Capacitary bounds of measures and ultracontractivity of time changed processes

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Abstract

Given a regular transient Dirichlet space on $L^2(X;m)$ and an associated m-symmetric Hunt process $\mathbf{M}$ on X, we show the equivalence of the capacitary isoperimetric inequality $\mu(K)^\kappa \le \Theta\ \mathrm{Cap}(K)$ for a Radon measure μ on X and the ultracontractivity $\check{p}_t(x,y) \le (H/t)^{1/(1-\kappa)}$ for the transition function $\check{p}_t$ of the time changed process of $\mathbf{M}$ on the support of μ by the corresponding additive functional. We shall also show how the constants Θ and H control each other. When the Dirichlet space is the Riesz potential space and $\mathbf{M}$ is the symmetric stable process on $\mathbb{R}^n$, we show further that the isoperimetric constant can be replaced by the d-bound $\sup_{x\in\mathbb{R}^n, r>0} \mu(B(x,r))r^{-d}$ of the measure μ.

Résumé

Etant donné un espace de Dirichlet régulier transient sur $L^2(X;m)$ et le processus associé de Hunt m-symétrique $\mathbf{M}$ sur X, nous montrons l'équivalence de l'inégalité isopérimétrique capacitaire $\mu(K)^\kappa \le \Theta\ \mathrm{Cap}(K)$ pour une mesure de Radon μ sur X et la ultracontractivité $\check{p}_t(x,y) \le (H/t)^{1/(1-\kappa)}$ pour la fonction de transition $\check{p}_t$ du processus sur le support de μ qui s'obtient de $\mathbf{M}$ après le changement de temps associé à la fonctionnelle additive correspondante. Nous alons aussi montrer comment les deux constantes Θ et H sont liées. Lorsque cet espace de Dirichlet est l'espace potentiel de Riesz et $\mathbf{M}$ est un processus stable symétrique dans $\mathbb{R}^n$, nous montrons en plus que la constante isopérimétrique peut être remplacée par la quantité $\sup_{x\in\mathbb{R}^n, r>0} \mu(B(x,r))r^{-d}$ pour la mesure μ.

Keywords: isoperimetric constant, ultracontractivity, time change

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1 Introduction

Let $(X, m, \mathcal{E}, \mathcal{F})$ be a general regular transient Dirichlet space and $\mathbf{M}$ be the associated m-symmetric Hunt process on X. For a given smooth Radon measure μ on X, let $\check{\mathbf{M}}$ be the Markov process living on the support F of μ obtained from the process $\mathbf{M}$ by the time change with respect to its positive continuous additive functional whose Revuz measure is μ.

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Let $\kappa \in (0,1)$. In this paper, we are concerned with the relationship between the capacitary isoperimetric bound

$$\mu(K)^\kappa \le \Theta\ \mathrm{Cap}(K) \qquad \forall K(\text{compact}) \subset X \tag{1.1}$$

of the measure μ and the ultracontractivity bound

$$\check{p}_t(x,y) \le \left(\frac{H}{t}\right)^{\frac{1}{1-\kappa}}, \qquad t > 0, \tag{1.2}$$

of the transition function $\check{p}_t$ of the time changed process $\check{\mathbf{M}}$. In Theorem 3.2 and Theorem 3.3, we shall show not only the equivalence of (1.1) and (1.2) but also some explicit mutual dependency of the isoperimetirc constant Θ and the heat constant H. By observing the behaviours of the time changed process $\check{\mathbf{M}}$ over F, we can thus detect certain isoperimetric characters of the measure μ.

To this end, we prepare in §2 the capacitary strong type inequality

$$\int_0^\infty \mathrm{Cap}(\{x \in X : |u(x)| \ge t\})d(t^2) \le 4\mathcal{E}(u,u) \quad \forall u \in \mathcal{F} \cap C_0(X), \tag{1.3}$$

the constant 4 on the right hand side being optimal. (1.3) has been shown by Vondraček[Vo 96] in the present general context but we will give an alternative simple proof of it.

By using this inequality, one can easily see the equivalence of the isoperimetric bound (1.1) to the Sobolev imbedding:

$$||u||^2_{L^{2/\kappa}(X;\mu)} \le S\mathcal{E}(u,u), \quad \forall\ u \in \mathcal{F}_e, \tag{1.4}$$

the Sobolev constant S and the isoperimetric constant Θ controlling each other explicitly as will be exhibited in Corollary 3.1. By the general time change theory for the Dirichlet form ([FOT 94, §6.2]), (1.4) can be converted into the Sobolev inequality holding for the Dirichlet form $(\check{\mathcal{E}}, \check{\mathcal{F}})$ of the time changed process $\check{\mathbf{M}}$ on $L^2(F;\mu)$

$$||\varphi||^2_{L^{2/\kappa}(F;\mu)} \le S\ \check{\mathcal{E}}(\varphi,\varphi) \qquad \forall \varphi \in \check{\mathcal{F}}_e, \tag{1.5}$$

with the same constant S as in (1.4).

The equivalence of (1.2) and (1.5) is well known as the Varopoulos theorem ([Va 85]) but we are more concerned with the mutual dependence of constants H and S. The mutual dependence of H and the constant N appearing in the Nash type inequality has been well studied by Carlen,Kusuoka and Stroock[CKS 87] and so we shall invoke the work by Bakry, Coulhon, Ledoux and Saloff-Coste[BCLS 95] concerning the relation between N and S to finish the proof of the stated assertions in §3.

In §4, we shall work with the symmetric 2α-stable process $\mathbf{M}$ on $\mathbb{R}^n$ for $0 < \alpha \le 1,\ 2\alpha < n$. The associated extended Dirichlet space coincides with the space $\dot{L}^{\alpha,2}(\mathbb{R}^n)$ of the Riesz potentials of functions in $L^2(\mathbb{R}^n)$. For a Radon measure μ on $\mathbb{R}^n$, we will be concerned with its d-bound defined by

$$v_d(\mu) = \sup_{x \in \mathbb{R}^n, r>0} \frac{\mu(B(x,r))}{r^d}.$$

For $n-2\alpha < d \le n$, we shall prove that isoperimetric constant Θ of the measure μ with respect to the Riesz capacity and for the exponent

$$\kappa = \frac{n-2\alpha}{d} \tag{1.6}$$

can be estimated by $v_d(\mu)$ from below and above with some explicit constants (see (4.7) and (4.9)). Combining this with the general results in §3, we shall see in Theorem 4.2 and Theorem 4.3 that the ultracontractivity bound (1.2) for κ of (1.6) of the time changed process is equivalent to the finiteness of $v_d(\mu)$ and that the heat constant H and the d-bound $v_d(\mu)$ control each other to a certain extent.

Theorem 4.2 also contains an assertion of imbedding of the space $\dot{L}^{\alpha,2}(\mathbb{R}^n)$ into $L^{\frac{2d}{n-2\alpha}}(F;\mu)$ which goes back to the work of Adams [A 73]. But the present estimate of the Sobolev constant S in terms of the d-bound of μ is more explicit than [A 73] (see (4.14)).

The trace Dirichlet space $(\check{\mathcal{E}},\check{\mathcal{F}})$ on a d-set F of the present Riesz potential space $\dot{L}^{\alpha,2}(\mathbb{R}^n)$ is related to the Besov space $B^{2,2}_{\{d-(n-2\alpha)\}/2}(F)$ over F recently studied in [FU 02] and [CK 02]. We shall discuss their relationship in §5. Especially the latter will be seen to be continuously imbedded into the former.

At the ends of §3 and §4, we shall also give some sufficient conditions for the gaugeability (cf. Takeda[T 02]) of the positive continuous additive functional with Revuz measure μ in terms of Θ and v_d respectively.

2 Capacitary strong type inequality

The capacitary strong type inequality was first established by V. Maz'ya [M 73] for the Sobolev space $W^{1,p}(\mathbb{R}^n)$, $1<p<\infty$, as

$$\int_0^\infty C_{1,p}(\{x\in\mathbb{R}^n : |u(x)|\ge t\})d(t^p) \le \frac{p^p}{(p-1)^p}\int_{\mathbb{R}^N}|\nabla u|^p dx, \quad u\in C_0^\infty(\mathbb{R}^n) \tag{2.1}$$

the constant on the right hand side being the best. It was then extended to a large class of function spaces on $\mathbb{R}^n$ including the Riesz and Bessel potential spaces ([AH 96]) and to a general function spaces with contractive p-norms as well ([Ka 92],[FU 02]).

When $p=2$, the constant appearing on the right hand side of (2.1) equals 4 and the integral on the right hand side is just the Dirichlet integral. Accordingly we see that, if a counterpart of the inequality (2.1) should ever hold for a general Dirichlet form, then 4 must be the optimal constant for the counterpart.

Let $(X,m,\mathcal{E},\mathcal{F})$ be a regular transient Dirichlet space. By this, we mean that X is a locally compact separable metric space, m is an everywhere dense positive Radon measure on X, and that $(\mathcal{E},\mathcal{F})$ is a regular transient Dirichlet form on $L^2(X;m)$. The 0-order capacity of a compact set $K\subset X$ is then defined by

$$\mathrm{Cap}(K)=\inf\{\mathcal{E}(u,u) : u\in\mathcal{F}\cap C_0(X),\ u(x)\ge 1,\ x\in K\} \tag{2.2}$$

and extended to any subsets of X as a Choquet capacity. $\mathcal{F}_e$ denotes the extended Dirichlet space. In what follows, any function $u\in\mathcal{F}_e$ will be always taken to be quasi-continuous (cf. [FOT 94]).

The following is the Dirichlet form version of the *capacitary strong type inequality* and the inequality is sharpe by the reason mentioned above.

Theorem 2.1.

$$\int_0^\infty \mathrm{Cap}(\{x\in X : |u(x)|\ge t\})d(t^2) \le 4\mathcal{E}(u,u) \quad \forall u\in\mathcal{F}\cap C_0(X). \tag{2.3}$$

Without loss of generality we can assume the transience of $\mathcal{E}$, because otherwise we may replace $\mathcal{E}$ and Cap by $\mathcal{E}_1$ and the 1-order capacity respectively.

This theorem was first proved by K. Hansson [H 79] in a little different setting and under the condition that the resolvent admits a continuous density with respect to m. Z. Vondraček [Vo 96] has succeeded to remove this condition in a general regular Dirichlet form setting but still by adopting Hansson's proof. Being suggested by a related inequality in A. Ben Amor[BA 02], we will present here an alternative simple proof of this theorem. We note that T. Kolsrud[Ko 84] and M. Rao[R 88] have also obtained the inequality (2.2) with less sharpe constants.

Proof of Theorem 2.1: Take $u \in \mathcal{F} \cap C_0(X)$ and let $N_t = \{x \in X : |u(x)| \geq t\}$, $t > 0$. Since N_t is a compact set, we can take the 0-order equilibrium potential $e(t) \in \mathcal{F}$ and the equilibrium measure μ_t of the set N_t. According to [FOT 94, §2.2],

$$\mathrm{Cap}(N_t) = \mu_t(N_t) = \mathcal{E}(e(t), e(t)), \quad \mathcal{E}(e(t), v) = \int_{N_t} v(x)\mu_t(dx) \quad \forall v \in \mathcal{F}_e.$$

For $0 < s \leq t$, $e(s) = 1$ q.e. on N_t and hence

$$\mathcal{E}(e(t), e(s)) = \mathrm{Cap}(N_t) = \mathcal{E}(e(t), e(t)) \tag{2.4}$$

and we have

$$||e(s) - e(t)||_{\mathcal{E}}^2 = \mathrm{Cap}(N_s) - \mathrm{Cap}(N_t),$$

which decreases to 0 as $s \uparrow t$ by the right-continuity of the Choquet capacity Cap on compact sets. Therefore $e(t)$ is $\mathcal{E}$-left-continuous and $\mathcal{E}$-measurable.

Denote by S_u the compact support of u. Since $N_t \subset S_u$ and N_t is empty for $t > ||u||_\infty$, we have the integrability of $||e(t)||_{\mathcal{E}}$:

$$\int_0^\infty ||e(t)||_{\mathcal{E}} dt = \int_0^\infty \sqrt{\mathrm{Cap}(N_t)} dt \leq ||u||_\infty \sqrt{\mathrm{Cap}(S_u)}.$$

Therefore (cf. [Y 69, Th.5.1]) the Bochner integral $\psi = \int_0^\infty e(t)dt$ makes sense in the space $(\mathcal{F}_e, \mathcal{E})$ and moreover

$$\mathcal{E}(\psi, v) = \int_0^\infty \mathcal{E}(e(t), v)dt, \quad v \in \mathcal{F}_e.$$

We turn to the proof of the inequality (2.2). Since $|u|/t \geq 1$ on N_t,

$$\begin{aligned}
\int_0^\infty \mathrm{Cap}(N_t)d(t^2) &= 2\int_0^\infty t\mathrm{Cap}(N_t)dt = 2\int_0^\infty t\mu_t(N_t)dt \\
&\leq 2\int_0^\infty t\cdot\frac{1}{t}\int_{N_t} |u(x)|\mu_t(dx)dt \\
&= 2\int_0^\infty \mathcal{E}(e(t), |u|)dt = 2\mathcal{E}(\psi, |u|). \\
&\leq 2\sqrt{\mathcal{E}(\psi, \psi)}\sqrt{\mathcal{E}(u, u)}.
\end{aligned}$$

We compute $\mathcal{E}(\psi, \psi)$. By the symmetry of $\mathcal{E}$,

$$\begin{aligned}
\mathcal{E}(\psi, \psi) &= \mathcal{E}\left(\int_0^\infty e(t)dt, \int_0^\infty e(s)ds\right) = \int_0^\infty \int_0^\infty \mathcal{E}(e(t), e(s))dtds \\
&= 2\int_0^\infty \int_0^s \mathcal{E}(e(t), e(s))dtds.
\end{aligned}$$

We then have from (2.4)

$$\begin{aligned}\mathcal{E}(\psi,\psi) &= 2\int_0^\infty \int_0^s \mathcal{E}(e(s),e(s))dtds = 2\int_0^\infty s\mathcal{E}(e(s),e(s))ds \\ &= 2\int_0^\infty t\,\mathrm{Cap}(N_t)dt = \int_0^\infty \mathrm{Cap}(N_t)d(t^2).\end{aligned}$$

Thus we get the desired inequality (2.3). □

3 Capacitary bounds of measures and ultracontractivity of time changed processes

We continue to work with a regular transient Dirichlet space $(X,m,\mathcal{E},\mathcal{F})$. Theorem 2.1 implies the following (cf. [AH 96, §7.2]):

Theorem 3.1. *Let μ be a Borel measure on X and $\kappa \in (0,1]$.*
(i) *If*

$$\mu(K)^\kappa \le \Theta \mathrm{Cap}(K), \quad \forall\ K \text{ compact}, \tag{3.1}$$

for some positive constant Θ, then μ is a smooth Radon measure and

$$||u||^2_{L^{2/\kappa}(X;\mu)} \le S\mathcal{E}(u,u), \quad \forall\ u \in \mathcal{F}_e, \tag{3.2}$$

for some positive constant $S \le (4/\kappa)^\kappa \Theta$.
(ii) *Conversely, if* (3.2) *holds for any $u \in \mathcal{F}\cap C_0(X)$ and for some positive constant S, then* (3.1) *holds for some positive constant $\Theta \le S$.*

Proof: (ii) is evident by taking the infimum in (3.2) for $u \in \mathcal{F}\cap C_0(X)$ such that $u \ge 1$ on K.

We assume (3.1). Obviously μ is then a smooth Radon measure. Let $u \in \mathcal{F}\cap C_0(X)$. Since the level set $N_t = \{x \in X : |u(x)| \ge t\}$ is compact for $t > 0$, we have, by using the level set representation of u with respect to μ,

$$\begin{aligned}\int_X |u(x)|^{2/\kappa}\mu(dx) &= \int_0^\infty \mu(N_t)d(t^{2/\kappa}) \\ &\le \int_0^\infty \Theta^{1/\kappa}\mathrm{Cap}(N_t)^{1/\kappa}d(t^{2/\kappa}) \\ &= \Theta^{1/\kappa}\int_0^\infty \mathrm{Cap}(N_t)^{(1/\kappa)-1}\mathrm{Cap}(N_t)d(t^{2/\kappa}).\end{aligned}$$

Since $|u(x)|/t \ge 1$ on N_t, we have $\mathrm{Cap}(N_t) \le \frac{1}{t^2}\mathcal{E}(u,u)$, and

$$\int_X |u(x)|^{2/\kappa}\mu(dx) \le \Theta^{1/\kappa}\mathcal{E}(u,u)^{(1/\kappa)-1}\int_0^\infty \mathrm{Cap}(N_t)\left(\frac{1}{t}\right)^{2/\kappa-2} d(t^{2/\kappa}).$$

By Theorem 2.1, we are led to

$$\begin{aligned}\int_X |u(x)|^{2/\kappa}\mu(dx) &\le \Theta^{1/\kappa}\mathcal{E}(u,u)^{(1/\kappa)-1}\frac{1}{\kappa}\int_0^\infty \mathrm{Cap}(N_t)d(t^2) \\ &\le \Theta^{1/\kappa}\left(\frac{4}{\kappa}\right)\mathcal{E}(u,u)^{1/\kappa}.\end{aligned}$$

We get (3.2) for $S = (4/\kappa)^\kappa \Theta$ and $u \in \mathcal{F} \cap C_0(X)$, which can be readily extended to $u \in \mathcal{F}_e$.

$\square$

For a measure μ on X, we introduce its *isoperimetric constant* and *Sobolev constant* respectively by

$$\Theta_\kappa(\mu) = \sup_K \frac{\mu(K)^\kappa}{\mathrm{Cap}(K)} \qquad \kappa \in (0,1], \tag{3.3}$$

$$S_\eta(\mu) = \sup_{u \in \mathcal{F} \cap C_0(X)} \frac{||u||^2_{L^\eta(\mu)}}{\mathcal{E}(u,u)} \qquad \eta \in [2,\infty). \tag{3.4}$$

The supremum in (3.4) can be taken for all $u \in \mathcal{F}_e$. $S_2(\mu)$ may be called the *Poincaré constant* of μ. Theorem 3.1 can be rephrased as follows:

Corollary 3.1. *For a measure μ on X and for $\kappa \in (0,1]$, $0 < \Theta_\kappa(\mu) < \infty$ if and only if $0 < S_{2/\kappa}(\mu) < \infty$. Moreover,*

$$\Theta_\kappa(\mu) \le S_{2/\kappa}(\mu) \le (4/\kappa)^\kappa \ \Theta_\kappa(\mu), \qquad \kappa \in (0,1]. \tag{3.5}$$

The number $(4/\kappa)^\kappa$ in the inequality (3.5) takes value in $(1,4]$ and decreases to 1 as $\kappa \downarrow 0$. Hence, the isoperimetric constant becomes more optimal to control the Sobolev constant when κ gets closer to 0. In the next section, we shall see that many d-measures on $\mathbb{R}^n$ admit finite isoperimeric constants for some $\kappa \in (0,1)$ with respect to the Riesz capacity $\dot{C}_{\alpha,2}$.

Suppose that a measure μ is of finite energy integral and that its potential $U\mu$ is m-essentially bounded. Then

$$\Theta_1(\mu) \le ||U\mu||_\infty. \tag{3.6}$$

In fact, we have for any $\varphi \in \mathcal{F} \cap C_0(X)$ and any compact set K

$$\int \varphi I_K d\mu = \mathcal{E}(\varphi, UI_K\mu) \le ||\varphi||_{\mathcal{E}} \cdot ||UI_K\mu||_{\mathcal{E}},$$

and

$$\mathcal{E}(UI_K\mu, UI_K\mu) = \int U\tilde{I_K}\mu \cdot I_K d\mu \le ||U\mu||_\infty \cdot \mu(K).$$

It then suffices to take the infimum for $\varphi \in \mathcal{F} \cap C_0(X)$ which is equal to 1 on K.

By Corollary 3.1 and (3.6), we are led to the bound of the Poincaré constant $S_2(\mu)$:

$$S_2(\mu) \le 4||U\mu||_\infty. \tag{3.7}$$

Vondračeck[Vo 96] first derived this bound from the capacitary strong type inequality (2.2). As a matter of fact, a better estimate is known in this case:

$$S_2(\mu) \le ||U\mu||_\infty. \tag{3.8}$$

At least three different proofs of (3.8) have been given by Stollmann-Voigt[SV 96], Fitzsimmons[F 00] and Ben Amor[BA 02]. The proof in [BA 02] seems to be simplest among them. The capacitary strong type inequality is less useful in this case.

The trace Sobolev inequality (3.2) is intrinsically related to the ultracontractivity of the transition semigroup of a time changed process. Therefore Corollary 3.1 indicates that the isoperimetric constant of a measure and the (heat) constant in the ultracontractive bound may control each other.

Let $\mathbf{M} = \{X_t, P_x\}$ be an m-symmetric Hunt process on X associated with the Dirichlet form $\mathcal{E}$ and $A = A_t$ be a PCAF of $\mathbf{M}$ whose Revuz measure is a given smooth Radon measure μ. Denote by F and $\tilde{F}$ the support of μ and A respectively. Then $\tilde{F} \subset F$ q.e., $\mu(F \setminus \tilde{F}) = 0$ and further $\tilde{F}$ is a quasi-support of μ, namely, if quasi-continuous functions coincide μ-a.e., then they coincide q.e. on $\tilde{F}$. Recall that each element $u \in \mathcal{F}_e$ is taken to be quasi-continuous in this paper.

We consider the time changed process $\check{\mathbf{M}} = (\check{X}_t, P_x)_{x \in \tilde{F}}$ defined by

$$\check{X}_t = X_{\tau_t} \qquad \tau_t = \inf\{s > 0 : A_s > t\}.$$

$\check{\mathbf{M}}$ is a μ-symmetric transient right process, whose Dirichlet form $(\check{\mathcal{E}}, \check{\mathcal{F}})$ on $L^2(F; \mu)$ and the extended Dirichlet space $\check{\mathcal{F}}_e$ can be described as follows (cf. [FOT 94, §6.2]) :

$$\check{\mathcal{F}}_e = \{\varphi = u|_F \, \mu - a.e. : u \in \mathcal{F}_e\} \qquad \check{\mathcal{F}} = \check{\mathcal{F}}_e \cap L^2(F; \mu) \tag{3.9}$$

$$\check{\mathcal{E}}(\varphi, \varphi) = \mathcal{E}\left(H_{\tilde{F}} u, H_{\tilde{F}} u\right) \quad \varphi = u|_F \in \check{\mathcal{F}}_e, \tag{3.10}$$

where

$$H_{\tilde{F}} u(x) = E_x\left(u\left(X_{\sigma_{\tilde{F}}}\right)\right) \quad x \in X,$$

E_x denoting the expectation with respect to P_x and $\sigma_{\tilde{F}}$ being the hitting time of the set $\tilde{F}$ by the sample path X_t. Two elements of $\check{\mathcal{F}}_e$ are regarded identical if they coincides μ-a.e. Since $\tilde{F}$ is a quasi-support of μ, the definition (3.10) of $\check{\mathcal{E}}$ makes sense.

The definition (3.10) can be described in a more analytic way. We introduce the closed subspace of $(\mathcal{F}_e, \mathcal{E})$ by

$$\mathcal{F}_{e, X \setminus \tilde{F}} = \{u \in \mathcal{F}_e : u = 0 \text{ q.e. on } \tilde{F}\},$$

and let $\mathcal{H}_{\tilde{F}}$ be its orthognal complement:

$$\mathcal{F}_e = \mathcal{F}_{e, X \setminus \tilde{F}} \oplus \mathcal{H}_{\tilde{F}}.$$

Then (c.f. [FOT 94, The. 4.3.2])

$$\mathcal{P} u = H_{\tilde{F}} u \qquad u \in \mathcal{F}_e.$$

where $\mathcal{P}$ denotes the orthogonal projection on the space $\mathcal{H}_{\tilde{F}}$. Thus we can restate (3.10) as follows (the Dirichlet principle):

$$\check{\mathcal{E}}(\varphi, \varphi) = \inf\{\mathcal{E}(u, u) : u \in \mathcal{F}_e, \ u = \varphi \ \mu\text{-a.e. on } F\}, \quad \varphi \in \check{\mathcal{F}}_e. \tag{3.11}$$

The first half of the next theorem is immediate from (3.2) and (3.11).

Theorem 3.2. *Suppose a measure μ satisfies $\Theta_\kappa(\mu) \in (0, \infty)$ for some $\kappa \in (0, 1)$. Then we have the following for $S = S_{2/\kappa}(\mu) (\in (\Theta_\kappa(\mu), (4/\kappa)^\kappa \Theta_\kappa(\mu)))$.*
(i)

$$||\varphi||^2_{L^{2/\kappa}(F;\mu)} \leq S \ \check{\mathcal{E}}(\varphi, \varphi) \qquad \forall \varphi \in \check{\mathcal{F}}_e. \tag{3.12}$$

(ii) *The transition function* $\check{p}_t$ *of the time changed process* $\check{\mathbf{M}}$ *on* F *satisfies*

$$\check{p}_t(x,y) \leq \left(\frac{H}{t}\right)^{\frac{1}{1-\kappa}}, \qquad t > 0, \tag{3.13}$$

for $\mu \times \mu-$a.e. $(x,y) \in F \times F$, *where* H *is some positive constant with*

$$H \leq \frac{1}{1-\kappa} \cdot S. \tag{3.14}$$

We know that (3.12) and (3.13) are equivalent by Voropoulos [Va 85]. But we are more concerned with dependence of constants Θ_κ and H.

In order to get the bound (3.14), we set

$$\kappa = (\nu - 2)/\nu \qquad (\nu = 2/(1-\kappa)). \tag{3.15}$$

Then (3.12) reads

$$||\varphi||^2_{L^{2\nu/(\nu-2)}} \leq S\ \check{\mathcal{E}}(\varphi,\varphi), \quad \varphi \in \check{\mathcal{F}}_e \tag{3.16}$$

which can be converted by a Hölder inequality into a Nash type inequality

$$||\varphi||_2^{2(1+\frac{2}{\nu})} \leq N\ \check{\mathcal{E}}(\varphi,\varphi)||\varphi||_1^{\frac{4}{\nu}}, \quad \varphi \in \check{\mathcal{F}}_e, \tag{3.17}$$

with $N = S$. Then, by a Nash argument adopted by [CKS 87],

$$||\check{p}_t||_{1\to\infty} \leq \left(\frac{H}{t}\right)^{\frac{\nu}{2}}, \qquad t > 0, \tag{3.18}$$

for $H = \frac{\nu}{2}\ S$ yielding (3.14).

Conversely, suppose that μ is a smooth Radon measure with support F and that the transition function $\check{p}_t$ of the time changed process $\check{\mathbf{M}}$ satisfies the ultracontractivity (3.18). Then, by Carlen-Kusuoka-Stroock[CKS 87](see also [SC 02]), we have the Nash type inequality (3.17) with

$$N = 2\left(1 + \frac{\nu}{2}\right)^{1+\frac{\nu}{2}} \cdot H.$$

On the other hand, the Nash type inequality (3.17) implies the Sobolev inequality (3.16) with

$$S = 24\ e^2\ \frac{\nu}{\nu-2}\ N,$$

by virtue of Bakry,Coulhon,Ledoux and Saloff-Coste[BCLS 95, Cor.4.4,Cor.7.3]. Combining these two bounds, we get the following converse to Theorem 3.2.

Theorem 3.3. *Suppose that* μ *is a smooth Radon measure with support* F *and that the transition function* $\check{p}_t$ *of the time changed process* $\check{\mathbf{M}}$ *on* F *with respect to the PCAF with Revuz measure* μ *satisfies the bound* (3.13) *for some* $\kappa \in (0,1)$, $H > 0$. *Then*

(i) *The Sobolev inequality* (3.12) *holds for some positive constant* S *with*

$$S \leq 48\ e^2\ \frac{1}{\kappa}\left(\frac{2-\kappa}{1-\kappa}\right)^{\frac{2-\kappa}{1-\kappa}} \cdot H. \tag{3.19}$$

(ii) μ *admits an isoperimetric constant* $\Theta_\kappa(\mu)$ *with a bound*

$$(4/\kappa)^{-\kappa}\ S \leq \Theta_\kappa(\mu) \leq S \tag{3.20}$$

by the constant S *of* (i).

The second assertion of Theorem 3.3 follows from Corollary 3.1 and the identity

$$S_\eta(\mu) = \sup_{\varphi \in \check{\mathcal{F}}_e} \frac{||\varphi||^2_{L^\eta(\mu)}}{\check{\mathcal{E}}(\varphi,\varphi)}, \quad \eta \in [2,\infty). \tag{3.21}$$

The inequalities $\geq$ and $\leq$ follow from (3.12) and Dirichlet principle (3.11) respectively.

Takeda's test (cf.[T 02],[C 02], [TU 02]) says that, under certain conditions on $\mathbf{M}$ (absolute continuity of the transition function with respect to m etc.) and on μ (a finite measure in the Kato class for instance), $S_2(\mu) < 1$ is necessary and sufficient for the *gaugeability* of the PCAF A_t associated with μ in the sense that

$$\sup_{x \in X} E_x(\exp(A_{\zeta-})) < \infty.$$

Therefore, under these additional conditions, we have the following from Corollary 3.1.

Theorem 3.4. *If*

$$\Theta_\kappa(\mu) < \infty, \ \exists \kappa \in (0,1) \qquad M = \mu(X) < \infty,$$

and if

$$M < (4 \, \Theta_\kappa(\mu))^{-\frac{1}{1-\kappa}},$$

then A is gaugeable.

We note the obvious bound $\Theta_1(\mu) \leq M^{1-\kappa} \, \Theta_\kappa(\mu)$.

4 d-bounds of measures on $\mathbb{R}^n$ and time changes of symmetric stable processes

In this section, we let $\mathbf{M} = (X_t, P_x)$ be the symmetric 2α-stable process on $\mathbb{R}^n$ for $0 < \alpha \leq 1$. The transition function of $\mathbf{M}$ is a convolution semigroup $\{\nu_t, t > 0\}$ of symmetric probability measures on $\mathbb{R}^n$ with

$$\hat{\nu}_t(x) \left(= \int_{\mathbb{R}^n} e^{i(x,y)} \nu_t(dy) \right) = e^{-tc|x|^{2\alpha}},$$

c being a fixed positive constant. For simplicity, we take $c = 1$. In case that $\alpha = 1$, $\mathbf{M}$ is the n-dimensional Brownian motion with variance of μ_t being equal to $2t$.

The Dirichlet form $(\mathcal{E},\mathcal{F})$ of $\mathbf{M}$ on $L^2(\mathbb{R}^n)$ is given by

$$\left\{ \begin{array}{lcl} \mathcal{E}(u,u) & = & \int_{\mathbb{R}^n} \hat{u}(x)\bar{\hat{v}}(x)|x|^{2\alpha}dx \\ \mathcal{F} & = & \{u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} |\hat{u}(x)|^2|x|^{2\alpha}dx < \infty\}. \end{array} \right. \tag{4.1}$$

In what follows, we assume that

$$0 < \alpha \leq 1, \qquad 2\alpha < n$$

so that $\mathbf{M}$ is transient. The extended Dirichlet space $(\mathcal{F}_e, \mathcal{E})$ of $\mathbf{M}$ can then be identified with the Riesz potential space $\dot{L}^{\alpha,2}(\mathbb{R}^n)$ described below. The Riesz potential of a measure ν on $\mathbb{R}^n$ is defined by

$$I_\alpha * \nu(x) = \gamma_\alpha \int_{\mathbb{R}^n} |x-y|^{-(n-\alpha)} \nu(dy), \quad \gamma_\alpha = \frac{\Gamma((n-\alpha)/2))}{\pi^{n/2} 2^\alpha \Gamma(\alpha/2)}.$$

When $\nu(x)$ is of the form $f(x)dx$, then $I_\alpha * \nu$ is denoted by $I_\alpha * f$. For $f \in L^2(\mathbb{R}^n)$, $I_\alpha * f(x)$ is absolutely convergent for a.e. $x \in \mathbb{R}^n$, and we may consider the function space

$$\dot{L}^{\alpha,2}(\mathbb{R}^n) = \{I_\alpha * f : f \in L^2(\mathbb{R}^n)\}. \tag{4.2}$$

For $f \in L^2(\mathbb{R}^n)$, we know that $I_\alpha * f \in L^{\frac{n-2\alpha}{2n}}(\mathbb{R}^n)$ by virtue of the Sobolev embedding theorem (cf. [S 70, pp119]) and hence $I_\alpha * f$ admits its Fourier transform as a tempered distribution. On the other hand, the Fourier transform of the kernel $\gamma_\alpha |x|^{-n+\alpha}$ as a tempered distribution is known to be equal to $|x|^{-\alpha}$. Consequently we have the identity (cf. [L 72, Th. 0.13])

$$I_\alpha \hat{*} f(x) = |x|^{-\alpha} \cdot \hat{f}(x) \qquad \text{a.e. } x \in \mathbb{R}^n.$$

If $I_\alpha * f = 0$ for $f \in L^2(\mathbb{R}^n)$, then the above identity implies that $\hat{f} = 0$ and so $f = 0$. Therefore the next inner product is well introduced on the space $\dot{L}^{\alpha,2}(\mathbb{R}^n)$:

$$(u,v)_{\dot{L}^{\alpha,2}(\mathbb{R}^n)} = (f,g)_{L^2(\mathbb{R}^n)} \qquad u = I_\alpha * f, \ v = I_\alpha * g, \ f,g \in L^2(\mathbb{R}^n). \tag{4.3}$$

The Riesz potential space equipped with the inner product (4.3) is thus a real Hilbert space. The capacity $\dot{C}_{\alpha,2}$ associated with this space is defined for a compact set $K \subset \mathbb{R}^n$ by

$$\dot{C}_{\alpha,2}(K) = \inf\{||f||^2_{L^2(\mathbb{R}^n)} : f \in L^2_+(\mathbb{R}^n), \ I_\alpha * f(x) \geq 1 \ \forall x \in K\}, \tag{4.4}$$

and extended to all subsets of $\mathbb{R}^n$ as a Choquet capacity.

Lemma 4.1. (i)

$$\mathcal{F}_e = \dot{L}^{\alpha,2}(\mathbb{R}^n), \qquad \mathcal{E}(u,v) = (u,v)_{\dot{L}^{\alpha,2}(\mathbb{R}^n)}, \quad u,v \in \mathcal{F}_e.$$

(ii) *For any compact set $K \subset \mathbb{R}^n$,*

$$\mathrm{Cap}(K) = \dot{C}_{\alpha,2}(K),$$

where Cap *is defined by* (2.2) *for the present Dirichlet form. It holds furthermore that*

$$\dot{C}_{\alpha,2}(K) = \inf\{||f||^2_{L^2(\mathbb{R}^n)} : f \in B_0^+(\mathbb{R}^n), \ I_\alpha * f(x) \geq 1 \ x \in K\}, \tag{4.5}$$

where $B_0^+(\mathbb{R}^n)$ denotes the space of non-negative bounded measurable functions on $\mathbb{R}^n$ vanishing outside some compact set.

Proof: (i) has been shown in [FOT 94, Example 1.5.2]. The proof of (ii) is given essentially in the proof of [AH 96, Prop. 2.3.13]. □

We call a closed subset F of $\mathbb{R}^n$ a (semi global) *d-set* for $0 < d \leq n$ if there exists a positive measure μ supported by F satisfying, for some constants $0 < c_1 \leq c_2$,

$$c_1 r^d \leq \mu(B(x,r)) \quad \forall x \in F, \forall r \in (0,1)$$

$$\mu(B(x,r)) \leq c_2 r^d \quad \forall x \in F, \forall r \in (0,\infty),$$

where $B(x,r)$ denotes the n-dimensional ball with center x and radius r. Such a measure is called a *d-measure.* It is known that the restriction of the d-dimensional Hausdorff measure to a d-set F is a d-measure (cf.[JW 84]).

For a d-measure μ, we will be concerned with its *d-bound* defined by

$$v_d(\mu) = \sup_{x \in \mathbb{R}^n, r>0} \frac{\mu(B(x,r))}{r^d} (\in [c_1, c_2]). \tag{4.6}$$

We consider a d-measure μ on a d set F with

$$n - 2\alpha < d \leq n.$$

Otherwise, $\dot{C}_{\alpha,2}(F) = 0$ and μ can not satisfy the isoperimetric inequality with respect to the present Dirichlet form. Since

$$\dot{C}_{\alpha,2}(B(x,r)) = \dot{c}_{\alpha,2}\, r^{n-2\alpha}, \qquad \dot{c}_{\alpha,2} = \dot{C}_{\alpha,2}(B(0,1)),$$

we can immediately obtain a lower bound of the isoperimetric constant for μ by its d-bound:

$$\dot{c}_{\alpha,2}^{-1}\, v_d(\mu)^{\frac{n-2\alpha}{d}} \;\leq\; \Theta_{\frac{n-2\alpha}{d}}(\mu). \tag{4.7}$$

In order to obtain an inequality in the opposite direction, we prepare a lemma.

Lemma 4.2. *For a Radon measure μ, suppose there exist constants $\kappa \in (0,1)$ and $A > 0$ such that*

$$||I_\alpha * \mu_K||_{L^2(\mathbb{R}^n)} \leq \sqrt{A}\, \mu(K)^{1-\frac{\kappa}{2}} \tag{4.8}$$

for any compact set $K \subset \mathbb{R}^n$. Here μ_K denotes $I_K \mu$. Then

$$\Theta_\kappa(\mu) \leq A.$$

Proof: For $f \in B_0^+(\mathbb{R}^n)$, we put $E = \{x \in \mathbb{R}^n : I_\alpha * f(x) \geq 1\}$. Since E is compact, we have from (4.8)

$$\mu(E) \leq \int_{\mathbb{R}^n} I_\alpha * f \; d\mu_E \leq ||f||_{L^2(\mathbb{R}^n)} ||I_\alpha * \mu_E||_{L^2(\mathbb{R}^n)} \leq \sqrt{A} ||f||_{L^2(\mathbb{R}^2)} \mu(E)^{1-\frac{\kappa}{2}},$$

and

$$\mu(E)^\kappa \leq A ||f||^2_{L^2(\mathbb{R}^n)}.$$

Taking the infimum for those functions $f \in B_0^+(\mathbb{R}^n)$ such that $I * f \geq 1$ on a compact set K, we get from (4.5)

$$\mu(K)^\kappa \leq A\, \dot{C}_{\alpha,2}(K).$$

□

Theorem 4.1. *For any Radon measure μ with finite d-bound, it holds that*

$$\Theta_{\frac{n-2\alpha}{d}}(\mu) \;\leq\; c(n,\alpha,d)\, v_d(\mu)^{\frac{n-2\alpha}{d}} \tag{4.9}$$

for

$$c(n,\alpha,d) = \frac{4d^2\, \gamma_\alpha^2\, v_n\, (n-\alpha)^2}{(n-2\alpha)^2 \{d - (n-2\alpha)\}^2}, \tag{4.10}$$

where v_n is the volume of the n dimensional unit ball.

Proof: By Lemma 4.2, it suffices to show that μ satisfies the Riesz potential bound (4.8) with

$$\kappa = \frac{n-2\alpha}{d}, \qquad A = c(n,\alpha,d)\; v_d(\mu)^{\frac{n-2\alpha}{d}} \tag{4.11}$$

for $c(n,\alpha,d)$ of (4.10).

Actually the inequality (4.8) holding for some positive constant A was essentially shown in the proof of [AH 96, Th. 7.2.2]. By making the computation employed there more detailed, we aim at deriving an expression of the constant A as explicitly as (4.10).

We first rewrite $I_\alpha * \mu_K$ as

$$I_\alpha * \mu_K(x) = (n-\alpha)\gamma_\alpha \int_0^\infty \frac{\mu_K(B(x,r))}{r^{n-\alpha}} \cdot \frac{dr}{r}$$

and use the Minkowski inequality to get

$$||I_\alpha * \mu_K||_2 \le (n-\alpha)\gamma_\alpha \int_0^\infty \frac{||\mu_K(B(\cdot,r))||_2}{r^{n-\alpha}} \cdot \frac{dr}{r}. \tag{4.12}$$

We have on the one hand,

$$\begin{aligned}
||\mu_K(\cdot.r))||_2^2 \;\; &= \int_{\mathbb{R}^n} \mu(K\cap B(x,r))^2 dx \\
&\le \mu(K) \int_{\mathbb{R}^n}\int_K I_{\{|x-y|<r\}}(y) d\mu(y) dx \\
&= \mu(K) \int_K \int_{\mathbb{R}^n} I_{\{|x-y|<r\}}(x) dx d\mu(y) \\
&= \mu(K) \int_K |B(y,r)| d\mu(y) = v_n r^n \mu(K)^2,
\end{aligned}$$

and on the other hand,

$$\begin{aligned}
||\mu_K(B(\cdot,r))||_2^2 \;\; &\le \sup_x \mu(B(x,r)) \int_{\mathbb{R}^n} \mu(K\cap B(x,r)) dx \\
&\le v_d(\mu) r^d v_n r^n \mu(K).
\end{aligned}$$

Splitting the interval $(0,\infty)$ of integration on the right hand side of (4.12) into two intervals $[R,\infty)$ and $(0,R)$, and substituting the preceding two bounds respectively, we get

$$||I_\alpha * \mu_K||_2 \le \gamma_\alpha (n-\alpha)(J_1(R) + J_2(R)) \tag{4.13}$$

with

$$J_1(R) = \sqrt{v_n}\; \mu(K)\; \frac{2}{n-2\alpha}\; \frac{1}{R^{\frac{n-2\alpha}{2}}},$$

$$J_2(R) = \sqrt{v_n v_d(\mu)}\; \sqrt{\mu(K)}\; \frac{2}{d-(n-2\alpha)}\; R^{\frac{d-(n-2\alpha)}{2}}.$$

Take $R = \eta\; \mu(K)^{1/d}$. Then

$$J_1 = \sqrt{v_n}\; \frac{2}{n-2\alpha}\; \eta^{-\frac{n-2\alpha}{2}} \cdot \mu(K)^{\frac{2d-(n-2\alpha)}{2d}},$$

$$J_2 = \sqrt{v_n v_d(\mu)}\ \frac{2}{d-(n-2\alpha)}\ \eta^{\frac{d-(n-2\alpha)}{2}} \cdot \mu(K)^{\frac{2d-(n-2\alpha)}{2d}}.$$

We then choose η minimizing the sum of the above two expressions, namely, $\eta = v_d(\mu)^{-1/d}$. Thus we obtain from (4.13)

$$||I_\alpha * \mu_K||_2 \le B\ v_d(\mu)^{\frac{n-2\alpha}{2d}}\mu(K)^{\frac{2d-(n-2\alpha)}{2d}}$$

with

$$B = \frac{2d\ \gamma_\alpha\ \sqrt{v_n}\ (n-\alpha)}{(n-2\alpha)\ \{d-(n-2\alpha)\}},$$

which equals the square root of the constant $c(n,\alpha,d)$ of (4.10).

□

As an example, take $n = 3, \alpha = 1, d = 2$. Then $c(3,1,2) = \dfrac{64}{3\pi^3}$ and hence any 2-measure μ on $\mathbb{R}^3$ have the isoperimetric bound

$$\frac{\Theta_{1/2}(\mu)}{v_2(\mu)^{1/2}} \le \frac{64}{3\pi^3} \approx 0.688$$

with respect to the Newtonian capacity $\dot{C}_{1,2}$ on $\mathbb{R}^3$. For the 2-dimensional Lebesgue measure μ_0 on a plane $F \subset \mathbb{R}^3$, it is known (cf. [M 85, pp116]) that

$$\frac{\Theta_{1/2}(\mu_0)}{v_2(\mu_0)^{1/2}} = \frac{1}{8} = 0.125.$$

By setting $\kappa = \frac{n-2\alpha}{d}$ in Corollary 3.1 and using (4.7) and (4.9), we get the bound of the Sobolev constant $S = S_{\frac{2d}{n-2\alpha}}(\mu)$ for μ in terms of its d-bound $v_d(\mu)$:

$$\dot{c}_{\alpha,2}^{-1}\ v_d(\mu)^{\frac{n-2\alpha}{d}}\ \le\ S\ \le\ (4d/(n-2\alpha))^{\frac{n-2\alpha}{d}}\ c(n,\alpha,d)\ v_d(\mu)^{\frac{n-2\alpha}{d}} \tag{4.14}$$

for the constant $c(n,\alpha,d)$ of (4.10).

By setting $\kappa = \frac{n-2\alpha}{d}$ in Theorem 3.1 and Theorem 3.2, we have

Theorem 4.2. *Suppose μ is a d-measure on $\mathbb{R}^n$ with $n-2\alpha < d \le n$. Then we have the following for S satisfying the bounds (4.14):*
(i)

$$||u||^2_{L^{\frac{2d}{n-2\alpha}}(\mathbb{R}^n;\mu)} \le S\,\mathcal{E}(u,u) \qquad \forall u \in \dot{L}^{\alpha,2}(\mathbb{R}^n). \tag{4.15}$$

(ii) *Let $\check{\mathbf{M}}$ be the time changed process on the support F of μ of $\mathbf{M}$ by the PCAF with Revuz measure μ. Then its transition function $\check{p}_t$ satisfies*

$$\check{p}_t(x,y) \le \left(\frac{H}{t}\right)^{\frac{d}{d-(n-2\alpha)}}, \qquad t > 0, \tag{4.16}$$

for $\mu\times\mu$-a.e. $(x,y) \in F\times F$, where H is some positive constant with

$$H \le \frac{d}{d-(n-2\alpha)}\ S. \tag{4.17}$$

Actually inequality (4.15) together with the bounds

$$c_3 \ v_d(\mu)^{\frac{n-2\alpha}{d}} \leq S \leq c_4 \ v_d(\mu)^{\frac{n-2\alpha}{d}}$$

holding for some positive constants c_3, c_4 independent of μ goes back to the work of Adams [A 73] (see also [M 85, 1.4.1]). Here we have made these contants c_3 and c_4 more explicit in (4.14).

We can also derive from Theorem 3.3 the following converse to Theorem 4.2.

Theorem 4.3. *Suppose that μ is a smooth Radon measure on $\mathbb{R}^n$ with support F and that the transition function $\tilde{p}_t$ of the time changed process $\tilde{\mathbf{M}}$ on F with respect to the PCAF with Revuz measure μ satisfies the bound* (4.16) *for some $d \in (n-2\alpha, n]$ and $H > 0$. Then*
(i) *The inequality* (4.15) *holds for some positive constant S with*

$$S \leq \frac{48de^2}{n-2\alpha} \left(\frac{2d-(n-2\alpha)}{d-(n-2\alpha)} \right)^{\frac{2d-(n-2\alpha)}{d-(n-2\alpha)}} \cdot H. \tag{4.18}$$

(ii) *μ is a d-measure whose d-bound $v_d(\mu)$ satisfies*

$$\frac{n-2\alpha}{4d} \ \left(\frac{S}{c(n,\alpha,d)} \right)^{\frac{d}{n-2\alpha}} \leq v_d(\mu) \leq (\dot{c}_{\alpha,2} \ S)^{\frac{d}{n-2\alpha}} \tag{4.19}$$

for the constant S of (i) *and for $c(n,\alpha,d)$ of* (4.10).

Any d-measure μ is not only smooth but in the Kato class. Since the present process $\mathbf{M}$ on $\mathbb{R}^n$ satisfies all conditions imposed by Takeda [T 02] (see also [C 02],[TU 02]), we see, for a finite d-measure μ, that $S_2(\mu) < 1$ is a necessary and sufficient condition for the gaugeability

$$\sup_{x\in\mathbb{R}^n} E_x(\exp(A_\infty)) < \infty \tag{4.20}$$

of the PCAF A with Revuz measure μ. By setting $\kappa = \frac{n-2\alpha}{d}$ in Theorem 3.4, we get the next theorem from Theorem 4.1.

Theorem 4.4. *Let μ be a d-measure on $\mathbb{R}^n$ for $n - 2\alpha < d \leq n$. Suppose $M = \mu(X)$ is finite. If*

$$M < \ (4 \ c(n,\alpha,d))^{-\frac{1}{d-(n-2\alpha)}} \ v_d(\mu)^{-\frac{n-2\alpha}{d-(n-2\alpha)}} \tag{4.21}$$

for $c(n,\alpha,d)$ of (4.10)*, then A is gaugeable.*

5 Relasionship to Besov spaces over d-sets

We continue to work under the setting of §4. We first note that, from the trace iniquality (4.15) for the Riesz potential space, we can get the same inequality for the Bessel potential space. The Bessel convolution kernel $G_\alpha(x), \ x \in \mathbb{R}^n$, is a positive integrable function with Fourier transform given by

$$\hat{G}_\alpha(x) = (1 + |x|^2)^{-\frac{\alpha}{2}}. \tag{5.1}$$

The Bessel potential space is defined by

$$\left\{\begin{array}{lcl} L^{\alpha,2}(\mathbb{R}^n) & = & \{G_\alpha * f : f \in L^2(\mathbb{R}^n)\} \\ (u,v)_{L^{\alpha,2}(\mathbb{R}^n)} & = & (f,g)_{L^2(\mathbb{R}^n)},\ u = G_\alpha * f,\ v = G_\alpha * g,\ f,g \in L^2(\mathbb{R}^n), \end{array}\right. \tag{5.2}$$

and hence

$$\left\{\begin{array}{lcl} L^{\alpha,2}(\mathbb{R}^n) & = & \{u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} |\hat{u}(x)|^2 (1+|x|^2)^\alpha dx < \infty\} \\ (u,v)_{L^{\alpha,2}(\mathbb{R}^n)} & = & \int_{\mathbb{R}^n} \hat{u}(x)\bar{\hat{v}}(x)(1+|x|^2)^\alpha dx. \end{array}\right. \tag{5.3}$$

A comparison with (4.1) gives

$$\mathcal{F} = L^{\alpha,2}(\mathbb{R}^n), \qquad \mathcal{E}(u,u) \le (u,u)_{L^{\alpha,2}(\mathbb{R}^n)}, \quad u \in \mathcal{F}. \tag{5.4}$$

Hence, Theorem 4.2 immediately implies the next theorem.

Theorem 5.1. *For $0 < 2\alpha \le n,\ n - 2\alpha < d \le n$, let μ be a d-measure. Then the following inequality holds for a constant S satisfying the bound* (4.14) *in terms of the d-bound of μ:*

$$||u||^2_{L^{\frac{2d}{n-2\alpha}}(\mathbb{R}^n;\mu)} \le S \cdot (u,u)_{L^{\alpha,2}(\mathbb{R}^n)} \quad \forall u \in L^{\alpha,2}(\mathbb{R}^n). \tag{5.5}$$

Let d and α be as in Theorem 5.1 and μ be the restriction of the d-dimensional Hausdorff measure on a d-set F. Define δ by

$$\alpha = \delta + \frac{n-d}{2} \tag{5.6}$$

so that

$$0 < \delta \le 1, \quad 2\delta < d, \quad \frac{2d}{n-2\alpha} = \frac{2d}{d-2\delta}, \quad \frac{d}{d-(n-2\alpha)} = \frac{d}{2\delta}.$$

We consider the Besov space $B^{2,2}_\delta(F)$ over F defined by

$$\left\{\begin{array}{lcl} (\varphi,\psi)_{B^{2,2}_\delta(F)} & = & \int_{F\times F\setminus d} \frac{(\varphi(x)-\varphi(y))(\psi(x)-\psi(y))}{|x-y|^{d+2\delta}}\mu(dx)\mu(dy) \\ B^{2,2}_\delta(F) & = & \{\varphi \in L^2(F;\mu) : (\varphi,\varphi)_{B^{2,2}_\delta(F)} < \infty\}. \end{array}\right. \tag{5.7}$$

$B^{2,2}_\delta(F)$ is a Dirichlet form on $L^2(F;\mu)$ equipped with the norm

$$||\varphi; B^{2,2}_\delta(F)||^2 = (\varphi,\varphi)_{L^2(F;\mu)} + (\varphi,\varphi)_{B^{2,2}_\delta(F)}.$$

Since the Bessel potential space $L^{\alpha,2}(\mathbb{R}^n)$ is known to be identical with the Besov space $B^{2,2}_\alpha(\mathbb{R}^n)$ on $\mathbb{R}^n$, a simple part of the Jonsson-Wallin trace theorem ([JW 84, Chap. V]) reads

$$B^{2,2}_\delta(F) = L_{\alpha,2}(\mathbb{R}^n)|_F \tag{5.8}$$

both the restriction and extension operators involved being continuous. This combined with the imbedding (5.5) readily leads us to the Sobolev inequality for the Besov space $B^{2,2}_\delta(F)$:

$$||u||_{L^{\frac{2d}{d-2\delta}}(F;\mu)} \le C\ ||u; B^{2,2}_\delta(F)||, \qquad u \in B^{2,2}_\delta(F), \tag{5.9}$$

holding for some positive constant C. The inequality (5.9) has been also obtained in [FU 02] without using the imbedding (5.5) but by deriving a bound of the measure μ in terms of the capacity for the space $B^{2,2}_\delta(F)$ from a metric property of the Bessel capacity on $\mathbb{R}^n$.

Denote by $\mathbf{M}^F$ the Hunt process on F associated with the regular Dirichlet form (5.7). (5.9) implies that its transition function p^F_t satisfies a short time ultracontractivity (cf. [D 89])

$$p^F_t(x,y) \le C\, t^{-\frac{d}{2\delta}}, \qquad 0 < t < 1,$$

for some positive constant C. In this sense, the process $\mathbf{M}^F$ behaves similarly to the time changed process $\check{\mathbf{M}}$ on F considered in Theorem 4.2.

The trace $(\check{\mathcal{F}}, \check{\mathcal{E}})$ on $L^2(F;\mu)$ of the present Dirichlet space $(\mathcal{F}, \mathcal{E})$ of the symmetric 2α-stable process $\mathbf{M}$ is transient because so is the latter (see [FOT 94, Th. 6.2.3]). To the contrary, the Besov space $(B^{2,2}_\delta(F), (\cdot,\cdot)_{B^{2,2}_\delta(F)})$ on $L^2(F;\mu)$ is recurrent when $\mu(F)$ is finite.

In view of (5.4), (5.8) and the Dirichlet principle (3.11), we have the following continuous imbedding:

Theorem 5.2.

$$B^{2,2}_\delta(F) \subset \check{\mathcal{F}}_e, \qquad \check{\mathcal{E}}(\varphi,\varphi) \le C\ ||\varphi; B^{2,2}_\delta(F)||^2,\ \forall \varphi \in B^{2,2}_\delta(F),$$

for some positive constant C.

Nevertheless, the preceding observation tells us that 0-order Dirichlet forms $\check{\mathcal{E}}$ and $(\cdot,\cdot)_{B^{2,2}_\delta(F)}$ are not necessarily equivalent.

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