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On the Works of Kiyosi Itô and Stochastic Analysis

Masatoshi Fukushima

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Abstract. A common feature that can be consistently found in the works of Professor Kiyosi Itô is a leap from the analysis in distribution family level toward the analysis and synthesis in sample paths level, which has turned analytic descriptions into thoroughly stochastic ones.

Keywords and phrases: Brownian motion, Lévy–Itô decomposition, Itô integral, Itô formula, stochastic differential equation, Wiener–Itô decomposition, one-dimensional diffusion, excursions, stochastic geometry, stochastic control, stochastic finance

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I feel very honored to have an opportunity to write a brief article on the works of Professor Kiyosi Itô on the happy occasion of his winning the first Gauss prize.

I began my study on probability theory as a graduate student of Professor Itô in 1959. It was just when Professor Itô in collaboration with Professor H.P. McKean succeeded to turn the analytic description of the structure of the most general one-dimensional diffusion discovered earlier by W. Feller into a thoroughly stochastic description. The probability seminar of Kyoto University conducted by Itô gathered many young researchers in Japan who were eager to learn Itô–McKean's work, and together with Itô, looked for its significant extensions to more general stochastic processes. Among them were N. Ikeda, M. Motoo, T. Hida, M. Nisio, H. Tanaka and others. Among graduate students were S. Watanabe and H. Kunita. As one of those who were fortunate enough to experience this exciting seminar atmosphere and still feel its lasting influences up to now,

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M. Fukushima
Emeritus Professor, Osaka University
(e-mail: `fuku@cg-s.bias.ne.jp`)

I shall try to explain some of the works of Professor Itô and their related developments hopefully in some detail.

The above mentioned phrase

turning an analytic description into a thoroughly stochastic one

has a more specific meaning, namely,

a leap from the analysis in distribution family level
to the analysis and synthesis in sample paths level,

which is consistent throughout the works of Professor Itô and of great importance in the history of the study of stochastic processes as well. The latter is much deeper than the former; taking expectations in a specific way of formulae and equations obtained in the latter will readily lead to those obtained in the former, but by taking expectations in different ways (for instance, after replacing constant times involved with random times), we may create new ones that cannot be achieved through the former method alone.

A stochastic process is a mathematical model of the motion of a particle changing randomly with the passage of time. Let me use two mathematical notations in order to make my explanations clearer. The time is denoted by t. The position of a particle at a specific time t is denoted by $X_t(\omega)$. ω is called a *sample* which parameterizes the randomness. By fixing a sample ω and moving the time t, we get the trajectory of the position $X_t(\omega)$ which is called the *sample path* or the *path* of the sample ω. For instance, a pollen grain suspended in water can be observed to perform an extremely irregular continuous curve, which is a sample path of the so-called Brownian motion. A graph plotting a stock price that varies in time may be considered as a sample path of a certain stochastic process.

In order to define a stochastic process in mathematically rigorous way, we need to consider a collection of samples ω and assign a quantity on it called a probability measure indicating the occurrence probability of each part of the collection. In this rigorous sense, N. Wiener ([46], 1923) gave for the first time a definition of the Brownian motion, A.N. Kolmogorov ([33], 1933) gave a systematic method to define more general stochastic processes and J.L. Doob ([4], 1938) gave a modification theorem for stochastic processes making it possible to evaluate expectations of functions of sample paths which may depend on uncountable number of times.

For a fixed t, the position $X_t(\omega)$ may be considered as a random variable taking various values with a certain probability distribution μ_t. Thus, there corresponds a distribution family $\{\mu_t\}$ with time parameter t to each stochastic process. For instance, $\{\mu_t\}$ for the Brownian motion is a collection of Gaussian distributions discovered by Gauss in his theory of errors. Being an object of mathematical analysis, it may be possible to characterize $\{\mu_t\}$ by Fourier analysis or partial differential equations. The investigations of stochastic processes started in this way. Among important classes of stochastic processes are the class called *additive processes* or *Lévy processes* characterized by the independence of disjoint increments and a much more general class called *Markov processes*. A Markov process with continuous sample path is called a *diffusion* (process).

Lévy–Khinchin ([35], 1937) gave a formula characterizing the Fourier transform of $\{\mu_t\}$ for the additive process and Kolmogorov ([32], 1931) characterized the transition probability distribution of a diffusion process as a solution of a parabolic partial differential equation involving diffusion coefficients and drift coefficients, which was then extended by Feller ([8], 1936) to the characterization of a Markov process admitting jumps by an equation involving an additional integral part with a kernel indicating jump rates.

Professor Kiyosi Itô entered the scene right after this. After graduating from Tokyo University, Department of Mathematics, in 1938, he worked in the Statistical Bureau of the Government until he got an academic position at Nagoya University in 1943. During this period, Professor Itô wrote two papers on additive process and Markov process. In both papers, the sample path level analysis and synthesis were accomplished rigorously and thoroughly, from which follow the Lévy–Khinchin formula and the Kolmogorov–Feller equation by taking expectations in specific manners. Reading these two papers once again, we feel as if we are looking at two precious stones being created instantly.

However Itô had a gemstone for the additive process, namely, the French book [35] of P. Lévy published in 1937. In this book, Lévy described a decomposition of the sample path of the additive process with his truly profound insight. But the book was not easy to follow partly because of the ambiguity and lack of rigor in the descriptions. Besides, as Professor Itô later reminded us occasionally, no copy machine was available at that time and he had to copy Lévy's book by hand in the library. Finally, by employing the rigorous framework of Kolmogorov and Doob, Itô was able to draw the Poisson random measure and a Brownian motion skillfully out of the sample path $X_t(\omega)$ of the additive process and then recover the original sample path by means of them ([14]). This is what is called the *Lévy–Itô decomposition theorem* nowadays.

Based on this success, Itô turned to the task of the path level construction of a Markov process using the basic ingredients of the additive process at a burst. Given diffusion coefficients, drift coefficients and a jump rate kernel in the Kolmogorov–Feller equation, he successfully constructed the sample path of a Markov process directly as the unique solution of a stochastic differential equation, being led by an intuitive picture of piecing Brownian paths and Poisson random measure together at each instant in accordance with the given data (*cf.* [45]). This result first appeared in his Japanese paper [15] in a hand-written mimeographed journal published in 1942. The journal was being issued from the Department of Mathematics, Osaka University, to promote research communications of mathematicians all over Japan at that time. In 1951, Itô wrote an English article [17] in the Memoirs of the American Mathematical Society as a refined and extended version of the Japanese paper in 1942.

In this Japanese paper, the notion of the integral of a random function with respect to the Brownian path called a *stochastic integral* or *Itô integral* as well as a transformation rule called *Itô formula* made their first appearance. Itô needed them in order to make sense of the stochastic differential equation and make possible systematic computations of its solution and related objects. A Brownian path is so irregular that each portion of it is of infinite length, which prevented him from defining an integral along a path in a traditional way initiated by Newton and Leibniz. But the path has a nice property that the sum of square of increments has a finite non-random limit locally. Owing to this property, the Itô integral and the Itô formula were well formulated. The Itô formula corresponds to the chain rule of the differentiation of a composite function in the Newton–Leibniz calculus but it has an additional term of the second order due to the mentioned property of the path, which in turns led him to the Kolmogorov second order differential equation by taking expectation in a specific way. This was the beginning of Itô calculus (*cf.* [38]).

I heard from the late Professor Gisiro Maruyama that, while he was in a military camp being drafted for the war, he repeatedly read Itô's Japanese paper under the light of a gatekeeper's box. In 1944, the late Professor Shizuo Kakutani wrote a pioneering paper [31] relating potential theory to Brownian path. Only Maruyama and Kakutani really understood the significance of Itô's work in Japan at that time. Japanese mathematics was isolated in the world during the war time and some postwar period. But, in those days, not many people all over the world except for those named above were able to appreciate Itô's works even under the assumption that they could have read and understood the Japanese sentences in Itô's paper in 1942. The stochastic process theory had hardly gained a comfortable citizenship as a branch of pure mathematics in the mathematical society not only in Japan but also in Western Europe and United States. Only a handful people in the world pioneered in the study of stochastic processes.

During this period, Professor Itô kept producing some other seminal works too. Among them was a paper [18] on the so-called *Wiener–Itô chaos decomposition theorem* of functionals of normal random measures, which particularly implies an important fact that any function of the Brownian path with finite variance admits a representation as an Itô integral.

The situation was changing radically at the time when Professor Itô returned from the Institute for Advanced Study at Princeton University to Kyoto University in 1956 and continued a collaboration with McKean in the study of the one-dimensional diffusions (*cf.* [19], [27]). Doob had given in his book [5] published in 1953 a profound analysis of sample paths of a martingale — a stochastic process modelling a fair game in a most abstract way. E.B. Dynkin and his school in Russia were developing a transformation theory of a general Markov process by means of functions of its sample path called additive functionals ([6]). The potential theoretic structure of a general Markov process was being revealed by G.A. Hunt [12]. All of those works had strong impacts on each other yielding later developments which took place sometimes with surprise. I shall explain some of them.

Feller had discovered in [10] that the local structure of the general one-dimensional diffusion can be characterized analytically by two quantities called a *canonical scale function* s and a *canonical measure* m (*cf.* [19], [6]). Itô showed in the book [29] of 1965 with McKean that the sample path $X_t(\omega)$ of a general diffusion can be obtained from the Brownian path firstly by changing the speed of time of the Brownian path by means of an additive functional produced by integrating Lévy's local time in [36] with the measure $m \circ s^{-1}$ and secondly by changing the position of the speed changed path by s^{-1}, thus establishing analysis and synthesis of the sample path $X_t(\omega)$ in terms of two elements representing figure and speed.

Using the framework of the Hille–Yosida theory on a semigroup of Markovian operators acting on the space of continuous functions (*cf.* [19], [6]), Feller [9] had also discovered the boundary classification and the most general boundary condition for the possible Markovian extension beyond the boundary of a given absorbed diffusion. In 1963, Itô–McKean [28] gave a stochastic construction of the extension subjected to Feller's general boundary condition starting with the reflected diffusion and using its local time at the boundary. Lévy [36] had studied Brownian excursions consisting of those parts of the Brownian path leaving from the boundary and returning to it (*cf.* [29]). In his unpublished lecture note [22] in 1969, Itô constructed the most general extension of an absorbed diffusion that jumps in from an exit boundary by making use of a Poisson point process taking values of (jump in) excursions.

Partly motivated by these developments, Itô investigated in his paper [23] in 1970 a Poisson point process taking values of excursions around a recurrent point attached to a general Markov process. This notion can be regarded as an infinite dimensional version of the Poisson random measure that appeared in his first paper in 1942, and as he himself recalled later, a manifestation of his habit of observing even finite dimensional objects from the infinite dimensional view point. I recall that, when I was staying in University of Illinois in 1970 and Professor Itô gave a talk on it in a symposium held there, Professor Doob admired him by saying that *you always look at things that way*.

In the one-dimensional diffusion case, its local generator given by a usual second order differential operator, as appears in the right hand side of the Kolmogorov differential equation, is only a special case of the Feller canonical form and Professor Itô might have conceived that the former is less intrinsic. In the Kyoto seminar in the 50's and the 60's, he rarely talked about the stochastic differential equation formulated in relation to the former and about the associated Itô calculus. In the meantime, Motoo–Watanabe wrote a paper [42] in 1965 where they gave, for a general Markov process, a profound analysis of the structure of the class of additive functionals with mean zero and finite variance accompanied by a special case of a new notion of stochastic integrals. At the same time, the Doob–Meyer decomposition theorem of submartingales was completed by P.A. Meyer.

These two works merged and developed into the Kunita–Watanabe paper [34] in 1967 from the Nagoya Mathematical Journal and a series of papers of Meyer [41] in the Strasbourg Seminar Notes published in the same year. In Kunita–Watanabe–Meyer theory, the notion of stochastic integral was formulated for a general semi-martingale and its transformation rule was derived under the name of the *Itô formula*. The Itô integral and the Itô calculus were revived in the magnificent framework of semimartingales.

Since then, many researchers including Professor Itô himself became much more concerned about Itô calculus and stochastic differential equations. The fundamental theorem due to Itô on the existence and uniqueness of the solution of the stochastic differential equation was extended and deepened from the 70's, accompanied by new formulations of the solutions and the uniqueness, by a research group including S. Watanabe (*cf.* [30], [13]).

By regarding stochastic integrals as operations acting on the space of stochastic differentials based on a semi-martingale, Itô himself systematically related his integral to Stratonovich's and others in [24] and [25] accompanying their associated calculus with applications to stochastic geometry. Stratonovich's version of the stochastic integral had been used in applications in the 60's and became important also in the later rapid progress of the stochastic analysis on differentiable manifolds, because its associated transformation rule was simply analogous to the classical chain rule (*cf.* [30], [13]).

In parallel to these developments, the Hunt theory was combined with an axiomatic potential theory [2] to create an active field of symmetric Markov processes ([11]), which can be regarded as a natural extension of the one-dimensional diffusion theory where the canonical measure m plays the role of symmetrizing measure (*cf.* [27], [10]). Further the paper [23] of Professor Itô in 1970 keeps motivating a lot of works on the excursion theory and its applications until now ([37], [44], [13], [43]).

So much for the birth and growth of Itô's theory and some of the related advances in pure mathematics, but it should be noticed that Itô's stochastic differential equations have been used directly in the stochastic control theory and filtering theory arose in the engineering sciences, which have in turn motivated and stimulated the mathematical study of the stochastic differential equations and related calculus since the 70's.

Professor Itô has a very strong interest in the applications of mathematics to physics from his student days, and indeed he has written several articles on turbulence and the Feynman integral ([16], [19, §3.10], [20], [21]). He has also enjoyed conversations with us expressing his deep understanding on the relation between probability theory and those matters in our daily life such as statistics of population growth. After the 70's, it became clearer for many people including Itô that the Itô theory itself is finding ever increasing ways to be applied to engineering, physics and biology. However it was entirely out of imagination of Professor Itô that his theory would be used to determine the option prices in finance. He was somehow puzzled by that.

It is true that the probability theory originated in the considerations of gambling. In the beginning of the last century, L. Bachelier [1] discovered the distribution family of the Brownian motion by linking its sample path to the stock price while he was working at the Paris Stock Exchange (*cf.* [29]). This was a little earlier than the publication of the famous Einstein paper [7] on a statistical mechanical consideration of Brownian motion. We can therefore naturally expect that the Itô theory as a basis of the sample path analysis of Brownian motion would be applied to some problems in finance eventually. Nevertheless, it is absolutely amazing that, by discovering a structural correspondence between the derivative in finance and the Itô theory, F. Black, M. Scholes and R. Merton succeeded in deriving a very useful formula for the option price in 1973 ([3], [39], [40]). It took more than 30 years between the birth of the Itô theory and this breakthrough in finance.

Rather than looking at the course of developments in pure mathematics and those in the associated applications as I have done above, the path can perhaps be better described in the following way. Itô's mathematical theory has been steadily penetrating into diverse human scientific activities which resound together and spread like a concerto being performed in a very long period of time. Certainly the motive of Itô's mathematical theory will continue to invite different tones by some new players in the 21st century.

Because my own research on stochastic analysis is in pure mathematics, the fact that my work has been chosen for the Gauss Prize for applications of mathematics is truely unexpected and deeply gratifying. I hope therefore to share this great honor and joy with my family, teachers, colleagues, and students in mathematics as well as with all those who took my work on stochastic analysis and extended it to areas far beyond my imagination.

Concurring with those words of Professor Kiyosi Itô, we would like to join in sharing this great joy with him.

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