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On Feller's boundary problem for Markov processes in weak duality

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Abstract

We give an affirmative answer to Feller's boundary problem going back to 1957 by obtaining a resolvent characterization for the duality preserving extensions of a pair of standard Markov processes in weak duality (minimal processes) to the boundary consisting of countably many points. Our resolvent characterization involves the resolvents for the minimal processes, the Feller measures that are intrinsic to the minimal processes as well as the restrictions to the boundary of the jumping and killing measures of the extension processes. Conversely, given killing rates on the boundary, we construct the corresponding duality preserving extensions of the minimal processes that admit no jumps between the boundary points and have the prescribed killing rate at the boundary, by repeatedly doing one-point extension one at a time using Itô's Poisson point processes of excursions.

Keywords: Standard process; Weak duality; Boundary theory; Extension process; Resolvent; Feller measures; Jumping measure; Killing measure; Time change; Darning

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1. Introduction

Let $\Lambda \subset \{1, 2, \ldots\}$ be a countable set, which can be finite or countably infinite. Consider a locally compact separable metric space E and a σ-finite measure m having full support on E. Let K be a closed subset of E expressible either as $K = \bigcup_{i\in\Lambda} K_i$ a countable union of locally finite disjoint compact subsets K_i, or as $K = K_i \cup \cdots \cup K_N$ where $\{K_i\}_{1\leqslant i\leqslant N}$ are disjoint, $K_1, \ldots, K_{N-1}$ are compact and $E \setminus K_N$ is relatively compact. Denoting $E \setminus K$ by E_0, we consider the topological space $E^* = E_0 \cup F$, where $F = \bigcup_i \{a_i\}$, obtained from E by regarding each set K_i as one point a_i. The restriction m_0 of the measure m to E_0 is extended to a measure on E^* by setting $m_0(F) = 0$.

Given a pair of standard processes X and $\widehat{X}$ on E that are in weak duality with respect to m, let X^0 and $\widehat{X}^0$ be their subprocesses on E_0 killed upon leaving E_0, respectively, which are known to be in weak duality with respect to m_0. We assume that X and $\widehat{X}$ are approachable to each set K_i but of no jumps from E_0 to K. Under some fairly general conditions on X, X^0, $\widehat{X}$ and $\widehat{X}^0$ formulated in Theorem 3.1 below, we shall successively apply the *darning* procedure established in our previous paper [4, Section 3] to each *hole* K_i to construct in Section 3 a pair of standard processes X^* and $\widehat{X}^*$ on E^*, which extend X^0 and $\widehat{X}^0$ on E_0, respectively, and which are in weak duality with respect to m_0. X^* and $\widehat{X}^*$ may admit killings on F but they have no jumps from F to F. More specifically, by identifying each K_i with a point a_i, we show in Theorem 3.1 that for every sequences of non-negative numbers $\{\kappa_i, \widehat{\kappa}_i,\ i \in \Lambda\}$ satisfying Eq. (1.1) below, there exist a duality preservation extensions $(X^*, \widehat{X}^*)$ of $(X^0, \widehat{X}^0)$. Here κ_i and $\widehat{\kappa}_i$ represent the killing rates of the extension processes X^* and $\widehat{X}^*$ at a_i. We point out here that the main result of Section 3, Theorem 3.1, is far from a straightforward application of results in our previous paper [4] on one-point extension. This is because in non-symmetric weak duality case, it is necessary for duality preserving one-point extension of $(X^0, \widehat{X}^0)$ to $E_0 \cup \{a_1\}$ to have suitable killings at a_1 with killing rates κ_1 and $\widehat{\kappa}_1$ related via the intrinsic quantities of the base processes $(X^0, \widehat{X}^0)$; i.e. the Feller measures of $(X^0$ and $\widehat{X}^0$ (see [4]). When applying one-point darning successively at a_i, at each stage the underlying base processes are changing and their associated intrinsic quantities (i.e. Feller measures) are different, while the condition (1.1) on killing rates $\{\kappa_i, \widehat{\kappa}_i;\ i \in \Lambda\}$ are prescribed in terms of the Feller measures of initial base processes $(X^0, \widehat{X}^0)$ only. A new technique is developed in this paper to overcome this difficulty.

A natural question arises in this connection:

How can we characterize the constructed processes X^* and $\widehat{X}^*$ in terms of X^0 and $\widehat{X}^0$?

Section 2 of this paper will be devoted to answering this question in a more general setting: For E and m as above and for a countable quasi-closed subset $F = \{a_i,\ i \in \Lambda\}$ of E such that $m(F) = 0$ and each point in F is separated from the other ones by a q.e. finely open set, we let $E_0 := E \setminus F$ and $m_0 := m|_{E_0}$. We assume that we are given a pair of standard processes X and $\widehat{X}$ on E which are in weak duality with respect to m and approachable to each point a_i but of no jumps from E_0 to F. Let X^0, $\widehat{X}^0$ be the subprocesses of X and $\widehat{X}$ on E_0, respectively, killed upon leaving E_0.

We view X and $\widehat{X}$ as most general duality preserving extensions of X^0 and $\widehat{X}^0$, respectively, from E_0 to $E = E_0 \cup F$. The processes X^0 and $\widehat{X}^0$ may be called the minimal processes in this sense. The objective of Section 2 is to characterize those extensions at the resolvent level (Theorem 2.6 and Proposition 2.7) using the quantities intrinsic to the minimal processes.

For the *quantities intrinsic* to the minimal processes, we mean the *Feller measures*

$$U^{ij}, \quad V^{(i)}, \quad U^{ij}_\alpha, \quad i, j \in \Lambda,$$

defined by (2.6), (2.7) and (2.8) below for the minimal processes. In order to characterize the Lévy system of the time changed process of X on F, these Feller measures of X^0 on F were first introduced in Fukushima, He, Ying [14] when X is a symmetric conservative diffusion and F is a closed (not necessarily countable) subset of E, and then in Chen, Fukushima, Ying [5] (where X is a general symmetric Markov process) and in Chen, Fukushima, Ying [6] (where X is a standard process with a weak dual $\widehat{X}$) for a general q.e. finely closed subset F of E.

These notions originated in W. Feller [11] where E_0 was countable, F was finite and U_α, U were defined by (2.8) and (2.9) below. Feller raised the question of finding the resolvent representation and giving the lateral condition on the generator for the general extension X of a minimal process X^0 in terms of $U_\alpha,\ U,\ V$. This was the original and prototype of the so called *boundary problem of Markov process* proposed soon after his discovery of the most general boundary conditions for the one dimensional diffusions.

In Section 2, we shall give an affirmative answer to Feller's question of finding resolvent characterization. The explicit representation of the resolvent given in Theorem 2.6 particularly implies that the extension X (resp. $\widehat{X}$) of X^0 (resp. $\widehat{X}^0$) is uniquely determined by the quantities J_{ij} (resp. $\widehat{J}_{ij} = J_{ji}$) and κ_i (resp. $\widehat{\kappa_i}$) which are the restrictions to F of the jumping measure and the killing measure of X (resp. $\widehat{X}$), respectively. We shall further see in Proposition 2.7 that, as a result of the duality of X and $\widehat{X}$, these characteristic quantities must satisfy the following equations:

$$\sum_{k\geqslant 1,\, k\neq i} \left(U^{ik} + J_{ik}\right) + V_i + \kappa_i = \sum_{k\geqslant 1,\, k\neq i} \left(U^{ki} + J_{ki}\right) + \widehat{V}_i + \widehat{\kappa}_i \quad \text{for every } i \in \Lambda. \tag{1.1}$$

There have been a lot of works on the resolvent representations of the Markovian extensions of X^0 to the 'boundary' F consisting of a finite number of points. See the paper by Rogers [20] and the references therein. To our knowledge, none of them incorporated the Feller measures suggested by [11] seriously into the descriptions, except for the work of Neveu [19] where a useful integral representation of the α-order Feller measure (2.8) was given (see Lemma 2.4 below and [4, Lemma 4.9]). However the notion of the Feller measures have survived in the study of the boundary theory of Dirichlet spaces by Fukushima [13], Kunita [17], Silverstein [21] and LeJan [18]. Indeed the main theorem of Section 2 (Theorem 2.6) will be proved not only by using the results established in [6] but also by incorporating and further developing the analysis initiated in [13] involving the α-order Feller measure.

The stated construction in Section 3 of a duality preserving extensions X^* and $\widehat{X}^*$ of X^0 and $\widehat{X}^0$ to $E^* = E_0 \cup F$ by darning finite or countably many holes $\{K_i\}$ can be regarded as a specific converse to Theorem 2.6 and Proposition 2.7. Indeed, the construction will be carried out in Theorem 3.1 for each choice of non-negative numbers $\{\kappa_i, \widehat{\kappa}_i,\ i \geqslant 1\}$, satisfying Eq. (1.1) with vanishing J_{ij}, $i, j \geqslant 1$, so that the restrictions to F of the killing measures of the constructed processes X^*, $\widehat{X}^*$ are $\{\kappa_i,\ i \in \Lambda\}$ and $\{\widehat{\kappa}_i,\ i \geqslant 1\}$, respectively, while both X^* and $\widehat{X}^*$ admit no jumps from F to F.

In Section 4, we shall present examples of the multidimensional censored stable processes and the multidimensional non-symmetric diffusions to illustrate the applicability of Theorem 3.1.

The characterization and the construction formulated in the present paper for Markov processes in weak duality of course apply to symmetric Markov processes. But when X is a symmetric right process on E and X^0 is its subprocess killed upon leaving $E_0 = E \setminus F$ for some quasi-closed subset F, it is also possible to give the lateral condition on the L^2-generator of X in terms of the intrinsic quantities of X^0. This is carried out in a subsequent paper [3] via reflected Dirichlet form of X^0 and a notion of flux functional that is introduced in [3].

2. Resolvent representation via Feller measures

In this section, we work under the setting of Chen, Fukushima, Ying [6] but for the special case that the 'boundary' set F is countable. To be more precise, let E be a locally compact separable metric space and m be a σ-finite Borel measure on E. We consider a pair of Borel standard processes $X = \{X_t, \zeta, \mathbf{P}_x,\ x \in E\}$ and $\widehat{X} = \{\widehat{X}_t, \widehat{\zeta}, \widehat{\mathbf{P}}_x,\ x \in E\}$ on E, which are in weak duality with respect to m in the sense that

$$\int_E G_\alpha f(x) g(x)\, m(dx) = \int_E f(x) \widehat{G}_\alpha g(x)\, m(dx) \quad \text{for } f, g \in \mathcal{B}^+(E),$$

where G_α and $\widehat{G}_\alpha$ are the α-resolvent of X and $\widehat{X}$, respectively. We assume that

(X.1) Every semi-polar set is m-polar for X.

Let $F = \{a_i, i \in \Lambda\}$ be a countable quasi-closed subset of E indexed by $\Lambda \subset \{1, 2, \ldots\}$ satisfying the next three conditions. Define $E_0 := E \setminus F$.

(X.2) $m(F) = 0$ and for each $i \in \Lambda$, there is a q.e. finely open subset $U \subset E$ such that $a_i \in U$ and $F \setminus \{a_i\} \subset E \setminus U$.

(X.3) $\mathbf{P}_m(\sigma_F < \infty, X_{\sigma_F} = a_i) > 0$ and $\widehat{\mathbf{P}}_m(\sigma_F < \infty, \widehat{X}_{\sigma_F} = a_i) > 0$ for every $i \in \Lambda$. Here for a subset $B \subset E$, $\sigma_B = \inf\{t > 0\colon\ X_t \in B\}$ denotes its first hitting time by X, with the convention of $\inf \emptyset = \infty$. The first hitting time of B by $\widehat{X}$ will be denoted by the same notation.

(X.4) X and $\widehat{X}$ admit no jumps from E_0 to a_i for every $i \geqslant 1$.

The first assumption in (X.2) is equivalent to the condition that X has no sojourn on F in the sense that

$$\mathbf{P}_x\left(\int_0^\infty \mathbf{1}_F(X_s)\, ds = 0\right) = 1 \quad \text{for q.e. } x \in E.$$

The second assumption on (X.2) implies the same property for $\widehat{X}$ under (X.1) in view of [6, Section 2].

Assumptions (X.1) and (X.3) imply that

$$a_i \text{ is not } m\text{-polar and } a_i \text{ is regular for itself for each } i \in \Lambda. \tag{2.1}$$

Indeed the first property is immediate from (X.3). The second follows from (X.1) and the general fact for X that the set of irregular points of a Borel set is semi-polar.

Remark 2.1. Condition (X.1) is imposed in [6] to ensure the existence of Lévy system for X and its time changed process on F in the original topology of E and F, respectively. This implies the existence of a *predictable* exit system for X on F. When F is a singleton or when $F = \{a_i, i \in \Lambda\}$ is a locally finite countable quasi-closed subset of E as supposed in this paper, condition (X.1) may be dropped by replacing it with the assumption that every a_i is a regular point for X and it is accessible from some other points in E_0. This is because under these assumption, each point $\{a_i\}$ is not semi-polar and hence there is continuous local time of X at each point a_i. This yields that there is a *predictable* exit system for X on F. See Fitzsimmons and Getoor [12] for more details on this.

Assumption (X.4) implies (see the proof of Proposition 4.1(iii) of [7]) that

$$X \text{ admits no jump from } a_i \text{ to } E_0 \text{ for every } i \in \Lambda. \tag{2.2}$$

Let $X^0 = \{X_t^0, \zeta^0, \mathbf{P}_x^0\}$ and $\widehat{X}^0 = \{\widehat{X}_t^0, \widehat{\zeta}^0, \widehat{\mathbf{P}}_x^0\}$ be the part processes of X and $\widehat{X}$, respectively, killed upon leaving E_0. The restriction of m to E_0 will be denoted by m_0. It is well known that X^0 and $\widehat{X}^0$ are in weak duality with respect to the measure m_0:

$$\int_{E_0} G_\alpha^0 f(x) g(x)\, m_0(dx) = \int_{E_0} f(x) \widehat{G}_\alpha^0 g(x)\, m_0(dx) \quad \text{for } f, g \in \mathcal{B}^+(E_0),$$

where G_α^0 and $\widehat{G}_\alpha^0$ are the α-resolvent of X and $\widehat{X}$, respectively.

We view X and $\widehat{X}$ as most general duality preserving extensions of X^0 and $\widehat{X}^0$, respectively, from E_0 to $E = E_0 \cup F$. The objectives of the present section is to characterize those extensions at the resolvent level using the quantities intrinsic to X^0 and $\widehat{X}^0$.

For functions u, v on E_0, we let $(u, v) := \int_{E_0} u(x) v(x)\, m_0(dx)$. The X^0-energy functional of an X^0-excessive measure η and an X^0-excessive function u is defined by

$$L^{(0)}(\eta, u) := \lim_{t \downarrow 0} \frac{1}{t} \langle \eta - \eta P_t^0, u \rangle, \tag{2.3}$$

where $\{P_t^0,\ t \geqslant 0\}$ is the transition semigroup of X^0. The dual notion $\widehat{L}^{(0)}$ is defined analogously.

Define for $x \in E_0$ and $i \in \Lambda$,

$$\varphi^{(i)}(x) := \mathbf{P}_x(\sigma_F < \infty, X_{\sigma_F} = a_i) \quad \text{and} \quad u_\alpha^{(i)}(x) := \mathbf{E}_x\big[e^{-\alpha\sigma_F}; X_{\sigma_F} = a_i\big]. \tag{2.4}$$

The functions $\widehat{\varphi}^{(i)}$, $\widehat{u}_\alpha^{(i)}$ are analogously defined for $\widehat{X}^0$. By (X.4) and [1, p. 59],

$$\varphi^{(i)}(x) = \mathbf{P}_x^0\big(\zeta^0 < \infty \text{ and } X_{\zeta^0-}^0 = a_i\big) \quad \text{and} \quad u_\alpha^{(i)}(x) = \mathbf{E}_x^0\big[e^{-\alpha\zeta^0}; X_{\zeta^0-}^0 = a_i\big]. \tag{2.5}$$

Analogous relations hold for $\widehat{X}$.

We can then introduce the Feller measure U^{ij} of X^0 on F by

$$U^{ij} := L^{(0)}\big(\widehat{\varphi}^{(i)} \cdot m_0, \varphi^{(j)}\big) \quad \text{for } i, j \geqslant 1 \text{ with } i \neq j, \tag{2.6}$$

and the supplementary Feller measure $V^{(i)}$ of X^0 on F by

$$V^{(i)} := L^{(0)}\big(\widehat{\varphi}^{(i)} \cdot m_0,\ 1-\varphi^{(i)}\big) \quad \text{for } i \in \Lambda. \tag{2.7}$$

For $\alpha > 0$, we also consider the α-order Feller measure U_α^{ij} defined by

$$U_\alpha^{ij} := \alpha\big(\widehat{u}_\alpha^{(i)}, \varphi^{(j)}\big) \quad \text{for } i, j \geqslant 1. \tag{2.8}$$

It holds then that

$$U^{ij} = \lim_{\alpha\to\infty} U_\alpha^{ij} \quad \text{for every } i \neq j. \tag{2.9}$$

The Feller measures $\widehat{U}$, $\widehat{V}$, $\widehat{U}_\alpha$, of $\widehat{X}^0$ on F are defined similarly.

We make some preliminary remarks. We put

$$\mathbf{H}_\alpha f(x) := \mathbf{E}_x\big[e^{-\alpha\sigma_F} f(X_{\sigma_F})\big] \quad \text{for } x \in E,$$

and regard $\mathbf{H}_\alpha$ as a linear operator mapping functions on F to functions on E. By the strong Markov property of X, for every function $v \in \mathcal{B}_b(E)$,

$$G_\alpha v(x) = G_\alpha^0 v(x) + \mathbf{H}_\alpha(G_\alpha v)(x) \quad \text{for } x \in E. \tag{2.10}$$

By choosing a strictly positive, bounded, m-integrable function v, we then have, for each $i \in \Lambda$,

$$\int_{E_0} u_\alpha^{(i)}(x)\, m_0(dx) \cdot G_\alpha v(a_i) \leqslant \int_E G_\alpha v(x)\, m(dx) \leqslant \int_E v(x)\, m(dx) < \infty.$$

As $G_\alpha v(a_i) > 0$, we get, along with an analogous consideration for $\widehat{X}$,

$$\big\{u_\alpha^{(i)}, \widehat{u}_\alpha^{(i)},\ i \in \Lambda\big\} \subset L^1(E_0, m_0), \quad i \in \Lambda. \tag{2.11}$$

We let $\widehat{\mathcal{H}}_\alpha$ to denote the operator that maps a function v on E_0 to a function $\widehat{\mathcal{H}}_\alpha v$ on F defined by

$$\widehat{\mathcal{H}}_\alpha v(a_i) := \int_{E_0} \widehat{u}_\alpha^{(i)}(x) v(x)\, m_0(dx), \quad i \in \Lambda.$$

In view of (2.11), $\widehat{\mathcal{H}}_\alpha v(a_i)$ is finite for $v \in \mathcal{B}_b(E_0)$ and $i \in \Lambda$. But it may not be bounded on F unless F is finite. For later use, we introduce the function space

$$\mathcal{B}_0(E_0) = \big\{v \in \mathcal{B}_b(E_0)\colon\ m_0\big(x \in E_0\colon\ v(x) \neq 0\big) < \infty\big\}.$$

$\widehat{\mathcal{H}}_\alpha$ can then be considered as a linear operator sending $\mathcal{B}_0(E_0)$ to $\mathcal{B}_b(F)$.

Now let K be the PCAF of X with Revuz measure

$$\mu_K(d\xi) = \sum_{i\in\Lambda} \delta_{a_i}(d\xi).$$

By (2.2) and the last formula in the proof of Theorem 3.3 of [6], we have

$$\int_F \mathbf{Q}^*_\alpha v(\xi) f(\xi)\mu_K(d\xi) = \int_{E_0} \widehat{\mathbf{E}}_y\big[e^{-\alpha\sigma_F} f(X_{\sigma_F})\big] v(y)\, m_0(dy),$$

where $\mathbf{Q}^*_\alpha v(\xi)$ is defined as $\int_0^\infty e^{-\alpha t}\mathbf{E}^*_\xi[v(X_t), t < R]\,dt$ in terms of the exit system $(\mathbf{P}^*_x, K)$ of X for the set F. This identity yields that

$$\mathbf{Q}^*_\alpha v(a_i) = \widehat{\mathcal{H}}_\alpha v(a_i), \quad \text{for } i \in \Lambda.$$

On the other hand, by [6, (3.13)] and (2.2),

$$G_\alpha v(a_i) = \mathbf{E}_{a_i}\left[\int_0^\infty e^{-\alpha s}\mathbf{Q}^*_\alpha v(X_s)\,dK_s\right] = \mathbf{E}_{a_i}\left[\int_0^{K_\infty} e^{-\alpha\tau_t}\mathbf{Q}^*_\alpha v(X_{\tau_t})\,dt\right],$$

where τ_t denotes the right-continuous inverse of K.

Define

$$\check{R}^\alpha g(a_i) =: \mathbf{E}_{a_i}\left[\int_0^{K_\infty} e^{-\alpha\tau_t} g(X_{\tau_t})\,dt\right], \quad g \in \mathcal{B}^+(F).$$

Note that $\check{R}^\alpha$ is the 0-order resolvent for the α-subprocess of X time-changed by τ_t. This observation together with (2.10) leads us to

Lemma 2.2 *(Feller decomposition). For $\alpha > 0$ and $v \in \mathcal{B}_b(E)$,*

$$G_\alpha v = G^0_\alpha v + \mathbf{H}_\alpha \check{R}^\alpha \widehat{\mathcal{H}}_\alpha v. \tag{2.12}$$

We define

$$\check{R}^\alpha(a_i, a_j) =: \check{R}^\alpha \mathbf{1}_{a_j}(a_i), \quad i, j \geqslant 1.$$

Then $\check{R}^\alpha g(a_i) = \sum_{j\geqslant 1}\check{R}^\alpha(a_i, a_j) g(a_j)$, $i \in \Lambda$. By taking a strictly positive bounded function v in (2.12) and noting (X.3), we see that $\check{R}^\alpha(a_i, a_j)$ is non-negative and finite for each $i, j \geqslant 1$.

Recall the α-order Feller measure U_α defined by (2.8).

Lemma 2.3.

(i) *For each $i, j \geqslant 1$, U^{ij}_α is finite and*

$$U^{ij}_\alpha = \alpha\big(\widehat{\varphi}^{(i)}, u^{(j)}_\alpha\big), \qquad U^{ij}_\alpha - U^{ij}_\beta = (\alpha-\beta)\big(\widehat{u}^{(i)}_\alpha, u^{(j)}_\beta\big), \quad \alpha, \beta > 0. \tag{2.13}$$

(ii) *For $\alpha, \beta > 0$,*

$$\big[\check{R}^\alpha - \check{R}^\beta + \check{R}^\beta(U_\alpha - U_\beta)\check{R}^\alpha\big]\widehat{\mathcal{H}}_\alpha v = 0, \quad v \in \mathcal{B}_b(E_0). \tag{2.14}$$

Proof. (i) The finiteness of U_α^{ij} follows from (2.11). By the Markov property of X and $\widehat{X}$, for every $\alpha, \beta \geqslant 0$, $\alpha \neq \beta$ and $i \in \Lambda$,

$$u_\alpha^{(i)} - u_\beta^{(i)} + (\alpha - \beta) G_\beta^0 u_\alpha^{(i)} = 0 \quad \text{and} \quad \widehat{u}_\alpha^{(i)} - \widehat{u}_\beta^{(i)} + (\alpha - \beta) \widehat{G}_\beta^0 \widehat{u}_\alpha^{(i)} = 0. \tag{2.15}$$

Here we make the convention that $u_0^{(i)} = \varphi^{(i)}$, $\widehat{u}_0^{(i)} = \widehat{\varphi}^{(i)}$. From this, the first identity of (2.13) follows easily. The left-hand side of the second identity of (2.13) equals

$$\begin{aligned}
&(\alpha - \beta)\big(\widehat{u}_\alpha^{(i)}, \varphi^{(j)}\big) + \beta\big(\widehat{u}_\alpha^{(i)} - \widehat{u}_\beta^{(i)}, \varphi^{(j)}\big) \\
&\quad = (\alpha - \beta)\big(\widehat{u}_\alpha^{(i)}, \varphi^{(j)}\big) - \beta(\alpha - \beta)\big(\widehat{G}_\beta^0 \widehat{u}_\alpha^{(i)}, \varphi^{(j)}\big) \\
&\quad = (\alpha - \beta)\big(\widehat{u}_\alpha^{(i)}, \varphi^{(j)} - \beta G_\beta^0 \varphi^{(j)}\big) \\
&\quad = (\alpha - \beta)\big(\widehat{u}_\alpha^{(i)}, u_\beta^{(j)}\big).
\end{aligned}$$

(ii) By restricting the resolvent equation

$$G_\alpha v - G_\beta v + (\alpha - \beta) G_\beta G_\alpha v = 0$$

to the set F and applying (2.12) and (2.15), we have

$$\begin{aligned}
0 &= \check{R}^\alpha \widehat{\mathcal{H}}_\alpha v - \check{R}^\beta \widehat{\mathcal{H}}_\beta v + (\alpha - \beta) \check{R}^\beta \widehat{\mathcal{H}}_\beta \big(G_\alpha^0 v + \mathbf{H}_\alpha \check{R}^\alpha \widehat{\mathcal{H}}_\alpha v\big) \\
&= \check{R}^\alpha \widehat{\mathcal{H}}_\alpha v - \check{R}^\beta \widehat{\mathcal{H}}_\alpha v + (\alpha - \beta) \check{R}^\beta \widehat{\mathcal{H}}_\beta \mathbf{H}_\alpha \check{R}^\alpha \widehat{\mathcal{H}}_\alpha v,
\end{aligned}$$

which combined with (2.13) leads us to (2.14). □

We next define for $\lambda > 0$ and $\alpha \geqslant 0$,

$$\check{R}_\lambda^\alpha g(a_i) := \mathbf{E}_{a_i}\left[\int_0^{K_\infty} e^{-\lambda t - \alpha \tau_t} g(X_{\tau_t})\, dt\right].$$

We claim that

$$\big[\check{R}^\alpha - \check{R}_\lambda^\beta + \check{R}_\lambda^\beta (U_\alpha - U_\beta) \check{R}^\alpha - \lambda \check{R}_\lambda^\beta \check{R}^\alpha\big] \widehat{\mathcal{H}}_\alpha v = 0 \quad \text{for } v \in \mathcal{B}_b(E_0). \tag{2.16}$$

To obtain this, observe first that

$$\check{R}^\beta g = \check{R}_\lambda^\beta g + \lambda \check{R}_\lambda^\beta \check{R}^\beta g \quad \text{for } \alpha, \lambda \geqslant 0.$$

It then follows from (2.14) that

$$0 = \big[\check{R}^\alpha - \check{R}_\lambda^\beta + \check{R}_\lambda^\beta (U_\alpha - U_\beta) \check{R}^\alpha\big] \widehat{\mathcal{H}}_\alpha v - \lambda \check{R}_\lambda^\beta \big[\check{R}^\beta - \check{R}^\beta (U_\alpha - U_\beta) \check{R}^\alpha\big] \widehat{\mathcal{H}}_\alpha v.$$

The last term above is equal to $\lambda \check{R}_\lambda^\beta \check{R}^\alpha \widehat{\mathcal{H}}_\alpha v$ by (2.14) again, which establishes (2.16).

For $\lambda > 0$, denote $\check{R}^0_\lambda$ by $\check{R}_\lambda$, which is the λ-resolvent of the time-changed process $Y = (Y_t, \check{\mathbf{P}}_{a_i})$ of X on F. On account of the condition (X.2), Y is a right continuous Markov process on the denumerable state space F such that each point $a_i \in F$ is stable in the sense that, denoting by τ the exit time of Y from a_i, $\check{\mathbf{P}}_{a_i}(\tau > 0) = 1$, $i \in \Lambda$.

The resolvent $\check{R}_\lambda$ can be regarded as a bounded linear operator on $\mathcal{B}_b(F)$, which is easily seen to be injective. So the generator $\check{\mathcal{A}}$ of Y is well-defined by

$$\mathcal{D}(\check{\mathcal{A}}) := \check{R}_\lambda\big(\mathcal{B}_b(F)\big) \quad \text{and} \quad \check{\mathcal{A}}u := \lambda u - f \quad \text{for } u = \check{R}_\lambda f \text{ with } f \in \mathcal{B}_b(F),$$

which is independent of the choice of $\lambda > 0$ (cf. [16]).

Lemma 2.4. *For $\alpha > 0$,*

$$\check{R}^\alpha\big(\widehat{\mathcal{H}}_\alpha v\big) \in \mathcal{D}(\check{\mathcal{A}}) \quad \textit{and} \quad -(\check{\mathcal{A}} - U_\alpha)\check{R}^\alpha\big(\widehat{\mathcal{H}}_\alpha v\big) = \widehat{\mathcal{H}}_\alpha v \quad \textit{for any } v \in \mathcal{B}_0(E_0). \tag{2.17}$$

When F is finite, the function $\widehat{\mathcal{H}}_\alpha v$ in the above can be replaced by any $f \in \mathcal{B}_b(F)$.

Proof. By virtue of the Feller–Neveu formula (4.5) of [6],

$$U^{ij}_\beta = \int_0^\infty \big(1 - e^{-\beta u}\big)\Theta^{ij}(du), \qquad \beta > 0,$$

for some σ-finite positive measure Θ^{ij} on $[0, \infty)$. This formula enables us to conclude that

$$\lim_{\beta \downarrow 0} U^{ij}_\beta = 0. \tag{2.18}$$

Notice that $U_\alpha \check{R}^\alpha \widehat{\mathcal{H}}_\alpha v$ is bounded on F for $v \in \mathcal{B}_0(E_0)$ because we have by (2.12) and (2.13)

$$\big|U_\alpha \check{R}^\alpha \widehat{\mathcal{H}}_\alpha v(a_i)\big| \leqslant \alpha\big(\widehat{\varphi}^{(i)}, \mathbf{H}_\alpha \check{R}^\alpha \widehat{\mathcal{H}}_\alpha |v|\big) \leqslant \alpha\big(\widehat{G}_\alpha 1, |v|\big) \leqslant \int_{E_0} \big|v(x)\big| m_0(dx) \quad \text{for } i \in \Lambda.$$

Therefore we can let $\beta \downarrow 0$ in (2.16) to obtain, for $v \in \mathcal{B}_0(E_0)$,

$$\big[\check{R}^\alpha - \check{R}_\lambda\big(I - U_\alpha \check{R}^\alpha + \lambda \check{R}^\alpha\big)\big]\widehat{\mathcal{H}}_\alpha v = 0. \tag{2.19}$$

This means that $\check{R}^\alpha(\widehat{\mathcal{H}}_\alpha v) \in \mathcal{D}(\check{\mathcal{A}})$ and we can obtain, by operating $\lambda - \check{\mathcal{A}}$ to both sides of (2.19),

$$(\lambda - \check{\mathcal{A}})\check{R}^\alpha \widehat{\mathcal{H}}_\alpha v - \big[I + U_\alpha \check{R}^\alpha - \lambda \check{R}^\alpha\big]\widehat{\mathcal{H}}_\alpha v = 0,$$

namely, the identity in (2.17).

When F consists of N points only, we can identify the operator in [ ] of (2.19) as an $(N \times N)$-matrix with entries $M(a_i, a_j)$. Then (2.19) becomes

$$\sum_{1 \leqslant k \leqslant N} M(a_i, a_k)\widehat{\mathcal{H}}_\alpha v(a_k) = 0 \quad \text{for } v \in \mathcal{B}_0(E_0) \text{ and } 1 \leqslant i \leqslant N,$$

and hence

$$\sum_{1\leqslant k\leqslant N} M(a_i, a_k)\widehat{u}_\alpha^{(k)}(y) = 0 \quad \text{for q.e. } y \in E_0 \text{ and } 1 \leqslant i \leqslant N. \tag{2.20}$$

In the remaining of this proof, for emphasis, for a subset $A \subset E$, we use $\widehat{\sigma}_A$ and $\widehat{\tau}_A$ to denote the first hitting time of A and the first exit time from A by the dual process $\widehat{X}$, respectively. Note that for $t \geqslant 0$ and $x \in E_0$,

$$\widehat{\mathbf{E}}_x\big[e^{-\alpha\widehat{\sigma}_F}\mathbf{1}_{\{\widehat{X}_{\widehat{\sigma}_F}=a_k\}}|\widehat{\mathcal{F}}_t\big] = e^{-\alpha\widehat{\sigma}_F}\mathbf{1}_{\{\widehat{X}_{\widehat{\sigma}_F}=a_k\}}\mathbf{1}_{\{t\geqslant\widehat{\sigma}_F\}} + e^{-\alpha t}\widehat{u}_\alpha^{(k)}\big(\widehat{X}_t\big)\mathbf{1}_{\{t<\widehat{\sigma}_F\}}.$$

It is well known that the bounded martingale $t \mapsto \widehat{\mathbf{E}}_x[e^{-\alpha\widehat{\sigma}_F}\mathbf{1}_{\{\widehat{X}_{\widehat{\sigma}_F}=a_k\}}|\widehat{\mathcal{F}}_t]$ is right continuous and has left limits $\widehat{\mathbf{P}}_x$-a.s. for every $x \in E_0$. Let $\{A_j, j \geqslant 1\}$ be an increasing sequence of compact sets so that $\bigcup_{j=1}^\infty A_j = E_0$. Since $\widehat{\tau}_{A_j}(=\widehat{\sigma}_{A_j^c} \wedge \widehat{\zeta})$ increases to $\widehat{\sigma}_F \wedge \widehat{\zeta}$ as $j \to \infty$, it follows from the quasi-left continuity of $\{\widehat{\mathcal{F}}_t, t \geqslant 0\}$ on $[0, \widehat{\zeta})$ that $\sigma\{\widehat{\mathcal{F}}_{\widehat{\tau}_{A_j}}, j \geqslant 1\} = \widehat{\mathcal{F}}_{\widehat{\sigma}_F}$ on $\{\widehat{\sigma}_F < \infty\}$. By martingale convergence theorem,

$$\lim_{j\to\infty}\widehat{\mathbf{E}}_x\big[e^{-\alpha\widehat{\sigma}_F}\mathbf{1}_{\{\widehat{X}_{\widehat{\sigma}_F}=a_k\}}|\widehat{\mathcal{F}}_{\widehat{\tau}_{A_j}}\big] = e^{-\alpha\widehat{\sigma}_F}\mathbf{1}_{\{\widehat{X}_{\widehat{\sigma}_F}=a_k\}}$$

$\widehat{\mathbf{P}}_x$-a.s. on $\{\widehat{\sigma}_F < \infty\}$ for every $x \in E_0$. This implies that for every $x \in E_0$, $\widehat{\mathbf{P}}_x$-a.s. on $\bigcup_j\{\widehat{\tau}_{A_j} < \widehat{\sigma}_F < \infty\}$,

$$\lim_{t\uparrow\widehat{\sigma}_F}\widehat{u}_\alpha^{(k)}(\widehat{X}_t) = \mathbf{1}_{\{a_k\}}\big(\widehat{X}_{\widehat{\sigma}_F}\big).$$

On the other hand, condition (X.3) and (X.4) imply that

$$\widehat{\mathbf{P}}_m\bigg(\bigcup_j\big\{\widehat{\tau}_{A_j} < \widehat{\sigma}_F < \infty \text{ and } \widehat{X}_{\widehat{\sigma}_F} = a_j\big\}\bigg) > 0 \quad \text{for every } 1 \leqslant j \leqslant N.$$

Thus it follows from (2.20) that

$$\sum_{1\leqslant k\leqslant N} M(a_i, a_k)\mathbf{1}_{\{a_j\}}(a_k) = 0 \quad \text{for every } 1 \leqslant i, j \leqslant N.$$

This implies that $M(a_i, a_k) = 0$ for every $1 \leqslant i, k \leqslant N$ and hence (2.17) holds for every $f \in \mathcal{B}_b(F)$. $\square$

We turn to the task of deriving an explicit expression of $\check{\mathcal{A}}$ from Theorem 5.6 of [6].

Denote by $(\check{N}, t)$ the Lévy system of the time changed process $Y = (Y_t, \check{\mathbf{P}}_{a_i})$ of X on F. Recall that the exit time of Y from a_i is denoted by τ. We have

$$\check{\mathbf{E}}_{a_i}\bigg[\sum_{s\leqslant\tau} f(X_{\tau-})g(X_\tau)\bigg] = \check{\mathbf{E}}_{a_i}\bigg[\int_0^\tau\int_F f(X_s)\check{N}g(X_s)\,ds\bigg].$$

Take $f = \mathbf{1}_{\{a_i\}}$ and $g = \mathbf{1}_{\{a_j\}}$ in this formula to obtain

$$\check{\mathbf{P}}_{a_i}(X_\tau = a_j) = \check{\mathbf{E}}_{a_i}(\tau)\check{N}(a_i, a_j). \tag{2.21}$$

Take then $f = \mathbf{1}_{\{a_i\}}$ and $g = \mathbf{1}_{\{\Delta\}}$ to obtain

$$\check{\mathbf{P}}_{a_i}(X_\tau = \Delta) = \check{\mathbf{E}}_{a_i}(\tau)\check{N}(a_i, \Delta). \tag{2.22}$$

We define

$$\pi_{ij} := \check{\mathbf{P}}_{a_i}(X_\tau = a_j) \quad \text{for } i \neq j, \qquad \pi_{i\Delta} = \check{\mathbf{P}}_{a_i}(X_\tau = \Delta) \quad \text{and} \quad q_i = 1/\check{\mathbf{E}}_{a_i}[\tau].$$

Then the generator $\check{\mathcal{A}}$ of Y admits the expression (cf. [16])

$$\check{\mathcal{A}}u(a_i) = q_i\left(\sum_{j\neq i}\pi_{ij}u(a_j) - u(a_i)\right). \tag{2.23}$$

Owing to [6, Theorem 5.6], we have

$$\check{N}(a_i, a_j) = U^{ij} + J_{ij} \quad \text{for } i \neq j \quad \text{and} \quad \check{N}(a_i, \Delta) = V_i + \kappa_i, \tag{2.24}$$

where U^{ij} and V_i are the Feller measure and the supplementary Feller measure of X^0 on F defined by (2.6) and (2.7), respectively, and

$$J_{ij} = J\big(\{a_i\}, \{a_j\}\big) \quad \text{and} \quad \kappa_i = \kappa\big(\{a_i\}\big). \tag{2.25}$$

Here J and κ are the jumping measure and the killing measure of X (see [6, (5.20)] for their definitions in terms of the Lévy system of X). Therefore we get from (2.21), (2.22) and (2.24) that

$$q_i\pi_{ij} = U^{ij} + J_{ij} \quad \text{and} \quad q_i\pi_{i\Delta} = V_i + \kappa_i.$$

From

$$\sum_{j\in\Lambda:\ j\neq i}\pi_{ij} + \pi_{i\Delta} = 1,$$

we have for every $i \in \Lambda$,

$$\sum_{j\in\Lambda:\ j\neq 1}\big(U^{ij} + J_{ij}\big) < \infty \tag{2.26}$$

and

$$q_i = \sum_{j\neq i}\big(U^{ij} + J_{ij}\big) + V_i + \kappa_i.$$

For later use, applying (2.26) to the dual process $\widehat{X}$ and noting that $\widehat{U}^{ij} = U^{ji}$ and $\widehat{J}_{ij} = J_{ji}$, we have for every $i \in \Lambda$,

$$\sum_{j\in\Lambda:\ j\neq 1} \left(U^{ji} + J_{ji}\right) < \infty. \tag{2.27}$$

We finally obtain from (2.23) the following theorem.

Theorem 2.5. *The generator $\check{\mathcal{A}}$ of the time changed process Y of X on F admits the following expression for $u \in \mathcal{D}(\check{\mathcal{A}})$:*

$$\check{\mathcal{A}}u(a_i) = \sum_{j\neq i}\left(U^{ij} + J_{ij}\right)\left(u(a_j) - u(a_i)\right) - (V_i + \kappa_i)u(a_i) \quad \textit{for } i \in \Lambda. \tag{2.28}$$

Let us denote by $\check{A}_{ij}$ the value of the right-hand side of (2.28) for $u = \mathbf{1}_{a_j}$; that is,

$$\check{\mathcal{A}}_{ij} = U^{ij} + J_{ij} \quad \text{for } i \neq j \quad \text{and} \quad \check{\mathcal{A}}_{ii} = -q_i.$$

The operator $\check{\mathcal{A}}$ will be viewed as a matrix with entries $(\check{\mathcal{A}}_{ij})_{i,j\in\Lambda}$. $\check{\mathcal{A}} - U_\alpha$ will also be viewed as a matrix with entries $(\check{\mathcal{A}}_{ij} - U_\alpha^{ij})_{i,j\geqslant 1}$.

Theorem 2.6 *(Representation of the resolvent of X).*

(i) *Suppose $F = \{a_i, a_2, \ldots, a_N\}$ is finite. Then $(N \times N)$-matrix $\check{\mathcal{A}} - U_\alpha$ is invertible and*

$$\check{R}^\alpha = -(\check{\mathcal{A}} - U_\alpha)^{-1} \quad \textit{and}$$
$$G_\alpha v = G_\alpha^0 v - \mathbf{H}_\alpha(\check{\mathcal{A}} - U_\alpha)^{-1}\widehat{\mathcal{H}}_\alpha v \quad \textit{for } v \in \mathcal{B}_b(E_0). \tag{2.29}$$

(ii) *When F is infinite, we define for each integer $N \geqslant 1$*

$$\check{A}_N = \{\check{A}_{ij}\}_{1\leqslant i,j\leqslant N} \quad \textit{and} \quad U_{\alpha,N} = \{U_\alpha^{ij}\}_{1\leqslant i,j\leqslant N}.$$

For $x \in E$ and $v \in \mathcal{B}_b(E_0)$, we also denote by $\mathbf{H}_{\alpha,N}(x)$ and by $\widehat{\mathcal{H}}_{\alpha,N}v$ the row vector $(u_\alpha^{(i)}(x))_{1\leqslant i\leqslant N}$ and the column vector $((\widehat{u}_\alpha^{(i)}, v))_{1\leqslant i\leqslant N}$, respectively. Then for $f \in \mathcal{B}_b(E_0)$,

$$\check{R}^\alpha = -\lim_{N\to\infty}\left(\check{\mathcal{A}}_N - U_{\alpha,N}\right), \quad \textit{and}$$
$$G_\alpha f(x) = G_\alpha^0 f(x) - \lim_{N\to\infty}\mathbf{H}_{\alpha,N}(x)\left(\check{A}_N - U_{\alpha,N}\right)^{-1}\widehat{\mathcal{H}}_{\alpha,N} f \quad \textit{for } x \in E_0. \tag{2.30}$$

Proof. (i) This part follows from Lemma 2.4 combined with Theorem 2.5.

(ii) Put $F_N = \{a_1, \ldots, a_N\}$ and $E_N = E_0 \cup F_N$. Let X^N be the part process of X on E_N, that is, X^N is the subprocess of X killed upon leaving E_N. Since $E_N = E_0 \cup F_N$ and X^0 is the part process of X^N on E_0, we see from (i) that the resolvent G_α^N of X^N and the 0-order resolvent $\check{R}^{N,\alpha}$ of the time-changed process on F_N of the α-subprocess $X^{N,\alpha}$ of X^N admit the expression

$$\check{R}^{N,\alpha} = -\big(\check{\mathcal{A}}^N - U_\alpha^N\big)^{-1}, \qquad G_\alpha^N f = G_\alpha^0 f - \mathbf{H}_{\alpha,N}\big(\check{\mathcal{A}}^N - U_\alpha^N\big)^{-1}\widehat{\mathcal{H}}_{\alpha,N} f. \tag{2.31}$$

Letting $N \to \infty$, we obtain (2.30). □

Let $\widehat{J}$ and $\widehat{\kappa}$ be the jumping measure and the killing measure of $\widehat{X}$, respectively. We put

$$\widehat{J}_{ij} = \widehat{J}\big(\{a_i\}, \{a_j\}\big), \quad i, j \geqslant 1,\ i \neq j, \quad \text{and} \quad \widehat{\kappa}_i = \widehat{\kappa}\big(\{a_i\}\big), \quad i \geqslant 1.$$

The next proposition is a consequence of the weak duality of X and $\widehat{X}$ and Theorem 2.5.

Proposition 2.7. *It holds that*

$$\widehat{J}_{ij} = J_{ji}, \quad \text{for } i, j \geqslant 1 \text{ with } i \neq j, \tag{2.32}$$

and

$$\sum_{k\in\Lambda:\ k\neq i} \big(U^{ik} + J_{ik}\big) + V_i + \kappa_i = \sum_{k\in\Lambda:\ k\neq i} \big(U^{ki} + J_{ki}\big) + \widehat{V}_i + \widehat{\kappa}_i \quad \text{for } i \in \Lambda, \tag{2.33}$$

where $\widehat{V}$ is the supplementary Feller measure for $\widehat{X}$.

Proof. In view of the proof of Theorem 2.6, we may and do assume that F is finite: $F = \{a_1, a_2, \dots, a_N\}$. From (2.8) and (2.13), we have

$$\widehat{U}_\alpha^{ij} = U_\alpha^{ji}, \quad 1 \leqslant i, j \leqslant N, \tag{2.34}$$

and, by letting $\alpha \to \infty$,

$$\widehat{U}^{ij} = U^{ji}, \quad 1 \leqslant i, j \leqslant N, \quad i \neq j. \tag{2.35}$$

Denote by $\check{S}^\alpha$ the 0-order resolvent of the time changed process of the α-subprocess of $\widehat{X}$ by means of the PCAF of $\widehat{X}$ with Revuz measure $\sum_{i=1}^N \delta_{a_i}$. Then analogously to (2.12), we have

$$\widehat{G}_\alpha v = \widehat{G}_\alpha^0 v + \widehat{\mathbf{H}}_\alpha \check{S}^\alpha \mathcal{H}_\alpha v,$$

which together with (2.12) and the duality of X, $\widehat{X}$ implies

$$\big(u, \mathbf{H}_\alpha \check{R}^\alpha \widehat{\mathcal{H}}_\alpha v\big) = \big(\widehat{\mathbf{H}}_\alpha \check{S}^\alpha \mathcal{H}_\alpha u, v\big), \quad u, v \in \mathcal{B}_b(E_0).$$

Hence, as in the proof of Lemma 2.2, we obtain $\check{R}^\alpha(a_i, a_j) = \check{S}^\alpha(a_j, a_i)$, $1 \leqslant i, j \leqslant N$, or, in the matrix form

$$\check{S}^\alpha = \big(\check{R}^\alpha\big)^{\mathrm{tr}}, \tag{2.36}$$

where M^{tr} stands for the transpose of a matrix M.

Let $\widehat{Y}$ be the time changed process of $\widehat{X}$ by means of the PCAF of $\widehat{X}$ with Revuz measure $\sum_{i=1}^N \delta_{a_i}$ and $\check{B}$ be its generator. Then analogous to (2.17) and (2.28), we have

$$-\big(\check{B} - \widehat{U}_\alpha\big)\check{S}^\alpha = I, \tag{2.37}$$

$$\check{B}u(a_i) = \sum_{j \neq i}\big(\widehat{U}^{ij} + \widehat{J}_{ij}\big)\big(u(a_j) - u(a_i)\big) - \big(\widehat{V}_i + \widehat{\kappa}_i\big)u(a_i), \quad 1 \leqslant i \leqslant N. \tag{2.38}$$

Since $-(\check{A} - U_\alpha)\check{R}^\alpha = -\check{R}^\alpha(\check{A} - U_\alpha) = I$, we get $-(\check{A} - U_\alpha)^{\mathrm{tr}}(\check{R}^\alpha)^{\mathrm{tr}} = I$. Consequently, (2.34), (2.36) and (2.37) imply

$$\check{B} = (\check{A})^{\mathrm{tr}}.$$

The desired identities (2.32) and (2.33) now follow from (2.28), (2.38) and (2.35). □

The duality (2.32) of J_{ij} follows also from a general theorem in [15].

Consider the special case that F consists of only one point; $F = \{a\}$. We define for $x \in E_0$

$$\varphi(x) = \mathbf{P}_x^0\big(\zeta^0 < \infty \text{ and } X^0_{\zeta^0-} = a\big) \quad \textit{and} \quad u_\alpha(x) = \mathbf{E}_x^0\big[e^{-\alpha\zeta^0}; X^0_{\zeta^0-} = a\big].$$

The functions $\widehat{\varphi}$, $\widehat{u}_\alpha$ are analogously defined for $\widehat{X}^0$. The value of the killing measure of X and $\widehat{X}$ at $\{a\}$ are denoted by κ and $\widehat{\kappa}$, respectively. Theorem 2.6 and Proposition 2.7 implies immediately the following.

Corollary 2.8. *When F is a one-point set $\{a\}$, we have for $f \in \mathcal{B}^+(E)$,*

$$G_\alpha f(a) = \frac{(\widehat{u}_\alpha, f)}{\alpha(\widehat{u}_\alpha, \varphi) + L^{(0)}(\widehat{\varphi} \cdot m_0, 1 - \varphi) + \kappa},$$

$$G_\alpha f(x) = G_\alpha^0 f(x) + u_\alpha(x) G_\alpha f(a) \quad \textit{for } x \in E_0. \tag{2.39}$$

Furthermore,

$$L^{(0)}(\widehat{\varphi} \cdot m_0, 1 - \varphi) + \kappa = \widehat{L}^{(0)}(\varphi \cdot m_0, 1 - \widehat{\varphi}) + \widehat{\kappa}. \tag{2.40}$$

In fact (2.39) and (2.40) have been obtained previously in [7, Theorem 4.2] and [4, Appendix].

3. Extending X^0 by darning countably many holes

Let E and m be as in previous sections. We consider a closed subset K of E such that either:

(K.1) $K = \bigcup_i K_i$, where $\{K_i\}$ are finite or countable disjoint compact sets which are locally finite in the sense that any compact set intersects only with finite many of K_i's; or

(K.2) $K = K_1 \cup \cdots \cup K_N$, where $\{K_i\}_{1 \leqslant i \leqslant N}$, are disjoint, $K_1, \ldots, K_{N-1}$ are compact and $E \setminus K_N$ is relatively compact.

We put $E_0 = E \setminus K$, $F := \bigcup_i \{a_i\}$ and let

$$E^* := E_0 \cup F$$

be the topological Hausdorff space obtained by adding to E_0 extra points $\{a_1, a_2, \ldots\}$, whose topology is prescribed as follows: for each $i \in \Lambda$, a subset U of E^* containing the point a_i is a neighborhood of a_i if there is an open set $\widetilde{U} \subset E$ containing K_i such that $\widetilde{U} \cap E_0 = U \setminus \{a_i\}$. In other words, E^* is obtained from E by identifying each closed set K_i with the point $\{a_i\}$ for every $i \in \Lambda$. We denote by m_0 the restriction of the measure m on E to E_0. The measure m_0 is then extended to E^* by setting $m_0(F) = 0$.

Consider a pair of Borel standard processes

$$X = \{X_t, \zeta, \mathbf{P}_x\} \quad \text{and} \quad \widehat{X} = \{\widehat{X}_t, \widehat{\zeta}, \widehat{\mathbf{P}}_x\}$$

on E which are in weak duality with respect to m. We shall assume that X satisfies the following conditions and that $\widehat{X}$ satisfies the corresponding counterpart conditions ($\widehat{\text{B}}$.1), ($\widehat{\text{B}}$.2), ($\widehat{\text{B}}$.3) and ($\widehat{\text{B}}$.5).

(B.1) X is m-irreducible. X satisfies condition

$$\mathbf{P}_x(\zeta < \infty \text{ and there exists } X_{\zeta-} \in E_\Delta) = \mathbf{P}_x(\zeta < \infty) \quad \text{for } x \in E, \tag{3.1}$$

and X admits no killings inside $U_0 \setminus K$ for some open neighborhood U_0 of K in E.

(B.2) $m_0(U \cap E_0)$ is finite for some neighborhood U of K. The set K_i is non-m-polar with respect to X for each $i \in \Lambda$.

(B.3) X admits no jumps from $E \setminus K_i$ to K_i for each $i \in \Lambda$.

(B.4) Every semi-polar set is m-polar for X.

(B.5) When X is transient, either:
- (a) The 0-order resolvent $G_{0+} f$ of X is lower semicontinuous for any non-negative Borel function f on E; or
- (b) X is m-symmetric and the associated Dirichlet form on $L^2(X; m)$ is regular.

For $\alpha > 0$ and $i \in \Lambda$, define the functions $\psi^{(i)}$, $v_\alpha^{(i)}$ on E by

$$\psi^{(i)}(x) = \mathbf{P}_x(\sigma_{K_i} < \infty) \quad \text{and} \quad v_\alpha^{(i)}(x) = \mathbf{E}_x\big[e^{-\alpha\sigma_{K_i}}\big] \quad \text{for } x \in E.$$

The corresponding functions for $\widehat{X}$ will be denoted by $\widehat{\psi}^{(i)}$, $\widehat{v}_\alpha^{(i)}$. By (B.1)–(B.2), ($\widehat{\text{B}}$.1)–($\widehat{\text{B}}$.2) and [4, Lemma 2.2(i)], we have

$$\psi^{(i)}(x) > 0 \quad \text{and} \quad \widehat{\psi}^{(i)}(x) > 0 \quad \text{for q.e. } x \in E_0,\ i \in \Lambda.$$

We also define, for $\alpha > 0$ and $i \in \Lambda$, the functions $\varphi^{(i)}$, $u_\alpha^{(i)}$ on E by

$$\varphi^{(i)}(x) = \mathbf{P}_x(\sigma_K < \infty, X_{\sigma_K} \in K_i) \quad \text{and} \quad u_\alpha^{(i)}(x) = \mathbf{E}_x\big[e^{-\alpha\sigma_K};\ X_{\sigma_K} \in K_i\big] \quad \text{for } x \in E.$$

The corresponding functions for $\widehat{X}$ will be denoted by $\widehat{\varphi}^{(i)}$ and $\widehat{u}_\alpha^{(i)}$, respectively.

We consider the subprocesses $X^0=(X^0_t,\zeta^0,\mathbf{P}_x)$ and $\widehat{X}^0=(\widehat{X}^0_t,\widehat{\zeta}^0,\widehat{\mathbf{P}}_x)$ of X and $\widehat{X}$ killed upon leaving E_0, respectively. The subprocesses X^0 and $\widehat{X}^0$ are in weak duality with respect to m_0 (cf. [6]). The resolvent of X^0 and $\widehat{X}^0$ are denoted by $\{G^0_\alpha,\alpha>0\}$ and $\{\widehat{G}^0_\alpha,\alpha>0\}$, respectively. By (B.3), ($\widehat{\mathrm{B}}$.3) and [1, p. 59], we have for $x\in E_0$ and $i\in\Lambda$,

$$\varphi^{(i)}(x)=\mathbf{P}^0_x\big(\zeta^0<\infty \text{ and } X^0_{\zeta^0-}\in K_i\big)\quad\text{and}\quad \widehat{\varphi}^{(i)}(x)=\widehat{\mathbf{P}}^0_x\big(\widehat{\zeta}^0<\infty \text{ and } \widehat{X}^0_{\widehat{\zeta}^0-}\in K_i\big),$$

and

$$u^{(i)}_\alpha(x)=\mathbf{E}^0_x\big[e^{-\alpha\zeta^0};X^0_{\zeta^0-}\in K_i\big]\quad\text{and}\quad \widehat{u}^{(i)}_\alpha(x)=\widehat{\mathbf{E}}^0_x\big[e^{-\alpha\widehat{\zeta}^0};\widehat{X}^0_{\widehat{\zeta}^0-}\in K_i\big].$$

Let us consider the next conditions on X^0:

(B^0.1) $G^0_{0+}f$ is lower semicontinuous on E_0 for any non-negative Borel function f on E_0. $\varphi^{(i)}(x)>0$ for any $x\in U_i\cap E_0$ for some neighborhood U_i of K_i for each $i\in\Lambda$.

(B^0.2) Either $E\setminus U$ is compact for any neighborhood U of K in E, or for any open neighborhood U_1 of K in E, there exists an open neighborhood U_2 of K in E with $\overline{U}_2\subset U_1$ such that $J_0(U_2\setminus K,E_0\setminus U_1)<\infty$, where J_0 denotes the jumping measures of X^0.

(B^0.3) $\lim_{x\to K_i}u^{(i)}_\alpha(x)=1$ for every $\alpha>0$ and $i\in\Lambda$.

Corresponding conditions on $\widehat{X}^0$ are designated by ($\widehat{\mathrm{B}}^0$.1), ($\widehat{\mathrm{B}}^0$.2) and ($\widehat{\mathrm{B}}^0$.3), respectively.

A strong Markov process X^* on E^* is said to be a q.e. extension of X^0 if the subprocess of X^* killed upon leaving E_0 coincides with X^0 for q.e. starting points $x\in E_0$.

Under the conditions stated in the above, we can construct extensions X^* (resp. $\widehat{X}^*$) on E^* of X^0 (resp. $\widehat{X}^0$) by applying successively the darning procedure based on [4, Theorem 3.1] to each hull K_i, $i\in\Lambda$. Recall that $\{U^{ik},\ i\neq k\}$ and $\{V^i\}$ are the Feller measure and supplementary Feller measure of X^0 defined by (2.6), (2.7), and $\{\widehat{U}^{ik},\ i\neq k\}$ and $\{\widehat{V}^i\}$ are the Feller measure and supplementary Feller measure of $\widehat{X}^0$ defined analogously.

Theorem 3.1. *Let us assume the conditions* (B.1)–(B.5) *on X and the corresponding counterpart conditions on $\widehat{X}$. We further assume for X^0 condition* (B^0.1) *as well as condition* (B^0.2) *when X^0 is not a diffusion and condition* (B^0.3) *when X^0 is not m_0-symmetric. Corresponding conditions on $\widehat{X}^0$ are also imposed.*

(i) *For every $i\in\Lambda$,*

$$\sum_{k\in\Lambda:\ k\neq i}U^{ik}<\infty\quad\textit{and}\quad\sum_{k\in\Lambda:\ k\neq i}U^{ki}<\infty. \tag{3.2}$$

(ii) *For any two sequences $\{\kappa_i,\ i\in\Lambda\}$ and $\{\widehat{\kappa}_i,\ i\in\Lambda\}$ of non-negative numbers satisfying*

$$\kappa_i+V^i+\sum_{k\in\Lambda:\ k\neq i}U^{ik}=\widehat{\kappa}_i+\widehat{V}^i+\sum_{k\in\Lambda:\ k\neq i}U^{ki}\quad\textit{for } i\in\Lambda, \tag{3.3}$$

there exist right processes X^* and $\widehat{X}^*$ on E^* satisfying the following properties and the law of such extension X^* and $\widehat{X}^*$ are unique.

(1) *X^* and $\widehat{X}^*$ are q.e. extensions of X^0 and $\widehat{X}^0$, respectively.*
(2) *X^* and $\widehat{X}^*$ are in weak duality with respect to m_0. For every $i \in \Lambda$, processes X^* and $\widehat{X}^*$ get killed at the point a_i at rate κ_i and $\widehat{\kappa_i}$, respectively. Every semi-polar set is m_0-polar for X^*.*
(3) *The sample path of X^* is cadlag and X^* is m_0-irreducible.*
(4) *$X^* = \{X_t^*, \zeta^*, \mathbf{P}_x^*,\ x \in E^*\}$ admits no sojourn on F, namely,*

$$\mathbf{P}_x^*\left(\int_0^\infty \mathbf{1}_F\left(X_s^*\right) ds = 0\right) = 1 \quad q.e.\ x \in E^*.$$

(5) *For every $i \in \Lambda$, point a_i is regular for itself with respect to X^* and $\{a_i\}$ is not m_0-polar for X^*.*
(6) *X^* admits no jumps from a_i to $E^* \setminus \{a_i\}$, $i \in \Lambda$, namely,*

$$\mathbf{P}_x^*\left(X_{t-}^* = a_i \text{ and } X_t^* \in E^* \setminus \{a_i\} \text{ for some } t > 0\right) = 0 \quad q.e.\ x \in E^*.$$

(7) *X^* is quasi-left continuous up to the life time for q.e. starting point $x \in E^*$.*
(8) *If $K = \bigcup_{i=1}^N K_i$ is the finite union of disjoint compact sets $\{K_i\}$, then X^* has the property* (3.1).
(9) *If X^0 is a diffusion, then so is X^* for q.e. starting points $x \in E^*$.*

The process $\widehat{X}^*$ also enjoy the counterpart properties corresponding to (3)–(9).

Proof. We shall give the proof only for the case where K is of the form (K.1) because the second case (K.2) can be treated in a simpler way.

(i) We first show that there exists some duality preserving extensions of X^0 and $\widehat{X}^0$ on E^*. This will yield (3.2) by (2.26), (2.27).

For existence, look at first the set K_1 and put $E_{01} = E \setminus K_1$, $m_{01} = m|_{E_{01}}$. Let X^{01} be the part of X on the set E_{01} with resolvent G_α^{01}. Since the approaching probability of X^{01} to K_1 equals $\psi^{(1)}$ and we have the inequality

$$\psi^{(1)} \geqslant \varphi^{(1)}, \quad G_1^{01}\psi^{(1)} \geqslant G_1^{(0)}\varphi^{(1)},$$

we can deduce from condition (B^0.1) that (X^{01}, K_1) satisfies the condition (C.2)$'$ stated in [4, Remark 3.2(ii)]. Conditions (C.3) and (C.4) of [4, Section 3] for (X^{01}, K_1) readily follow from (B^0.2), (B.3) and (B^0.3). The pair (X, K_1) clearly satisfies (B.1)–(B.5) of [4, Section 3]. Corresponding properties are also satisfied by the dual process $\widehat{X}$ for K_1.

Let $E^{*1} = E_{01} \cup \{a_1\}$ be the space obtained from E by regarding the set K_1 as one point $\{a_1\}$. m_{01} is extended to E^{*1} by setting $m_{01}(\{a_1\}) = 0$. Let L^{01} and $\widehat{L}^{01}$ be the energy functional of X^{01} and $\widehat{X}^{01}$, respectively. Let δ_1 and $\widehat{\delta}_1 \geqslant 0$ be such that

$$\delta_1 + L^{01}\left(\widehat{\psi}^{(1)} \cdot m_{01}, 1 - \psi^{(1)}\right) = \widehat{\delta}_1 + \widehat{L}^{01}\left(\psi^{(1)} \cdot m_{01}, 1 - \widehat{\psi}^{(1)}\right). \tag{3.4}$$

Theorem 3.1 of [4] guarantees the m_{01}-integrability of $v_\alpha^{(1)}$ and $\widehat{v}_\alpha^{(1)}$ together with the existence of pair of processes X^{*1} and $\widehat{X}^{*1}$ on E^{*1} that are q.e. extensions of X^{01} and $\widehat{X}^{01}$, respectively, and satisfy the properties (i.2)–(i.9) in [4, Theorem 3.1] and their dual ones. In particular,

$$K_i \subset E_{01} \text{ is not } X^{*1}\text{-polar for every } i \geqslant 2, \tag{3.5}$$

and X^{*1} and $\widehat{X}^{*1}$ have killing rate δ_1 and $\widehat{\delta}_1$ at the point a_1, respectively (see [4, Remark 3.2(i)]). Note that the killing rate δ_1 and $\widehat{\delta}_1$ of X^{*1} and $\widehat{X}^{*1}$ are described in terms of the Lévy system of X^{*1} and $\widehat{X}^{*1}$, respectively (see (2.25) and Corollary 2.8).

Applying the analogous argument consecutively to each K_i, we get for $i \in \Lambda$,

$$v_\alpha^{(i)} \text{ and } \widehat{v}_\alpha^{(i)} \text{ are integrable on } E \setminus K_i \text{ with respect to } m_{0i} = m|_{E\setminus K_i}. \tag{3.6}$$

We next pick up $K_2 \subset E^{*1}$ and put $E^{*1}_{02} = E^{*1} \setminus K_2$, $m_{02} = m_{01}|_{E^{*1}_{02}}$. Let X^{02} be the part of X^{*1} on the set E^{*1}_{02} with resolvent G^{02}_α. For the approaching probability

$$\psi^{12}(x) = \mathbf{P}^{*1}_x(\sigma_{K_2} < \infty), \quad x \in E^{*1}_{02},$$

of X^{*1} to K_2, we clearly have

$$\psi^{12} \geqslant \varphi^{(2)} \quad \text{and} \quad G^{02}_1\psi^{12} \geqslant G^0_1\varphi^{(2)},$$

which combined with (B^0.1) implies that the pair (X^{02}, K_2) satisfies (C.2)$'$ of [4, Section 3] again. This pair also satisfies the integrability condition (C.1) of [4, Section 3], because, for the α-order approaching probability

$$v^{12}_\alpha(x) = \mathbf{E}^{*1}_x\left[e^{-\alpha\sigma_{K_2}}\right], \quad x \in E^{*1}_{02},$$

of X^{*1} to K_2, we have the bound

$$\begin{aligned} v^{12}_\alpha(x) &= \mathbf{E}^{*1}_x\left[e^{-\alpha\sigma_{K_2}}; \sigma_{K_2} < \sigma_{a_1}\right] + \mathbf{E}^{*1}_x\left[e^{-\alpha\sigma_{K_2}}; \sigma_{K_2} > \sigma_{a_1}\right] \\ &\leqslant v^{(2)}_\alpha + \mathbf{E}^{*1}_x\left[e^{-\alpha\sigma_{a_1}}; \sigma_{a_1} < \sigma_{K_2}\right]\mathbf{E}^{*1}_{a_1}\left[e^{-\alpha\sigma_{K_2}}\right] \\ &\leqslant v^{(2)}_\alpha(x) + v^{(1)}_\alpha(x), \end{aligned}$$

which is m_{02}-integrable on E^{*1}_{02} by virtue of (3.6). Conditions (C.3) and (C.4) of [4, Section 3] for the pair follow from (B^0.2), (B.3) and (B^0.3) as in the previous step.

We already know in the first step that the pair (X^{*1}, K_2) satisfies (B.1)–(B.4) of [4, Section 3]. In particular K_2 is not X^{*1}-polar in view of (3.5). Corresponding properties are also satisfied by the dual process $\widehat{X}^{*1}$ for K_2.

Let $E^{*2} = E^{*1}_{*2} \cup \{a_2\}$ be the space obtained from E^{*1} by regarding the set K_2 as one point $\{a_2\}$. The measure m_{02} is extended to E^{*2} by setting $m_{02}(\{a_2\}) = 0$. Let L^{02} and $\widehat{L}^{02}$ be the energy functional of X^{02} and $\widehat{X}^{02}$, respectively. Let δ_2 and $\widehat{\delta}_2 \geqslant 0$ be such that

$$\delta_2 + L^{02}\left(\widehat{\psi}^{(12)} \cdot m_{02}, 1 - \psi^{(12)}\right) = \widehat{\delta}_2 + \widehat{L}^{02}\left(\psi^{(12)} \cdot m_{02}, 1 - \widehat{\psi}^{(12)}\right). \tag{3.7}$$

Theorem 3.1 of [4] guarantees the existence of pair of processes X^{*2} and $\widehat{X}^{*2}$ on E^{*2} that are q.e. extensions of X^{02} and $\widehat{X}^{02}$, respectively, and satisfy the properties (i.2)–(i.9) in [4, Theorem 3.1] and their dual ones. In particular, X^{*2} and $\widehat{X}^{*2}$ have killing rates δ_j and $\widehat{\delta}_j$ at the point a_j for $j \in \{1, 2\}$, respectively (see [4, Remark 3.2(i)]), and

$$K_i \subset E_{02} \text{ is not } X^{*2}\text{-polar for every } i \geqslant 3. \tag{3.8}$$

We next pick up $K_3 \subset E^{*2}$ and put $E^{*2}_{03} = E^{*2} \setminus K_3$, $m_{03} = m_{02}|_{E^{*2}_{03}}$. Let X^{03} be the part of X^{*2} on the set E^{*2}_{03} with resolvent G^{03}_α. For the approaching probability

$$\psi^{23}(x) = \mathbf{P}^{*2}_x(\sigma_{K_3} < \infty), \quad x \in E^{*2}_{03},$$

of X^{*2} to K_3, we clearly have the bound

$$\psi^{23} \geqslant \varphi^{(3)}, \quad G^{03}_1\psi^{23} \geqslant G^0_1\varphi^{(3)},$$

which combined with (B^0.1) implies that the pair (X^{03}, K_3) satisfies (C.2)$'$ of [4, Section 3] again. The pair also satisfies the integrability condition (C.1) of [4, Section 3], because, for the α-order approaching probability

$$v^{23}_\alpha(x) = \mathbf{E}^{*2}_x\big[e^{-\alpha\sigma_{K_3}}\big], \quad x \in E^{*2}_{03},$$

of X^{*2} to K_3, we have the bound

$$\begin{aligned}
v^{23}_\alpha(x) &= \mathbf{E}^{*2}_x\big[e^{-\alpha\sigma_{K_3}};\ \sigma_{K_3} < \sigma_{a_1\cup a_2}\big] + \mathbf{E}^{*2}_x\big[e^{-\alpha\sigma_{K_3}};\ \sigma_{K_3} > \sigma_{a_1\cup a_2}\big] \\
&\leqslant v^{(3)}_\alpha + \mathbf{E}^{*2}_x\big[e^{-\alpha\sigma_{a_1}}; \sigma_{a_1} < \sigma_{a_2},\ \sigma_{a_1} < \sigma_{K_3}\big]\mathbf{E}^{*2}_{a_1}\big[e^{-\alpha\sigma_{K_3}}\big] \\
&\quad + \mathbf{E}^{*2}_x\big[e^{-\alpha\sigma_{a_2}}; \sigma_{a_2} < \sigma_{a_1},\ \sigma_{a_2} < \sigma_{K_3}\big]\mathbf{E}^{*2}_{a_2}\big[e^{-\alpha\sigma_{K_3}}\big] \\
&\leqslant v^{(3)}_\alpha(x) + v^{(2)}_\alpha(x) + v^{(1)}_\alpha(x),
\end{aligned}$$

which is m_{03}-integrable on E^{*2}_{03} by virtue of (3.6). Conditions (C.3) and (C.4) of [4, Section 3] for the pair follow from (B^0.2), (B.3) and (B^0.3) as in the previous step.

We already know in the second step that the pair (X^{*2}, K_3) satisfies (B.1)–(B.4) of [4, Section 3]. In particular K_3 is not X^{*2}-polar in view of (3.8). Corresponding properties are also satisfied by the dual process $\widehat{X}^{*2}$ for K_3.

Let $E^{*3} = E^{*2}_{03} \cup \{a_3\}$ be the space obtained from E^{*2} by regarding the set K_3 as one point $\{a_3\}$. The measure m_{03} is extended to E^{*3} by setting $m_{03}(\{a_3\}) = 0$. Let L^{03} and $\widehat{L}^{03}$ be the energy functional of X^{03} and $\widehat{X}^{03}$, respectively. Let δ_3 and $\widehat{\delta}_3 \geqslant 0$ be such that

$$\delta_3 + L^{02}\big(\widehat{\psi}^{(23)} \cdot m_{03}, 1 - \psi^{(23)}\big) = \widehat{\delta}_3 + \widehat{L}^{03}\big(\psi^{(23)} \cdot m_{03}, 1 - \widehat{\psi}^{(23)}\big). \tag{3.9}$$

Theorem 3.1 of [4] guarantees the existence of pair of processes X^{*3} and $\widehat{X}^{*3}$ on E^{*3} that are q.e. extensions of X^{03} and $\widehat{X}^{03}$, respectively, and satisfy the properties (i.2)–(i.9) in [4, Theorem 3.1] and their dual ones. In particular, X^{*3} and $\widehat{X}^{*3}$ have killing rates δ_j and $\widehat{\delta}_j$ at the point a_j for $j \in \{1, 2, 3\}$, respectively (see [4, Remark 3.2(i)]), and

$$K_i \subset E_{03} \text{ is not } X^{*3}\text{-polar for every } i \geqslant 4. \tag{3.10}$$

Repeating this, we get a sequence of a pair of right processes $(X^{*k}, \widehat{X}^{*k})$ on

$$E^{*k} := \left(E \Big\backslash \left(\bigcup_{j=1}^k K_j\right)\right) \cup \left(\bigcup_{j=1}^k \{a_j\}\right)$$

in weak duality such that $X^{*(k+1)}$ and $\widehat{X}^{*(k+1)}$ are the q.e. extension to $E^{*(k+1)}$ of the subprocesses of X^{k*} and $\widehat{X}^{k*}$ killed upon the hitting times of K_{k+1}, respectively, and satisfy the corresponding properties (i.2)–(i.9) of [4, Theorem 3.1].

Since $\{K_i\}$ are assumed to be locally finite, we may find, after renumbering $\{K_i\}$ if necessary, relatively compact open subsets $\{D_k\}_{k\geqslant 1}$ increasing to E such that for every $k \in \Lambda$,

$$\bigcup_{i=1}^{k} K_i \subset D_k \quad \text{and} \quad \bigcup_{i=k+1}^{\infty} K_i \subset E \setminus D_k.$$

We may also assume that the constructed processes $\{X^{*k}, \widehat{X}^{*k}\}_{k\geqslant 1}$ are defined on a common probability space.

Let τ_k and τ_k denote the first exit time of X^{k*} and $\widehat{X}^{k*}$ from D_k, respectively. Then we have for q.e. $x \in E^*$ and for every $j \geqslant k$,

$$X_t^{*j} = X_t^{*k} \quad \text{for } t < \tau_k \quad \text{and} \quad \widehat{X}_t^{*j} = \widehat{X}_t^{*k} \quad \text{for } t < \widehat{\tau}_k.$$

Define $\zeta^* := \lim_{k\to\infty} \tau_k$, $\widehat{\tau}^* := \lim_{k\to\infty} \widehat{\tau}_k$, and

$$X_t^* := \begin{cases} \lim_{k\to\infty} X_t^{*k} & \text{if } t < \zeta^*, \\ \partial & \text{if } t \geqslant \zeta^*, \end{cases} \quad \text{and} \quad \widehat{X}_t^* := \begin{cases} \lim_{k\to\infty} \widehat{X}_t^{*k} & \text{if } t < \widehat{\zeta}^*, \\ \partial & \text{if } t \geqslant \widehat{\zeta}^*. \end{cases}$$

It is easy to see that X^* and $\widehat{X}^*$ are a pair of right processes on E^* that are dual to each other with respect to measure m_0. They are q.e. extensions of X^0 and $\widehat{X}^0$, respectively, satisfy the properties (2)–(9) of the theorem and have killing rates δ_j and $\widehat{\delta}_j$ at a_j, respectively, for every $j \in \Lambda$.

By abusing the notion, we note that

$$\varphi^{(i)}(x) = \mathbf{P}_x^0\big(\zeta^0 < \infty \text{ and } X^0_{\zeta^0-} = a_i\big) \quad \text{and} \quad \widehat{\varphi}^{(i)}(x) = \widehat{\mathbf{P}}_x^0\big(\widehat{\zeta}^0 < \infty \text{ and } \widehat{X}^0_{\widehat{\zeta}^0-} = a_i\big).$$

Thus (3.2) now follows from (2.26), (2.27). Moreover, we have from (2.33) that for every $i \in \Lambda$,

$$\delta_i + V_i + \sum_{k:\, k\neq i} U^{ik} = \widehat{\delta}_i + \widehat{V}_i + \sum_{k:\, k\neq i} U^{ki}. \tag{3.11}$$

(ii) The uniqueness follows immediately from Theorem 2.6. For existence, let $\{\kappa_i\}$ and $\{\widehat{\kappa}_i\}$ be two sequence of non-negative constants that satisfy (3.3). In the construction of extensions of X^0 and $\widehat{X}^0$ in (i), let δ_1 be the smallest constant that is no less than κ_1 so that there is $\widehat{\delta}_1 \geqslant 0$ satisfying (3.4). Then let δ_2 be the smallest constant that is no less than κ_2 so that there is $\widehat{\delta}_2 \geqslant 0$ satisfying (3.7). Continue this way, we get two sequence of non-negative constants $\{\delta_i\}$ and $\widehat{\delta}_i$ and a pair of right processes $(X^*, \widehat{X}^*)$ that are duality preserving q.e. extensions of $(X^0, \widehat{X}^0)$ on E^* that have killing rates δ_j and $\widehat{\delta}_j$ at a_j for $j \in \Lambda$ and for each $j \in \Lambda$, δ_j is the smallest constant that is no less than κ_j so that there is $\widehat{\delta}_j \geqslant 0$ satisfying

$$\begin{aligned} &\delta_j + L^{0(j-1)}\big(\widehat{\psi}^{((j-1)j)} \cdot m_{0j}, 1 - \psi^{((j-1)j)}\big) \\ &\quad = \widehat{\delta}_j + \widehat{L}^{0(j-1)}\big(\psi^{((j-1)j)} \cdot m_{0j}, 1 - \widehat{\psi}^{((j-1)j)}\big). \end{aligned} \tag{3.12}$$

These $\{\delta_i\}$ and $\widehat{\delta_i}$ have to satisfy (3.11). For each $i \in \Lambda$, we deduce from (3.3) that

$$0 \leqslant \delta_i - \kappa_i = \widehat{\delta_i} - \widehat{\kappa_i}.$$

This together with the minimality of δ_i in (3.11) implies that $\delta_i = \kappa_i$ and $\widehat{\delta_i} = \widehat{\kappa_i}$. This establishes the existence part of (ii). $\square$

Remark 3.2. In view of Remark 2.1, it might be possible to drop the assumption (B.4) from Theorem 3.1. We are grateful to Ron Getoor for his comment on this possibility.

4. Examples

In Section 5 of Chen and Fukushima [4], we have given several examples on one-point extensions of a pair of right processes, which include one-dimensional Brownian motion, reflected and circular Brownian motions, and skew Brownian motion, diffusions on half-lines merging at one point, multidimensional Brownian motions, multidimensional censored stable processes, and multidimensional non-symmetric diffusions. All these examples can be extended to allow darning finite or countably many holes. In the following, we will confine ourselves to the last two cases to illustrate the applicability of Theorem 3.1 of this paper.

4.1. Multidimensional censored stable processes

In this subsection, we consider a censored stable process X^0 on an Euclidean open set D studied in [2]. The process X^0 is symmetric with respect to the Lebesgue measure in D. It is of pure jump type and admits no killings inside E_0.

Let D be an open *n-set* in $\mathbb{R}^n$, that is, there exists a constant $C_1 > 0$ such that

$$m\big(B(x,r)\big) \geqslant C_1 r^n \quad \text{for all } x \in D \text{ and } 0 < r \leqslant 1.$$

Here m is the Lebesgue measure on $\mathbb{R}^n$, $B(x,r) := \{y \in \mathbb{R}^n\colon\ |x-y| < r\}$ and $|\cdot|$ is the Euclidean metric in $\mathbb{R}^n$. Note that bounded Lipschitz domains in $\mathbb{R}^n$ are open n-set and any open n-set with a closed subset having zero Lebesgue measure removed is still an n-set. Fix $0 < \alpha < 2$ and an n-set D (which can be disconnected) in $\mathbb{R}^n$. Define

$$W^{\alpha/2,2}(D) := \left\{u \in L^2(D; dx)\colon \int\limits_{D\times D} \frac{(u(x)-u(y))^2}{|x-y|^{n+\alpha}}\,dx\,dy < \infty\right\},$$

$$\mathcal{E}(u,v) := \mathcal{A}_{n,\alpha} \int\limits_{D\times D} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+\alpha}}\,dx\,dy \quad \text{for } u, v \in \mathcal{F},$$

with

$$\mathcal{A}_{n,\alpha} = \frac{\alpha 2^{\alpha-1}\Gamma(\frac{\alpha+n}{2})}{\pi^{n/2}\Gamma(1-\frac{\alpha}{2})}.$$

When $D = \mathbb{R}^n$, $(\mathcal{E}, W^{\alpha/2,2}(\mathbb{R}^n))$ is just the Dirichlet form on $L^2(\mathbb{R}^n, dx)$ of the symmetric α-stable process on $\mathbb{R}^n$.

We refer the reader to [2] for the following facts. The bilinear form $(\mathcal{E}, W^{\alpha/2,2}(D))$ is a regular irreducible Dirichlet form on $L^2(\overline{D}; 1_D(x)\,dx)$ and the associated Hunt process X on $\overline{D}$ may be called a *reflected α-stable process*. Thus condition (B.5) is satisfied. It is shown in [8] that X has Hölder continuous transition density functions $p(t,x,y)$ with respect to the Lebesgue measure dx on $\overline{D}$ and therefore X can be refined to start from every point in $\overline{D}$. X admits no killing inside $\overline{D}$. Further, X admits no jump from D to ∂D nor from ∂D to ∂D. Let $K = \bigcup_{i\in\Lambda} K_i$ be the union of a finite or countable many disjoint non-trivial compact subsets of ∂D which are locally finite. Then conditions (B.1)–(B.4) are satisfied with $E = \overline{D}$.

We assume each compact set $K_i \subset \partial D$ has finite and strictly positive d_i-dimensional Hausdorff measure when $n \geqslant 2$ and is non-empty when $n = 1$. Note that the subprocess of X killed upon hitting ∂D is the *censored α-stable process* in D that is studied in details in [2]. Let

$$\sigma_K := \inf\{t \geqslant 0\colon\ X_t \in K\} \wedge \zeta,$$

where ζ is the lifetime time of X. It follows from [2, Theorem 2.5 and Remark 2.2(i)] that

$$\phi^{(i)}(x) := \mathbf{P}_x(\sigma_K < \infty \text{ and } X_{\sigma_K-} \in K_i) > 0 \quad \text{for every } x \in \overline{D} \setminus K \tag{4.1}$$

if and only if $\alpha > n - d_i$ when $n \geqslant 2$ and $\alpha > 1$ when $n = 1$. Moreover, with

$$\varphi^{(i)}(x) := \mathbf{P}_x(\sigma_K < \infty \text{ with } X_{\sigma_K-} \in K_i),$$

we have

$$\lim_{x\in D,\ x\to K_i} \varphi^{(i)}(x) = 1 \quad \text{for every } i \in \Lambda.$$

Since X spends zero Lebesgue amount of time on ∂D, from this we can show that

$$\lim_{x\in \overline{D}\setminus K,\ x\to K_i} \varphi^{(i)}(x) = 1 \quad \text{for every } i \in \Lambda.$$

From now we assume that each K_i satisfies the Hausdorff dimensional condition proceeding (4.1) so that $\alpha > n - d_i$ when $n \geqslant 2$ and $\alpha > 1$ when $n = 1$.

Let $X^0 = (X_t^0, \mathbf{P}_x^0, \zeta^0)$ of X killed upon hitting K. By an argument similar to that of [10, Theorem 2.4], one show that X^0 has a symmetric transition density function $p^0(t,x,y)$, which can be represented as

$$p^D(t,x,y) = p(t,x,y) - \mathbf{E}_x\big[p(t-\tau_D, X_{\tau_D}, y); \tau_D < t\big] \quad \text{for } t > 0 \text{ and } x, y \in \overline{D} \setminus K.$$

Moreover, the density function $p^0(t,x,y)$ is continuous on $(0,\infty) \times (\overline{D} \setminus K) \times (\overline{D} \setminus K)$. Thus the conditions $(\mathrm{B}^0.1)$, $(\mathrm{B}^0.2)$ are satisfied. Hence we can apply Theorem 3.1 to get, for each choice of non-negative numbers κ_i, $i \in \Lambda$, the symmetric extension of X^0 possessing them as killing rates by darning the holes $\{K_i,\ i \in \Lambda\}$.

4.2. Multidimensional non-symmetric diffusions

In this subsection, we apply Theorem 3.1 to give an example of multiple-points extensions of non-symmetric diffusions in Euclidean domains. This example is mentioned in [7, Section 6.2] and in [4, Section 5.5].

Let U be a domain in $\mathbb{R}^n$ ($n \geqslant 3$) and m be the Lebesgue measure on U. Let K be a closed subset of U expressible either as (K.1) or (K.2) for $E = U$ stated in the beginning of the preceding section.

Denote $U \setminus K$ by D. Assume that ∂D is regular for Brownian motion, or, equivalently, for $\frac{1}{2}\Delta$. Let

$$\mathcal{L} = \frac{1}{2}\nabla \cdot (a\nabla) + b \cdot \nabla + q = \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right) + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + q,$$

where $a : \mathbb{R}^n \to \mathbb{R}^d \otimes \mathbb{R}^n$ is a measurable, symmetric $(n \times n)$-matrix-valued function which satisfies the uniform elliptic condition

$$\lambda^{-1} I_{n\times n} \leqslant a(\cdot) \leqslant \lambda I_{n\times n}$$

for some $\lambda \geqslant 1$, $b = (b_1, \ldots, b_n) : \mathbb{R}^n \to \mathbb{R}^n$ are measurable functions which could be singular such that $|b|^2 \in \mathbf{K}(\mathbb{R}^n)$ and q is a non-positive measurable function in $\mathbf{K}(\mathbb{R}^n)$ vanishing in a neighborhood of ∂D. Here $\mathbf{K}(\mathbb{R}^n)$ denotes the Kato class functions on $\mathbb{R}^n$. We refer the reader to [9] for its definition. We only mention here that $L^p(\mathbb{R}^n, dx) \subset \mathbf{K}(\mathbb{R}^n)$ for $p > n/2$.

Let $\widehat{q} =: q + \sum_{i=1}^n \frac{\partial b_i}{\partial x_i}$. We assume that $\widehat{q}$ satisfies the condition that

$$\widehat{q} \in \mathbf{K}\big(\mathbb{R}^n\big), \quad \widehat{q} \leqslant 0 \text{ on } \mathbb{R}^n \quad \text{and} \quad \widehat{q} = 0 \text{ in a neighborhood of } \partial D.$$

Under the above condition, the Dirichlet form $(\mathcal{E}, \mathcal{F})$ generated by $(C_c^\infty(U), \mathcal{L})$ is regular on U and satisfies the (generalized) sector condition. Let X be the diffusion in U associated with $(\mathcal{E}, \mathcal{F})$, which can start from every point in U (see [9]). It is clear that X has a weak dual diffusion $\widehat{X}$ in U with respect to the Lebesgue measure m on U whose generator is

$$\widehat{\mathcal{L}} = \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right) - \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + \widehat{q}$$

satisfying zero Dirichlet boundary condition on ∂U. Clearly $\widehat{\mathcal{L}}$ is the dual operator of Dirichlet $\mathcal{L}$ on U. As $(\mathcal{F}, \mathcal{E})$ satisfies the sector condition, it follows from [22] that every semi-polar is m-polar for X; that is, the condition (B.4) is satisfied. Observe that conditions (B.1)–(B.3) and their dual ones are trivially satisfied, while (B.5) and its dual version are satisfied by [9, Lemma 5.7 and Theorem 5.11].

Let X^0 and $\widetilde{X}^0$ be the subprocess of X and $\widehat{X}$, respectively, killed upon leaving D. We assume that each K_i has positive Newtonian capacity and so it is non-polar with respect to X. Then conditions (B^0.1) and (B^0.3) and their dual ones are satisfied by [9, Lemma 5.7 and Theorem 5.11] and the fact that every point in ∂D is regular for Brownian motion.

We can now apply Theorem 3.1 to get, for each choice of $\{\kappa_i, \widehat{\kappa}_i,\ i \in \Lambda\}$ satisfying (3.2), the duality preserving extensions X^* and $\widehat{X}^*$ of X^0 and $\widehat{X}^0$, respectively, to $D^* = D \cup \{a_i\colon 1 \geqslant 1\}$.

Here D^* is the extension of D obtained by regarding each set K_i to be the one point a_i. The process X^* (resp. $\widehat{X}^*$) is a diffusion on D^* but killed at each a_i with killing measure κ_i (resp. $\widehat{\kappa_i}$).

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