Dirichlet form $(\mathcal{E}^Y,\mathcal{F}^Y)$ of Y on $L^2(D;m)$ is then described as

$$\begin{cases} \mathcal{F}^Y = \left\{ u = u_0 + \sum_{i=1}^{\ell} c_i \widehat{\varphi}_i : \ u_0 \in H_e^1(D) \cap L^2(D; m), \ c_i \in \mathbb{R} \right\}, \\ \mathcal{E}^Y(u, v) = \frac{1}{2}\mathbf{D}(u, v), \qquad u, v \in \mathcal{F}^Y, \end{cases}$$

where $\widehat{\varphi}_i,\ 1 \le i \le \ell$, are defined by (5.1).

Let $\mathcal{A}$ be the L^2-generator of Y, that is, $\mathcal{A}$ is a self-adjoint operator on $L^2(D;m)$ such that $u \in \mathcal{D}(\mathcal{A}),\ \mathcal{A}u = v \in L^2(D;m)$ if and only if $u \in \mathcal{F}^Y$ with $\mathcal{E}^Y(u,w) = -(v,w)$ for every $w \in \mathcal{F}^Y$. In view of Proposition 3.2, the condition (7.3.4) of [**CF2**] is fulfilled by Y. Therefore Theorem 7.7.3 (vii) of [**CF2**] is well applicable in getting the following characterization of $\mathcal{A}$:

$$u \in \mathcal{D}(\mathcal{A}) \quad \text{if and only if} \quad u \in \mathcal{D}(\mathcal{L}) \text{ and } \mathcal{N}(u)(\widehat{\partial}_i) = 0,\ 1 \le i \le \ell.$$

In this case, $\mathcal{A}u = \mathcal{L}u$.

Here $\mathcal{L}$ is a linear operator defined as follows: $u \in \mathcal{D}(\mathcal{L}),\ \mathcal{L}u = v \in L^2(D;m)$ if and only if $u \in \mathrm{BL}(D) \cap L^2(D;m)$ and $\mathbf{D}/2(u,w) = -(v,w)$ for every $w \in H_e^1(D) \cap L^2(D;m)$, or equivalently, for every $w \in C_c^1(\overline{D})$. $\mathcal{N}(u)(\widehat{\partial}_i)$ is the *flux of* u *at* $\widehat{\partial}_i$ defined by

$$\mathcal{N}(u)(\widehat{\partial}_i) = \frac{1}{2}\mathbf{D}(u, \widehat{\varphi}_i) + (\mathcal{L}u, \widehat{\varphi}_i), \quad 1 \le i \le \ell.$$

It can be readily verified that $u \in \mathcal{D}(\mathcal{L})$ if and only if $u \in \mathrm{BL}(D) \cap L^2(D;m)$, Δu in the Schwartz distribution sense is in $L^2(D)$ and

$$\mathbf{D}(u, w) + \int_D \Delta u(x) \cdot w(x) dx = 0 \quad \text{for every } w \in C_c^1(\overline{D}). \tag{6.1}$$

In this case, $\mathcal{L}u(x) = 1/2f(x)\Delta u(x),\ x \in D$. The equation (6.1) can be interpreted as the requirement that the *generalized normal derivative* of u vanishes on ∂D. Thus we have

THEOREM 6.1. *$u \in \mathcal{D}(\mathcal{A})$ if and only if $u \in \mathrm{BL}(D) \cap L^2(D;m)$, Δu in the Schwartz distribution sense belongs to $L^2(D)$, the equation* (6.1) *is satisfied and*

$$\left(\mathcal{N}(u)(\widehat{\partial}_i) =\right) \frac{1}{2}\mathbf{D}(u, \widehat{\varphi}_i) + \frac{1}{2}\int_D \Delta u(x)\widehat{\varphi}_i(x) dx = 0, \quad 1 \le i \le \ell. \tag{6.2}$$

In this case,

$$\mathcal{A}u(x) = \frac{1}{2f(x)}\,\Delta u(x), \quad \text{a.e. on } D. \tag{6.3}$$

Suppose $u \in \mathcal{D}(\mathcal{A})$ is smooth on $\overline{D}$. Then $\partial u/\partial \boldsymbol{n} = 0$ on ∂D due to the condition (6.1) so that the zero flux condition (6.2) at $\widehat{\partial}_j$ can be expressed as

$$\lim_{r\uparrow\infty} \int_{D\cap\partial B_r(\mathbf{0})} u_r(x)\widehat{\varphi}_i(x) d\sigma_r(dx) = 0, \quad 1 \le i \le \ell, \tag{6.4}$$

where σ_r is the surface measure on $\partial B_r(\mathbf{0})$.

The last part of Section 7.6 (4°) of [**CF2**] has treated a very special case of the above where $D = \mathbb{R}^d$, $d \geq 3$, and Y is the one-point reflection at the infinity of $\mathbb{R}^d$ of a time changed Brownian motion on $\mathbb{R}^d$.

In [**F3**], the L^2-generator of any symmetric diffusion extension Y of a one-dimensional minimal diffusion X is identified. In this case, the Dirichlet form of Y admits its reproducing kernel which enables us to identify also the C_b-generator of Y, recovering the general boundary condition due to Feller and Itô–McKean.

7. Extensions of more general time changed RBMs.

All the results in Sections 4–6 except for (6.3) hold for more general time changed RBMs than X^f. Let $Z = (Z_t, \mathbf{Q}_x)$, f, $X = X^f = (X_t, \zeta, \mathbf{P}_x)$, $X^* = (X_t^*, \mathbf{P}_x^*)$ be as in Section 4.

We consider a positive finite measure μ on $\overline{D}$ charging no polar set with full quasi-support $\overline{D}$ relative to the Dirichlet form $(\mathcal{E}, \mathcal{F})$ of (2.2). Let A^μ be the PCAF of Z with Revuz measure μ and $X^\mu = (X_t^\mu, \zeta^\mu, \mathbf{P}_x^\mu)$ be the time changed process of Z by A^μ. The Markov process X^μ is μ-symmetric and its Dirichlet form $(\mathcal{E}^{X^\mu}, \mathcal{F}^{X^\mu})$ on $L^2(\overline{D}; \mu)$ is given by

$$\mathcal{E}^{X^\mu} = \frac{1}{2}\mathbf{D}, \quad \mathcal{F}^{X^\mu} = H_e^1(D) \cap L^2(\overline{D}; \mu). \tag{7.1}$$

PROPOSITION 7.1. *It holds that*

$$\mathbf{Q}_x(A_\infty^\mu < \infty) = 1 \quad \textit{for q.e. } x \in \overline{D}, \tag{7.2}$$

$$\mathbf{P}_x^\mu(\zeta^\mu < \infty,\ X_{\zeta^\mu -}^\mu = \partial_i) = \varphi_i(x) > 0 \quad \textit{for q.e. } x \in \overline{D} \textit{ and } 1 \leq i \leq N. \tag{7.3}$$

PROOF. Fix a strictly positive bounded integrable function h_0. By the transience of Z and [**CF2**, Theorem A.2.13 (v)], $G_{0+}^Z h_0(x) < \infty$ for q.e. $x \in \overline{D}$. For integer $k \geq 1$, let

$$\Lambda_k := \left\{x \in \overline{D} : G_{0+}^Z h_0(x) \leq 2^k\right\} \quad \text{and} \quad h(x) = \sum_{k=1}^\infty 2^{-2k} \mathbf{1}_{\Lambda_k}(x) h_0(x).$$

Then h is a strictly positive bounded integrable function on $\overline{D}$ with $G_{0+}^Z h(x) \leq \mathbf{1}$ q.e. on $\overline{D}$. From [**CF2**, (4.1.3)], we have

$$\int_{\overline{D}} \mathbf{E}^{\mathbf{Q}_x}\left[A_\infty^\mu\right] h(x)dx = \langle G_{0+}^Z h, \mu\rangle \leq \mu(\overline{D}) < \infty. \tag{7.4}$$

It follows that $\mathbf{E}^{\mathbf{Q}_x}\left[A_\infty^\mu\right] < \infty$ a.e $x \in \overline{D}$ and hence q.e. $x \in \overline{D}$ by [**CF2**, Theorem A.2.13 (v)], yielding (7.2). (7.3) follows from (7.2) and Proposition 3.1. □

Since $m(dx) = f(x)dx$ has its quasi-support $\overline{D}$ relative to $(\mathcal{E}, \mathcal{F})$, the Dirichlet form $(\mathcal{E}^X, \mathcal{F}^X)$ of (4.4) shares the common quasi-notion with $(\mathcal{E}, \mathcal{F})$ ([**CF2**, Theorem 5.2.11]). Hence the quasi-support of μ relative to $(\mathcal{E}^X, \mathcal{F}^X)$ is still $\overline{D}$.

The Dirichlet form $(\mathcal{E}^*, \mathcal{F}^*)$ on $L^2(\overline{D}^*, m)$ of X^* is quasi-regular. According to the quasi-homeomorphism method already used in Section 4, we may assume it to be regular. The measure μ on $\overline{D}$ is extended to $\overline{D}^*$ by setting $\mu(F)=0$. We claim that the quasi-support of μ relative to this Dirichlet form equals $\overline{D}^*$ by using a criteria [**CF2**, Theorem 3.3.5].

Assume that $u \in \mathcal{F}^*$ is $\mathcal{E}^*$-quasi-continuous and that $u=0$ μ-a.e. Then $u\big|_{\overline{D}}$ is $\mathcal{E}^X$-quasi-continuous ([**CF2**, Theorem 3.3.8]) so that $u=0$ q.e. on $\overline{D}$. According to the same reference, there exists a Borel m-polar set $C \subset \overline{D}$ relative to X^* such that $u(x)=0$ for every $x \in \overline{D}\setminus C$. Since u is continuous along the path of X^* ([**CF2**, Theorem 3.1.7]), we have for each $1 \le i \le N$

$$\mathbf{P}^*_m\left(u(\partial_i)=\lim_{t\uparrow\sigma_F} u(X^*_t),\ \sigma_C=\infty,\ \sigma_F<\infty,\ X^*_{\sigma_F}=\partial_i\right)=\mathbf{P}_m(\zeta<\infty, X_{\zeta-}=\partial_i)>0,$$

and so u vanishes on F and hence q.e. on $\overline{D}^*$, as was to be proved.

THEOREM 7.2. *There exists a unique μ-symmetric conservative diffusion $\widetilde{X}^{*,\mu}$ on $\overline{D}^*$ which is a q.e. extension of X^μ in the sense that the part of the former on $\overline{D}$ coincides in law with the latter for q.e. starting points $x \in \overline{D}$. The extended Dirichlet space of $\widetilde{X}^{*,\mu}$ equals* $(\mathrm{BL}(D), \mathbf{D}/2)$ *the reflected Dirichlet space of X^μ.*

PROOF. Let B^0_t and B_t be the PCAFs of X and X^*, respectively, with Revuz measure μ. According to [**CF2**, Proposition 4.1.10]

$$B^0_t = B_{t\wedge\sigma_F}. \tag{7.5}$$

Let $\widetilde{X}^\mu$ and $\widetilde{X}^{*,\mu}$ be the time changed processes of X and X^* by means of B^0_t and B_t, respectively. The Markov process $\widetilde{X}^\mu$ is then the part of $\widetilde{X}^{*,\mu}$ on $\overline{D}$ by (7.5). Since X^* is recurrent, so is $\widetilde{X}^{*,\mu}$ in view of [**CF2**, Theorem 5.2.5]. Therefore $\widetilde{X}^{*,\mu}$ is a μ-symmetric conservative diffusion extension of $\widetilde{X}^\mu$.

On the other hand, the Dirichlet form of $\widetilde{X}^\mu$ on $L^2(\overline{D};\mu)$ is identical with (7.1) the Dirichlet form of X^μ on $L^2(\overline{D};\mu)$, and consequently $\widetilde{X}^{*,\mu}$ is a q.e. extension of X^μ. The last statement follows from the invariance of extended and reflected Dirichlet spaces under time changes by fully supported PCAFs.

The uniqueness of such a μ-symmetric conservative Markovian extension of X^μ to $\overline{D}^*$ follows from [**CF2**, Theorem 7.7.3]. $\square$

Similarly, all results in Section 4 and 5 with μ in place of $dm = f dx$ remain valid except for (6.3).

REMARK 7.3. One can give an alternative proof of Theorem 7.2 without invoking the time change of X^* but still using the quasi-regularity of $(\mathcal{E}^*, \mathcal{F}^*)$. Indeed, the following proposition combined with (7.3) and [**CF2**, Theorem 7.7.3] readily yields Theorem 7.2.

Each function in $\mathcal{F}^*_e$ is taken to be $\mathcal{E}^*$-quasi continuous. Define

$$\widehat{\mathcal{F}} = \mathcal{F}^*_e \cap L^2(\overline{D};\mu) \quad \text{and} \quad \widehat{\mathcal{E}}(u,v)=\mathcal{E}^*(u,v)=\frac{1}{2}\mathbf{D}(u,v) \text{ for } u,v \in \widehat{\mathcal{F}}. \tag{7.6}$$

Proposition 7.4. (i) *$(\widehat{\mathcal{E}}, \widehat{\mathcal{F}})$ is a quasi-regular Dirichlet form on $L^2(\overline{D}^*; \mu)$.*

(ii) *Its associated strong Markov process $\widehat{X}$ on $\overline{D}^*$ is a μ-symmetric conservative diffusion which is a q.e. extension of X^μ.*

(iii) *Each ∂_j is non-$\widehat{\mathcal{E}}$-polar.*

Proof. (i) As $\overline{D}$ is a quasi-support of μ, $u = 0$ μ−a.e. for $u \in \widehat{\mathcal{F}}$ implies $u = 0$ a.e. on $\overline{D}$ and $\mathbf{D}(u, u) = 0$. This together with the transience of $(\mathcal{F}^*_e, \mathcal{E}^*)$ implies that $(\widehat{\mathcal{E}}, \widehat{\mathcal{F}})$ is a well defined Dirichlet form on $L^2(\overline{D}^*; \mu)$.

Since $(\mathcal{E}^*, \mathcal{F}^*)$ is a quasi-regular Dirichlet form on $L^2(\overline{D}^*; m)$, by [**CF2**, Remark 1.3.9], there is an increasing sequence of compact subsets $\{F_k\}$ of $\overline{D}^*$ so that

(a) there is an increasing sequence of compact subsets $\{F_k\}$ of $\overline{D}^*$ so that $\cup_{k\geq 1}\mathcal{F}^*_{F_k}$ is $\mathcal{E}^*_1$-dense in $\mathcal{F}^*$.

(b) there is an $\mathcal{E}^*_1$-dense of countable set $\Lambda_0 := \{f_j; j \geq 1\}$ of bounded functions of $\mathcal{F}^*$ so that $\{f_j; j \geq 1\} \subset C(\{F_k\})$ and they separate points of $\cup_{k\geq 1}F_k$.

By the contraction of the Dirichlet form, we may and do assume without loss of generality that for every integer $n \geq 1$ and $f \in \Lambda_0$, $((-n) \vee f) \wedge n \in \Lambda_0$. We claim that $\cup_{k\geq 1}\mathcal{F}^*_{F_k,b} \subset \cup_{k\geq 1}\widehat{\mathcal{F}}_{F_k,b}$ is $\widehat{\mathcal{E}}_1$-dense in $\widehat{\mathcal{F}}_b$. Let $u \in \widehat{\mathcal{F}}_b$. Since $\widehat{\mathcal{F}}_b = \mathcal{F}^*_b$, there are $u_k \in \mathcal{F}^*_{F_k}$ so that $u_k \to u$ in $\mathcal{E}^*_1$-norm. Using truncation if needed, we may and do assume $\|u_k\|_\infty \leq \|u\|_\infty + 1$. Taking a subsequence if needed, we may also assume that u_k converges to u $\mathcal{E}^*$-q.e. on $\overline{D}^*$. Since μ is a finite smooth measure, we conclude that u_k is $\widehat{\mathcal{E}}_1$-convergent to u. This proves the claim. As $\widehat{\mathcal{F}}_b$ is $\widehat{\mathcal{E}}_1$ dense in $\widehat{\mathcal{F}}$, it follows that $\{F_k\}$ is an $\widehat{\mathcal{E}}$-nest on $\overline{D}^*$.

A similar argument shows that $\Lambda_0 \subset \widehat{\mathcal{F}}_b = \mathcal{F}^*_b$ is $\widehat{\mathcal{E}}_1$-dense in $\widehat{\mathcal{F}}_b$ and hence in $\widehat{\mathcal{F}}$. This proves the assertion (i).

(ii) Since $1 \in \widehat{\mathcal{F}}$ and $\mathbf{D}(1, 1) = 0$, the associated μ-symmetric diffusion $\widehat{X}$ on $\overline{D}^*$ is recurrent and conservative. For $R > r$, take $\psi \in C^\infty_c(\overline{D})$ with $\psi = 1$ on $B_{R+1}(\mathbf{0})$. Then, for any bounded $u \in \widehat{\mathcal{F}}$, $\psi u \in H^1_e(D)$ and so

$$\{v \in \widehat{\mathcal{F}} : v = 0 \text{ q.e. on } \overline{D}^* \setminus B_R(\mathbf{0})\} = \{v \in H^1_e(D) \cap L^2(\overline{D}; \mu) : v = 0 \text{ q.e. on } \overline{D} \setminus B_R(\mathbf{0})\},$$

namely, the part of $\widehat{\mathcal{E}}$ on $\overline{D} \cap B_R(\mathbf{0})$ coincides with the part of $\mathcal{E}^{X^\mu}$ on $\overline{D} \cap B_R(\mathbf{0})$. By letting $R \to \infty$, we see that the part of $\widehat{\mathcal{E}}$ on $\overline{D}$ coincides with $\mathcal{E}^{X^\mu}$, proving (ii).

(iii) The non-$\widehat{\mathcal{E}}$-polarity of ∂_j follows from (ii) and (7.3). □

8. Appendix: equivalence and quasi-homeomorphism.

In dealing with boundary problems for symmetric Markov processes, it is convenient to introduce an equivalence of Dirichlet spaces following [**FOT**, A.4] as will be stated below.

We say that a quadruplet $(E, m, \mathcal{E}, \mathcal{F})$ is a *Dirichlet space* if E is a Hausdorff topological space with a countable base, m is a σ-finite positive Borel measure on E and $\mathcal{E}$ with domain $\mathcal{F}$ is a Dirichlet form on $L^2(E; m)$. The inner product in $L^2(E; m)$ is denoted by $(\cdot, \cdot)_E$. For a given Dirichlet space $(E, m, \mathcal{E}, \mathcal{F})$, the notions of an $\mathcal{E}$-*nest*, an

$\mathcal{E}$-polar set, an *$\mathcal{E}$-quasi-continuous numerical function* and '*$\mathcal{E}$-quasi-everywhere*' ('$\mathcal{E}$-q.e.' in abbreviation) are defined as in [**CF2**, Definition 1.2.12]. The *quasi-regularity* of the Dirichlet space is defined just as in [**CF2**, Definition 1.3.8]. We note that the space $\mathcal{F}_b = \mathcal{F} \cap L^\infty(E;m)$ is an algebra.

REMARK 8.1. In Section 1.2 and the first half of Section 1.3 of [**CF2**], it is assumed that

$$\mathrm{supp}[m] = E. \tag{8.1}$$

We need not assume it. Generally, if we let $E' = \mathrm{supp}[m]$, then $E \setminus E'$ is $\mathcal{E}$-polar according to the definition of the $\mathcal{E}$-polarity. If $(E, m, \mathcal{E}, \mathcal{F})$ is quasi-regular, so is $(E', m\big|_{E'}, \mathcal{E}, \mathcal{F})$ accordingly. Therefore we may assume (8.1) if we like by replacing E with E'.

Given two Dirichlet spaces

$$(E, m, \mathcal{E}, \mathcal{F}), \qquad (\tilde{E}, \tilde{m}, \tilde{\mathcal{E}}, \tilde{\mathcal{F}}), \tag{8.2}$$

we call them *equivalent* if there is an algebraic isomorphism Φ from $\mathcal{F}_b$ onto $\tilde{\mathcal{F}}_b$ preserving three kinds of metrics: for $u \in \mathcal{F}_b$

$$\|u\|_\infty = \|\Phi u\|_\infty, \ (u,u)_E = (\Phi u, \Phi u)_{\tilde{E}}, \ \mathcal{E}(u,u) = \tilde{\mathcal{E}}(\Phi u, \Phi u).$$

One of the two equivalent Dirichlet spaces is called a *representation* of the other.

The underlying spaces E, $\tilde{E}$ of two Dirichlet spaces (8.2) are said to be *quasi-homeomorphic* if there exist $\mathcal{E}$-nest $\{F_n\}$, $\tilde{\mathcal{E}}$-nest $\{\tilde{F}_n\}$ and a one to one mapping q from $E_0 = \cup_{n=1}^\infty F_n$ onto $\tilde{E}_0 = \cup_{n=1}^\infty \tilde{F}_n$ such that the restriction of q to each F_n is a homeomorphism onto $\tilde{F}_n$. $\{F_n\}$, $\{\tilde{F}_n\}$ are called the *nests attached to the quasi-homemorphism q*. Any quasi-homeomorphism is quasi-notion-preseving.

We say that the equivalnce Φ of two Dirichlet spaces (8.2) *is induced by a quasi-homeomorphism q* of the underlying spaces if

$$\Phi u(\tilde{x}) = u(q^{-1}(\tilde{x})), \quad u \in \mathcal{F}_b, \quad \tilde{m}\text{−a.e. } \tilde{x}.$$

Then $\tilde{m}$ is the image measure of m and $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ is the image Dirichlet form of $(\mathcal{E}, \mathcal{F})$.

THEOREM 8.2. *Assume that two Dirichlet spaces* (8.2) *are quasi-regular and that they are equivalent. Let $X = (X_t, \mathbb{P}_x)$ (resp. $\tilde{X} = (\tilde{X}_t, \tilde{\mathbb{P}}_x)$) be an m-symmetric right process on E (resp. an $\tilde{m}$-symmetric right process on $\tilde{E}$) properly associated with $(\mathcal{E}, \mathcal{F})$ on $L^2(E;m)$ (resp. $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ on $L^2(\tilde{E};\tilde{m})$). Then the equivalence is induced by a quasi-homeomorphism q with attached nests $\{F_n\}$, $\{\tilde{F}_n\}$ such that $\tilde{X}$ is the image of X by q in the following sense: there exist an m-inessential Borel subset N of E containing $\cap_{n=1}^\infty F_n^c$ and an $\tilde{m}$-inessential Borel subset $\tilde{N}$ of $\tilde{E}$ containing $\cap_{n=1}^\infty \tilde{F}_n^c$ so that q is one to one from $E \setminus N$ onto $\tilde{E} \setminus \tilde{N}$ and*

$$\tilde{X}_t = q(X_t), \qquad \tilde{\mathbb{P}}_{\tilde{x}} = \mathbb{P}_{q^{-1}\tilde{x}}, \qquad \tilde{x} \in \tilde{E} \setminus \tilde{N}. \tag{8.3}$$

PROOF. Since both Dirichlet spaces in (8.2) are assumed to be quasi-regular, they are equivalent to some regular Dirichlet spaces and the equivalences are induced by some quasi-homeomorphisms q_1, q_2 in view of [**CF2**, Theorem 1.4.3]. Since two Dirichlet spaces in (8.2) are also assumed to be equivalent, so are the corresponding two regular Dirichlet spaces, the equivalence being induced by a quasi-homeomorphism q_3 on account of [**FOT**, Theorem A.4.2] combined with [**CF2**, Theorem 1.2.14]. Hence the equivalence of the quasi-regular Dirichlet spaces in (8.2) is induced by the quasi-homeomorphism $q = q_1 \circ q_3 \circ q_2^{-1}$ between E and $\tilde{E}$. Let $\{F_n\}, \{\tilde{F}_n\}$ be the nests attached to q.

According to [**CF2**, Theorem 3.1.13], we may assume without loss of generality that both X and $\tilde{X}$ are Borel right processes. Further the $\mathcal{E}$-polarity is equivalent to the m-polar for X. By virtue of [**CF2**, Theorem A.2.15], we can therefore find an m-inessential Borel set $N_1 \subset E$ containing $\cap_{n=1}^{\infty} F_n^c$. Consider the set $\tilde{N}_1 \subset \tilde{E}$ defined by $q(E \setminus N_1) = \tilde{E} \setminus \tilde{N}_1$. $\tilde{N}_1$ is an $\tilde{\mathcal{E}}$-polar Borel set and q is one to one from $E \setminus N_1$ onto $\tilde{E} \setminus \tilde{N}_1$.

Define the process $\widehat{X} = (\widehat{X}_t, \widehat{\mathbb{P}}_{\tilde{x}})_{\tilde{x} \in \tilde{E} \setminus \tilde{N}_1}$ by

$$\widehat{X}_t = q(X_t), \qquad \widehat{\mathbb{P}}_{\tilde{x}} = \mathbb{P}_{q^{-1}\tilde{x}}, \qquad \tilde{x} \in \tilde{E} \setminus \tilde{N}_1.$$

On account of [**FFY**, Lemma 3.1], we can then see that $\widehat{X}$ is an $\tilde{m}$-symmetric Markov process on $\tilde{E} \setminus \tilde{N}_1$ properly associated with the Dirichlet form $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ on $L^2(\tilde{E}; \tilde{m})$. Since the $\tilde{m}$-symmetric Borel right process $\tilde{X}$ is also properly associated with the Dirichlet form $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ on $L^2(\tilde{E}; \tilde{m})$, the same method as in the proof of [**CF2**, Theorem 3.1.12] combined with [**CF2**, Theorem A.2.15] leads us to finding an $\tilde{m}$-inessential Borel set $\tilde{N}$ containing $\tilde{N}_1$ for $\tilde{X}$ such that the Markov processes $\tilde{X}\big|_{\tilde{E}\setminus\tilde{N}}$ and $\widehat{X}\big|_{\tilde{E}\setminus\tilde{N}}$ are identical in law. It now suffices to define the set N by $E \setminus N = q^{-1}(\tilde{E} \setminus \tilde{N})$. □

Remark 8.3. Owing to the works of Albeverio, Ma, Röckner and Fitzsimmons, the quasi-regularity of a Dirichlet form has been known to be not only a sufficient condition but also a necessary one for the existence of a properly associated right process. It is further shown in [**CMR**] that a Dirichlet form is quasi-regular if and only if it is quasi-homeomorphic to a regular Dirichlet form on a locally compact separable metric space. These facts are formulated by Theorem 1.5.3 and Theorem 1.4.3, respectively, of [**CF2**] under the assumption (8.1) which is not needed actually. But we may assume it without loss of generality as will be seen below.

Indeed, let E be a Lusin space, m be a σ-finite measure on E and X be an m-symmetric Borel right process on E. Then, for $E_0 = \text{supp}[m]$, $E \setminus E_0$ is an m-negligible open set so that it is m-polar for X by [**CF2**, Theorem A.2.13 (iii)]. Hence, by [**CF2**, Theorem A.2.15], there exists a Borel set $E_1 \subset E_0$ such that $E \setminus E_1$ is m-inessential for X. E_1 is the support of $m\big|_{E_1}$ because, for any $x \in E_1$ and any neighborhood $O(x)$ of x, $m(O(x) \cap E_1) = m(O(x)) - m(O(x) \cap (E \setminus E_1)) > 0$. Hence it suffices to replace E by E_1.

In Theorem 5.1, the extension process Y is assumed to live on a Lusin space E into which $\overline{D}$ is homeomorphically embedded as an open subset. In this particular case, the above set E_1 can be choosen to contain $\overline{D}$ on account of the proof of [**CF2**, Theorem A.2.15]. Therefore, in Theorem 5.1 (resp. Remark 5.4), we can assume more strongly that $\overline{D}$ (resp. I) is homeomorphically embedded into the state space E of Y as a dense open subset.

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Zhen-Qing Chen
Department of Mathematics
University of Washington
Seattle
WA 98195, USA
E-mail: zqchen@uw.edu

Masatoshi Fukushima
Branch of Mathematical Science
Osaka University
Toyonaka
Osaka 560-8531, Japan
E-mail: fuku2@mx5.canvas.ne.jp