Feller's Contributions to the One-Dimensional Diffusion Theory and Beyond

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1 Intrinsic forms of infinitesimal generators

For an open interval $I \subset \mathbb{R}$, we denote by $C_b(I)$ the space of bounded continuous functions on I. Consider a second order elliptic differential operator

$$(1)\qquad \begin{aligned} &\mathscr{L}u(x) = a(x)u''(x) + b(x)u'(x) + c(x)u(x), \quad x \in I, \\ &\text{for} \quad a,\, b,\, c \in C_b(I) \quad \text{with} \quad a > 0,\, c \le 0 \quad \text{on} \quad I. \end{aligned}$$

This differential operator $\mathscr{L}$ can be converted by

$$ds = e^{-B}\,d\xi, \quad dm = \frac{1}{a}e^{B}\,d\xi, \quad dk = -\frac{c}{a}e^{B}\,d\xi$$

where $B(\xi) = \int_\gamma^\xi \frac{b(\eta)}{a(\eta)}d\eta$, $\gamma \in I$, into the operator $\mathcal{A}$ of the *canonical form*

$$(2)\qquad \mathcal{A}u(x) = \frac{dD_s u - u\,dk}{dm}(x), \qquad x \in I.$$

The triplet (s, m, k) appearing in the expression (2) consists of

(3) *canonical scale* s : a strictly increasing continuous function on I,

(4) *speed measure* m : a positive Radon measure on I with full support,

(5) *killing measure* k : a positive Radon measure on I.

We take the domain $\mathscr{D}(\mathcal{A})$ of $\mathcal{A}$ to be

$$(6)\qquad \mathscr{D}(\mathcal{A}) = \left\{ u \in C_b(I) \;\middle|\; \begin{array}{l} du \text{ is absolutely continuous with respect to } ds, \\ du/ds \text{ has a version } D_s u \text{ of bounded variation,} \\ D_s u - u\,dk \text{ is absolutely continuous w. r. t. } m \\ \text{with a Radon–Nikodým derivative in } C_b(I) \end{array} \right\}.$$

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The operator (2) has an obvious advantage over (1) in that it is a *topological invariant*: if the interval I is mapped onto another interval J by any strictly increasing continuous function, then $(\mathcal{A}, \mathscr{D}(\mathcal{A}))$ is transformed into an operator based on J of the same type as (2), while $\mathscr{L}$ in (1) is not – unless the map is differentiable.

Let us consider a Markov process $X = (X_t, \zeta, \mathbb{P}_x)$ on I whose transition function $P_t(x,A) = \mathbb{P}_x(X_t \in A)$ makes the space $C_b(I)$ invariant. X then induces a semigroup on $C_b(I)$: $P_s P_t u = P_{s+t} u,\ 0 < s,t,\ u \in C_b(I)$. Define the *infinitesimal generator* of this transition semigroup by

$$\mathcal{G}u(x) = \lim_{t \downarrow 0} \frac{1}{t}(P_t u(x) - u(x)), \quad x \in I. \tag{7}$$

The domain of the operator $\mathcal{G}$ consists of those functions $u \in C_b(I)$ for which the right hand side converges for every $x \in I$ with the limit function in $C_b(I)$.

In 1931, A. N. Kolmogorov showed that, under certain regularity conditions on the transition function P_t, $\mathcal{G}u$ admits an expression $\mathscr{L}u$ of (1) with $c = 0$ for smooth functions u. The imposed regularity conditions included a *local property of the transition function*:

(8) $P_t(x, U(x)^c) = o(t), \quad t \downarrow 0,$ for any $x \in I$ and for any neigborhood $U(x)$ of x,

that is equivalent to the following *local property of the generator* $\mathcal{G}$:

(9) If $u, v \in \mathscr{D}(\mathcal{G})$ coincide in a neighborhood of $x \in I$, then $\mathcal{G}u(x) = \mathcal{G}v(x)$.

In [Feller 1936c],[1] W. Feller had called a Markov process satisfying condition (8) a *continuous process* and constructed, among other things, the fundamental solution of the parabolic differential equation with (1) on its right-hand side. He allowed (1) to contain a coefficient c of arbitrary sign in order to solve Kolmogorov's backward and forward equations simultaneously.

Soon after moving to Princeton in 1950, Feller resumed his study of the one-dimensional diffusion taking a remarkable lead on the subject. In a series of papers [Feller 1954a, Feller 1955a, Feller 1957a, Feller 1958], he successfully derived the canonical form (2) with $k = 0$ of the infinitesimal generator $\mathcal{G}$ defined by (7) from its local property (9) and its *strong minimum property*:

(10) If u in the domain of $\mathcal{G}$ has a local minimum at $x \in I$, then $\mathcal{G}u(x) \geq 0$.

In view of the definition (7), the last property is obviously satisfied by $\mathcal{G}$ when X is, for instance, conservative ($P_t 1 = 1$).

Thus Feller's method of deriving the canonical form is purely analytic, and yet Feller had clear picture of the sample paths of Markov processes behind his analytic presentations. In [Feller 1954a], he defined a diffusion process to be a Markov process satisfying the local property (8) as in [Feller 1936c], but added a sentence

[1] The references [Feller 19*nn*] and [*Feller 19*nn*] (the star indicating that the respective paper is not

> The functions X_t are continuous with probability one if, and only if, the process is of a local character.

by quoting a paper by D. Ray [13] that was not yet published at that time.
He also wrote in [Feller 1954b] that

> [...] the continuity of X_t is a topological property, whereas a continuous change of scale may change differentiable functions into nondifferentiable ones. It now appears possible to generalize the differential equation $u_t(t,x) = \mathscr{L}_x u(t,x)$ and to reformulate all the results of [Feller 1952a] in a topological invariant form.

From 1955 to 1956, K. Itô stayed in Princeton as a research fellow where he started his collaboration with H. McKean, a Ph.D. student of Feller. In his lecture at Kyoto in 1985 recorded in a video film, Itô recollected the period as follows.

> When I was in Princeton, Feller calculated repeatedly, for a one-dimensional diffusion with a simple generator $\mathscr{L}u = au'' + bu'$, the quantities like
>
> $$s(x) = \mathbb{P}_x(\sigma_\alpha > \sigma_\beta), \quad e(x) = \mathbb{E}_x[\sigma_\alpha \wedge \sigma_\beta], \quad x \in (\alpha, \beta) \subset I,$$
>
> as a harmonic function and a solution of a Poisson equation for $\mathscr{L}$. At first, I wondered why he was repeating so simple computations as exercise, not for objects in higher dimensions. However, in this way, he was bringing out intrinsic topological invariants. I understood that the one-dimensional diffusion is a topological concept but I was not as thoroughgoing as Feller. When Feller told me about this, he said he once listened to a lecture by Hilbert who mentioned: 'Study a quite simple case very much profoundly, then you will truly understand a general case'.

However, Feller was unable to attain by his analytic method the canonical form (2) in that generality involving the killing measure k. A reason for this seems to be the following. The true *minimum property* for the generator $\mathcal{G}$ of a Markov process X allowing a possible finite life-time ($\mathbb{P}_x(\zeta < \infty) > 0$) is

$$\text{If } u \text{ in the domain of } \mathcal{G} \text{ has a local minimum at } x \in I \text{ and } u(x) \leq 0, \text{ then } \mathcal{G}u(x) \geq 0, \tag{11}$$

which is much weaker than (10). But Feller employed an even weaker principle called a *weak minimum property*:

$$\text{If } u \text{ in the domain of } \mathcal{G} \text{ has a local minimum at } x \in I \text{ and } u(x) = 0, \text{ then } \mathcal{G}u(x) \geq 0, \tag{12}$$

contained in these Selecta) refer to Feller's bibliography, while [n] points to the list of references at the end of this essay.

allowing not only killing but also creation of paths so that the corresponding measure k, if it exists, could be a signed measure.

The canonical form (2) was eventually established by McKean [12] by a probabilistic method using results due to Ray [13] and E. B. Dynkin [3] on a *strong Markov process* (a Markov process satisfying the strong Markov property) with continuous sample paths, which is nowadays adopted as a definition of a *diffusion process*. Here the so called *Dynkin formula* expressing $\mathcal{G}u(x)$ in terms of exit times of X from neighborhoods of x played a key role.

To be more precise, let $I = (r_1, r_2)$ and let us start with a minimal diffusion X^0 on I. A Markov process $X^0 = (X_t^0, \zeta^0, \mathbb{P}_x^0)$ on I is called a *minimal diffusion* if

(d.1) X^0 is a Hunt process on I,

(d.2) X^0 is a diffusion process: X_t^0 is continuous in $t \in (0, \zeta^0)$ almost surely,

(d.3) X^0 is irreducible: $\mathbb{P}_x^0(\sigma_y < \infty) > 0$ for any $x, y \in I$.

Denote the one-point compactification of I by $I_\partial = I \cup \{\partial\}$. X_t^0 takes values in I_∂. For $B \subset I_\partial$, we define

$$\sigma_B = \inf\left\{t > 0 : X_t^0 \in B\right\}, \quad \inf\emptyset = \infty, \quad \tau_B = \sigma_{I_\partial \setminus B}.$$

We write σ_B as σ_b when $B = \{b\}$ is a one-point set. $\{\partial\}$ plays the role of cemetery for X^0: $\zeta^0 = \sigma_\partial$, $X_t^0 = \partial$ for any $t \geq \zeta^0$. Condition (**d.1**) means that X^0 is a strong Markov process whose sample paths X_t^0 are right continuous and have left limits on $[0, \infty)$, and are absorbed upon approaching $\{\partial\}$: $\lim_{n\to\infty} \tau_{J_n} = \zeta^0$ whenever $\{J_n\}$ are subintervals of I with $\overline{J}_n \subset I$, $J_n \uparrow I$. X^0 is *minimal* in this sense. Under (**d.1**) and (**d.2**), the condition (**d.3**) is equivalent to the requirement for each point $a \in I$ to be *regular* in the sense that

$$\mathbb{E}_a[e^{-\alpha\sigma_{a+}}] = \mathbb{E}_a[e^{-\alpha\sigma_{a-}}] = 1, \quad \text{for } \alpha > 0,$$

where $E_a[e^{-\alpha\sigma_{a\pm}}] = \lim_{b\to\pm a} E_a[e^{-\alpha\sigma_b}]$.

Let $\{R_\alpha^0; \alpha > 0\}$ be the *resolvent* of a minimal diffusion X^0:

$$R_\alpha^0 f(x) = \mathbb{E}_x^0\left[\int_0^\infty e^{-\alpha t} f(X_t^0)dt\right].$$

Denote by $\mathscr{B}_b(I)$ the space of all bounded Borel measurable functions in I. Then

$$R_\alpha^0(\mathscr{B}_b(I)) \subset C_b(I)$$

due to the above regularity of each point of I. R_α^0 is a one-to-one map from $C_b(I)$ into itself because of the resolvent equation $R_\alpha^0 - R_\beta^0 + (\alpha - \beta)R_\alpha^0 R_\beta^0 = 0$ and

$$\lim_{\alpha\to\infty} \alpha R_\alpha^0 f(x) = f(x), \quad x \in I,\ f \in C_b(I).$$

Thus, the generator $\mathcal{G}^0$ of X^0 is well defined by

$$\text{(13)} \qquad \begin{cases} \mathscr{D}(\mathcal{G}^0) = R_\alpha^0(C_b(I)), \\ (\mathcal{G}^0 u)(x) = \alpha u(x) - f(x) \quad \text{for} \quad u = R_\alpha^0 f,\ f \in C_b(I),\ x \in I; \end{cases}$$

$\mathcal{G}^0$ so defined is independent of $\alpha > 0$. Let us call $\mathcal{G}^0$ the *C_b-generator* of X^0.

In [12, Chapter 4], it was proved that, for a given minimal diffusion X^0, there exist a canonical scale s, a speed measure m and a killing measure k such that, if we define $(\mathcal{A}, \mathscr{D}(\mathcal{A}))$ by (2) and (6) using this triplet (s,m,k), then

$$\mathscr{D}(\mathcal{G}^0) \subset \mathscr{D}(\mathcal{A}) \quad \text{and} \quad \mathcal{G}^0 u = \mathcal{A}u \quad \text{for any} \quad u \in \mathscr{D}(\mathcal{G}^0), \tag{14}$$

namely, $\mathcal{G}^0$ is a restriction of $\mathcal{A}$. In particular, $u = R^0_\alpha f$ for $\alpha > 0$, $f \in C_b(I)$, satisfies an inhomogeneous equation

$$(\alpha - \mathcal{A})u = f. \tag{15}$$

The triplet (s,m,k) is unique up to a multiplicative constant in the sense that, for another such triplet $(\widetilde{s}, \widetilde{m}, \widetilde{k})$, there exists a constant $c > 0$ such that $d\widetilde{s} = c\,ds$, $d\widetilde{m} = c^{-1}\,dm$ and $d\widetilde{k} = c^{-1}\,dk$.

We call (s,m,k) satisfying (14) a *triplet attached to* the minimal diffusion X^0.

The triplet (s,m,k) attached to the minimal diffusion X^0 was constructed in [12, §4.4] probabilistically as follows: For $r_1 < a < b < r_2$ and $J = (a,b)$, consider the hitting probabilities and mean exit time

$$p_{ab}(\xi) = \mathbb{P}^0_\xi(\sigma_a < \sigma_b), \quad p_{ba}(\xi) = \mathbb{P}^0_\xi(\sigma_b < \sigma_a), \quad e_{ab}(\xi) = \mathbb{E}^0_\xi[\tau_J], \quad \xi \in J,$$

and define

$$\begin{cases} s(d\xi) = s_{ab}(d\xi) = p_{ab}(\xi)p_{ba}(d\xi) - p_{ba}(\xi)p_{ab}(d\xi) \\ k(d\xi) = k_{ab}(d\xi) = D_s p_{ab}(d\xi)/p_{ab}(\xi) \\ m(d\xi) = m_{ab}(d\xi) = -\{D_s e_{ab}(d\xi) - e_{ab}(\xi)k_{ab}(d\xi)\}, \quad a < \xi < b. \end{cases}$$

For another choice of $\widetilde{a}$, $\widetilde{b}$ with $r_1 < \widetilde{a} < a < b < \widetilde{b} < r_2$, we have

$$s_{\widetilde{a}\widetilde{b}}(d\xi) = c s_{ab}(d\xi),\ k_{\widetilde{a}\widetilde{b}}(d\xi) = c^{-1}k_{ab}(d\xi),\ m_{\widetilde{a}\widetilde{b}}(d\xi) = c^{-1}m_{ab}(d\xi),\ a < \xi < b,$$

for a constant $c > 0$ depending on a, b, $\widetilde{a}$, $\widetilde{b}$, so that a universal triplet (s,m,k) can be introduced on I. Here, $f(d\xi)$ denotes the measure induced by a function $f(\xi)$ of bounded variation.

In [4, Chap. 15,16], Dynkin gave similar but slightly different representations of the infinitesimal generators of one-dimensional diffusions.

2 Classification of boundaries and lateral conditions

Let $X^0 = (X^0_t, \zeta^0, \mathbb{P}^0_x)$ be a minimal diffusion on an interval $I = (r_1, r_2)$ with attached triplet (s,m,k). Denote by I^* an interval of the type $[r_1, r_2]$ or $[r_1, r_2)$ or $(r_1, r_2]$, and $X = (X_t, \zeta, \mathbb{P}_x)$ be a Markov process on I^* satisfying conditions (**d.1**), (**d.3**) with I^* in place of I. We call X a *Markovian extension* of X^0, if the subprocess of X being killed upon hitting $I^* \setminus I$ is identical with X^0 in law. If X further satisfies condition (**d.2**), then we call it a *diffusion extension* of X^0. The *boundary problem of the one-dimensional diffusion* that Feller first raised and partly solved in [Feller 1952a, Feller 1957a] can be phrased as follows:

$$\text{Characterize and construct all possible Markovian extensions of } X^0 \text{ in terms of } (s,m,k) \text{ and other intrinsic quantities.} \tag{16}$$

For any diffusion extension X of X^0, its C_b-generator $\mathcal{G}$ is well defined by (13) with I^* in place of I. Just as $\mathcal{G}^0$, $\mathcal{G}$ can been seen to be a restriction of $\mathcal{A}$ and, therefore, the problem is equivalent to finding a *lateral condition* Σ imposed on the functions $u \in \mathscr{D}(\mathcal{A})$ such that

$$\mathscr{D}(\mathcal{G}) = \mathscr{D}(\mathcal{A}) \cap \Sigma. \tag{17}$$

Since the resolvent R_α of X satisfies (15) along with R^0_α and the difference $R_\alpha - R^0_\alpha$ is a positive kernel, the problem can be also reduced to the study of behavior of positive solutions u of the homogeneous equation

$$(\alpha - \mathcal{A})u = 0. \tag{18}$$

We write $j = m + k$ and define for $r_1 < c < r_2$

$$\lambda_1 = \int_{r_1}^{c} s(dx) \int_{x}^{c} j(dy), \quad \mu_1 = \int_{r_1}^{c} j(dx) \int_{x}^{c} s(dy), \quad r_1 < c < r_2.$$

The left boundary r_1 of I is called

$$\begin{array}{rlll} \textit{regular} & \text{if} & \lambda_1 < \infty, & \mu_1 < \infty, \\ \textit{exit} & \text{if} & \lambda_1 < \infty, & \mu_1 = \infty, \\ \textit{entrance} & \text{if} & \lambda_1 = \infty, & \mu_1 < \infty, \\ \textit{natural} & \text{if} & \lambda_1 = \infty, & \mu_1 = \infty. \end{array}$$

An analogous classification of r_2 is in force. This classification goes back to the paper [Feller 1952a], where Feller considered a very simple differential operator $\mathscr{L}u = u'' + bu'$ on I so that

$$ds = e^{-B} d\xi, \quad dj = dm = e^{B} d\xi, \quad dk = 0, \tag{19}$$

for $B(\xi) = \int_c^\xi b(\eta) d\eta$, $c \in I$.

In this special case, he made the classification in terms of (19) in accordance with the boundary behavior of positive solutions of the homogeneous equation (18). It was so nice and intrinsic that it was carried over straightforwardly into the case of a general pair (s,m) by [Feller 1957a] and a general triplet (s,m,k) by Itô–McKean [12] where the usage of the terms were somewhat modified, though.

The boundary problem of the one-dimensional diffusion was essentially settled by Itô–McKean [11, 12].

First of all, the generator $\mathcal{G}^0$ of the minimal diffusion X^0 can be characterized simply as follows ([Feller 1952a], [5]):

(20) $\quad \mathscr{D}(\mathcal{G}^0) = \{u \in \mathscr{D}(\mathcal{A}) : \ u(r_i) = 0 \quad \text{whenever } r_i \text{ is regular or exit}, \ i = 1,2\}.$

Next, suppose r_1 is regular and r_2 is natural. Then the most general diffusion extension X of X^0 is an extension to $I^* = [r_1, r_2)$ whose generator $\mathcal{G}$ is characterized as follows ([12], [5]):

$u \in \mathscr{D}(\mathcal{G})$ if and only if

$$u \in C_b(I^*), \quad \frac{dD_s u - u\,dk}{dm} \in C_b(I^*), \tag{21}$$

$$D_s u(r_1) - u(r_1)k^*(r_1) = \mathcal{G}u(r_1)m^*(r_1). \tag{22}$$

where $\mathcal{G}u(r_1)$ denotes the value of the function $(dD_s u - u\,dk)/dm$ (which is in $C_b(I^*)$) at r_1. Here, $k^*(r_1)$ and $m^*(r_1)$ are non-negative constants indicating the killing and sojourn at r_1 of the sample paths of X. $D_s u(r_1)$ indicates the reflection of paths at r_1. We may think that the killing measure k and the speed measure m are extended to $[r_1, r_2)$ by assigning the above point masses at r_1. Boundary conditions analogous to (20) and (22) can be also found in Dynkin [4, Chapters 15, 16].

Let X^0 be the standard absorbing Brownian motion on $I = (0, \infty)$. In this case, $ds = dx$, $dm = 2dx$, $dk = 0$, $\mathcal{A}u = \frac{1}{2}u''$ and $\mathscr{D}(\mathcal{A}) = C_b^2(I)$, the boundary 0 is regular and ∞ is natural. The most general Markovian extension X of X^0 is then an extension to $[0, \infty)$ whose generator $\mathcal{G}$ is characterized as follows ([Feller 1952a], [11]):

$u \in \mathscr{D}(\mathcal{G})$ if and only if $u \in C_b^2[0, \infty)$ and

$$p_1 u(0) - p_2 u'(0) + p_3 \mathcal{G}u(0) = \int_I [u(\ell) - u(0)] p_4(d\ell), \tag{23}$$

where $\mathcal{G}u(0) = \frac{1}{2}u''(0+)$. Here p_1, p_2, p_3 are non-negative constants and p_4 is a measure on I with

$$p_1 + p_2 + p_3 + \int_I (\ell \wedge 1) p_4(d\ell) = 1 \tag{24}$$

and

$$p_4(I) = \infty \quad \text{in case} \quad p_2 = p_3 = 0. \tag{25}$$

Itô and McKean ([11, 12]) call X *Feller's Brownian motion*.

The general boundary condition (23) with p_4 being a finite measure on I appeared already in [Feller 1952a], where Feller also mentioned that the reflection rate p_2 should vanish when r_1 is exit and r_2 is natural. The diffusion extension of X^0 corresponding to the boundary condition (23) with $p_3 = 0$, $p_4 = 0$ is called an *elastic barrier process*, while a Markovian extension of X^0 corresponding $p_1 = p_2 = p_3 = 0$ with finite measure p_4 is called an *elementary return process*. In [Feller 1954b], he stated a plan to construct a general Markovian extension X of X^0 as a limit of a sequence of elementary return processes. Later, Itô recollected in his article [10]

> There was once an occasion when McKean tried to explain to Feller my work on the stochastic differential equations along with the above mentioned idea of tangent. It seemed to me that Feller did not fully understand its significance, but when I explained Lévy's local time to Feller, he immediately appreciated its relevance to the study of the one-dimensional diffusion. Indeed, Feller later gave us a conjecture that the Brownian motion on $[0,\infty)$ with an elastic boundary condition could be constructed from the reflecting barrier Brownian motion by killing it at the time when its local time at the origin exceeds an independent exponentially distributed random time, which was eventually substantiated in my joint paper with McKean.

Thus the most general Markovian extension X on $[0,\infty)$ of the absorbing Brownian motion on I subjected to the boundary condition (23) was constructed in [11] purely probabilistically starting with the reflecting Brownian motion X^r on $[0,\infty)$, making a killing at 0 as described above, and making the sample paths to jump in I from 0 according to the infinite measure p_4 by using an increasing Lévy process.

Furthermore, a similar construction was carried out in [11, §17] for the Brownian motion on $\mathbb{R}$ with *two sided barriers* at 0. Let X^0 be the standard absorbing Brownian motion on $\mathbb{R}\setminus\{0\}$. Then the most general Markovian extension X of X^0 to $\mathbb{R}$ ought to have the generator $\mathcal{G}$ characterized as follows:

$u \in \mathscr{D}(\mathcal{G})$ if and only if

$$u \in C_b^2(\mathbb{R}\setminus\{0\}), \quad u''(0-) = u''(0+)$$

and

$$p_1 u(0) + p_{-2} u^-(0) - p_{+2} u^+(0) + p_3 \mathcal{G} u(0\pm) = \int_{\mathbb{R}\setminus\{0\}} [u(\ell) - u(0)] p_4(d\ell), \tag{26}$$

where $u^-(0)$ (resp. $u^+(0)$) denotes the left (resp. right) derivative at 0, and $\mathcal{G}u(0\pm) = \frac{1}{2}u''(0\pm)$. Here p_1, $p_{\pm 2}$, p_3 are non-negative constants and p_4 is a measure on $\mathbb{R}\setminus\{0\}$ subject to

$$p_1 + p_{-2} + p_{+2} + p_3 + \int_{\mathbb{R}\setminus\{0\}} (|\ell| \wedge 1) p_4(d\ell) = 1 \tag{27}$$

and

$$p_4(\mathbb{R}\setminus\{0\}) = \infty \quad \text{in case} \quad p_{\pm 2} = p_3 = 0. \tag{28}$$

Thus, what is nowadays called a *skew Brownian motion* was studied already in [11] by allowing a killing and a sojourn at 0 as well as a jump into I from 0.

Actually, a boundary condition with two-sided barriers, similar to (26), had appeared in [Feller 1957a, (12.6a), (12.6b)] in order to describe analytically all possible Markovian extensions of an absorbing Brownian motion on a finite interval (β_1, β_2) on the circle $\mathbb{S}$ that is obtained by identifying β_1 and β_2. The corresponding Markov process on $\mathbb{S}$ was also constructed in [11, §17] by projecting a periodic process on $\mathbb{R}$ onto $\mathbb{S}$.

3 Some later developments and Feller measures

Along with his study of the boundary problem of the one-dimensional diffusion, Feller wrote an influential paper [Feller 1957d] in which a boundary problem for a minimal (time continuous) Markov process X^0 on a discrete countable state space E was formulated in an analogous manner to (16). In particular, he introduced what is called a *Feller measure* U (intuitively speaking, measuring the excursion rate from an entrance boundary point to an exit boundary point) as well as a *supplementary Feller measure* τ (measuring the excursion rate from an entrance boundary point never returning back to exit boundary points in finite time) as global intrinsic quantities for X^0 and indicated their possible roles in the study of the boundary problem for a general Markov process. Here the terms *entrance* and *exit* are used in a similar sense to Itô–McKean [12] but differently from Feller's sense mentioned in §2.

The study of the one-dimensional diffusion initiated by [Feller 1952a] was almost finalized until the middle of 1960's. In what follows, I briefly mention some of closely related developments of general features after 1970. Particularly, we will find that the supplementary Feller measure plays important roles even in the study of the one-dimensional diffusion.

3.1 Excursion point processes

Let X^0 (resp. $X^r = (X_t^r, \mathbb{P}_x^r)$) be the absorbing (resp. reflecting) Brownian motion on $(0,\infty)$ (resp. $[0,\infty)$). The transition function of X^0 is denoted by $\{P_t^0,\ t>0\}$. A σ-finite measure $\{\nu_t,\ t>0\}$ is called an X^0-*entrance law* if $\nu_s P_t^0 = \nu_{s+t},\ s,t>0$.

Let $\{\ell_t,\ t\geq 0\}$ be the local time of X^r at the origin 0 normalized in a way that its Revuz measure relative to the (symmetrizing) speed measure $dm = 2dx$ is the Dirac mass at 0. Then its inverse S_t is a subordinator and, by associating with each of its jumping times an excursion of X^r around 0, we can get an excursion-valued point process $(p,\ \mathbb{P}_0^r)$. According to the general theory due to Itô [9], p is a *Poisson point process* whose characteristic measure n called the *excursion law* is a σ-finite measure on the excursion path space

$$W = \Big\{ w : [0,\zeta(w)) \to (0,\infty) \text{ continuous}, \\ 0<\zeta(w),\ w(0+)=0,\ w(\zeta-)\in\{0\}\cup\{\infty\}\Big\},$$

determined by

$$\begin{aligned}(29)\qquad &\int_W f_1(w(t_1))f_2(w(t_2))\cdots f_n(w(t_n))\,n(dw) \\ &= \mu_{t_1} f_1 P^0_{t_2-t_1} f_2 \ldots P^0_{t_{n-1}-t_{n-2}} f_{n-1} P^0_{t_n-t_{n-1}} f_n, \qquad 0<t_1<t_2<\cdots<t_n,\end{aligned}$$

for some X^0-entrance law $\{\mu_t,\ t>0\}$.

In this case μ_t is known to be

$$(30)\qquad \mu_t(dx) = \frac{1}{(2\pi t^3)^{1/2}} x e^{-x^2/(2t)}\, m(dx)$$

so that $\mu_t(dx)/m(dx)$ is the density of the approaching time distribution of X^0 to 0 starting at $x \in (0,\infty)$. Therefore the excursion law n is intrinsically determined by the absorbing Brownian motion X^0, and we can recover from X^0 the reflecting Brownian motion X^r starting at 0 by piecing together the excursions around 0 using the Poisson point process p with characteristic measure n. More than that, we can construct from X^0 Feller's Brownian motion subjected to the boundary condition (23) at once by modifying the excursion law n in a way to allow jumps to the cemetery with rate p_1/p_2 and to $d\ell \subset (0,\infty)$ with rate $p_4(d\ell)/p_2$.

Actually, when X^0 is a minimal diffusion on $(0,\infty)$ with 0 being an exit boundary, Itô [8] constructed and characterized all jump in extensions of X^0 in this way. In this case, no continuous entry from 0 is possible and such extensions of X^0 with finite jump in measures was analytically described in [Feller 1952a] already. A typical example of such a minimal diffusion on $(0,\infty)$ is the celebrated *Feller's branching diffusion* corresponding to $\mathscr{L}u(x) = \beta x u''(x)$, $ds = dx$, $dm = dx/(\beta x)$, $x > 0$, as is presented in Section 3.1 of the essay by Ellen Baake and Anton Wakolbinger in these Selecta.

Next, we consider a general minimal diffusion $X^0 = (X_t^0, \zeta^0, \mathbb{P}_x^0)$ on $(0,\infty)$ with attached triplet (s,m,k) for which 0 is a regular boundary and also a unique diffusion extension $X^r = (X_t^r, \mathbb{P}_x^r)$ of X^0 to $[0,\infty)$ with no killing nor sojourn at 0. Define the approaching probability of X^0 to 0 in finite time by $\varphi(x) = \mathbb{P}_x^0(\zeta^0 < \infty,\ X^0_{\zeta^0-} = 0)$, $x \in (0,\infty)$. When X^0 is the absorbing Brownian motion, $\varphi = 1$ and the entrance law (30) trivially satisfies the equation

$$\int_0^\infty \mu_t \, dt = \varphi \cdot m, \tag{31}$$

but $\varphi \neq 1$ in general. Nevertheless, there exists a unique X^0-entrance law $\{\mu_t,\ t > 0\}$ satisfying (31) due to a general theorem by P. J. Fitzsimmons ([6]). It has been known (see [2, §7.5] and references therein) that the excursion point process $(p, \mathbb{P}_0^r)$ produced by the local time of X^r at 0 is a certain Poisson point process $\widetilde{p}$ being stopped at an independent exponentially distributed random time with rate $L^0(\varphi, 1-\varphi)$ which is the weight of the *supplementary Feller measure* for the one-point set $\{0\}$ intrinsically determined by X^0 that will be explained in the next subsection. Further the characteristic measure n of $\widetilde{p}$ is determined by (29) and (31). Thus, we can recover X^r from X^0 intrinsically just in the same way as above.

3.2 Symmetric Markov processes

Let X^0 be a minimal diffusion on an interval $I = (r_1, r_2)$ with attached triplet (s,m,k). The boundary r_i is called *approachable* if $|s(r_i)| < \infty$, $i = 1,2$. As is shown in [5], X^0 is m-symmetric in the sense that its transition function $\{P_t,\ t > 0\}$ satisfies

$$\int_I P_t f \cdot g \, dm = \int_I g \cdot P_t f \, dm, \quad t > 0,\ f,g \in \mathscr{B}_+(I). \tag{32}$$

Furthermore, the Dirichlet form $(\mathscr{E}^0, \mathscr{F}^0)$ of X^0 on $L^2(I;m)$ can be described in terms of the triplet as follows:

$$\mathscr{F}^0 = \mathscr{F}_0^{s,k}(I) \cap L^2(I;m), \quad \mathscr{E}^0(u,v) = \mathscr{E}_I^{s,k}(u,v), \tag{33}$$

where

$$(34)\qquad \mathscr{F}^{s,k}(I)=\Big\{u : du \text{ is absolutely continuous w. r. t. } ds \text{ and } \mathscr{E}_I^{s,k}(u,u)<\infty\Big\},$$

$$(35)\quad \mathscr{E}_I^{s,k}(u,v)=\int_I D_s u(x) D_s v(x)\,ds(x)+\int_I u(x)v(x)\,dk(x),$$

$$(36)\qquad \mathscr{F}_0^{s,k}(I)=\Big\{u\in\mathscr{F}^{s,k}(I) : u(r_i)=0 \text{ whenever } r_i \text{ is approachable}\Big\}.$$

For a general topological measure space (E,m), the notion of a *Dirichlet form* $(\mathscr{E},\mathscr{F})$ on the space $L^2(E;m)$ was introduced by A. Beurling and J. Deny [1]. M. L. Silverstein, one of the last Ph.D. students of Feller, then introduced the associated fundamental notions of an *extended Dirichlet space* and *reflected Dirichlet space* as well in [14]. The significance of the extended Dirichlet space $(\mathscr{F}_e,\mathscr{E})$ of $(\mathscr{E},\mathscr{F})$ is in that it is an invariant under time changes. It always holds that

$$(37)\qquad \mathscr{F}=\mathscr{F}_e\cap L^2(E;m).$$

If an m-symmetric Hunt process X associated with a regular Dirichlet form $(\mathscr{E},\mathscr{F})$ on $L^2(E;m)$ is changed into $\widetilde{X}$ by a time substitution with respect to its positive continuous additive functional (PCAF) whose Revuz measure $\widetilde{m}$ is of full quasi-support, then $\widetilde{X}$ becomes $\widetilde{m}$-symmetric while its extended Dirichlet space is unchanged so that we only need to replace m by $\widetilde{m}$ in (37) to obtain the Dirichlet form of $\widetilde{X}$ on $L^2(E;\widetilde{m})$ (cf. [2, Chapter 5]). In the special case of the above minimal diffusion X^0, the extended Dirichlet space of $(\mathscr{E}^0,\mathscr{F}^0)$ equals $(\mathscr{F}_0^{s,k}(I),\mathscr{E}_I^{s,k})$ that depends only on s and k. Concurring with a famous saying of Feller, we may say that

> the symmetric Hunt process X travels according to a road map indicated by the Beurling-Deny formula for the extended Dirichlet space $(\mathscr{F}_e,\mathscr{E})$ and with speed indicated by the symmetrizing measure m.

When the quasi-support F of the Revuz measure $\widetilde{m}$ of a time changing PCAF of X is a proper subset of E, things are not that simple. The time-changed process $\widetilde{X}$ on F is then $\widetilde{m}$-symmetric, and the extended Dirichlet space $(\widetilde{\mathscr{F}_e},\widetilde{\mathscr{E}})$ of the Dirichlet form of $\widetilde{X}$ on $L^2(F;\widehat{m})$ is also changed from the original one $(\mathscr{F}_e,\mathscr{E})$ according to the following rule ([2, Chapter 5]): $\widetilde{\mathscr{F}_e}$ is the restriction of $\mathscr{F}_e$ to F and $\widetilde{\mathscr{E}}(u,v)$, $u,v\in\widetilde{\mathscr{F}_e}$, is the sum of the due restriction of $\mathscr{E}(u,v)$, $u,v\in\mathscr{F}_e$, to F and the integral form

$$(38)\qquad \frac{1}{2}\int_{F\times F\setminus\Delta}(u(\xi)-u(\eta))(v(\xi)-v(\eta))\,U(d\xi d\eta)+\int_F u(\xi)v(\xi)\,V(d\xi).$$

Here U is a measure on $F\times F$ off the diagonal $\Delta\subset F\times F$, and V is a measure on F specified by (39) below. Let $X^0=(X_t^0,\zeta^0,\mathbb{P}_x^0)$ be the subprocess of X on $E_0=E\setminus F$ obtained by killing upon the hitting time σ_F of F. Define $H_F f(x)=\mathbb{E}_x^0[f(X^0_{\zeta^0-});\,\zeta^0<\infty]$, $x\in E_0$, for $f\in\mathscr{B}_+(F)$. Denote by $L^0(f,g)$ the *energy functional* of a purely excessive function f and an excessive function g relative to X^0 (cf.

[6], [2, §5.4]). Then, for $f,g \in \mathscr{B}_+(F)$ with $fg = 0$,

$$(39)\qquad \begin{aligned} \int_{F\times F} f(\xi)g(\eta)U(d\xi d\eta) &= L^0(H_F f, H_F g), \\ \int_F f(\xi)V(d\xi) &= L^0(H_F f, 1 - H_F 1_F). \end{aligned}$$

We call U and V the *Feller measure* and the *supplementary Feller measure*, respectively, because they are analogous to the quantities U, τ introduced by [Feller 1957d].

Let us consider the case where X is a minimal diffusion on $\mathbb{R}$ with attched triplet (s,m,k), $F = [0,\infty)$ and $\widetilde{m}$ is a positive Radon measure on $\mathbb{R}$ with support F. For simplicity, we assume that ∞ is non-approachable: $s(\infty) = \infty$. The PCAF of X with Revuz measure $\widetilde{m}$ is then given by $A_t = \int_F \ell_t(x)\widetilde{m}(dx)$ where $\ell_t(x)$ is the local time of X with Revuz measure δ_x relative to m. The subprocess X^0 of X on $E_0 = (-\infty, 0)$ is the minimal diffusion on E_0 with attached triplet consisting of the restrictions to E_0 of s, m and k. Then the measure $H_F(x,\cdot)$, $x \in E_0$, is concentrated on the one-point set $\{0\}$ so that the Feller measure U vanishes and the *supplementary Feller measure V* has the expression

$$(40)\qquad V = L^0(\varphi, 1-\varphi)\delta_0,$$

where L^0 is the energy functional for X^0 and $\varphi(x) = \mathbb{P}^0_x(\zeta^0 < \infty, X^0_{\zeta^0-} = 0)$, $x \in E_0$.

The time-changed process $\widetilde{X}$ of X by means of A is therefore an $\widetilde{m}$-symmetric diffusion on $[0,\infty)$ whose extended Dirichlet space $(\widetilde{\mathscr{F}_e}, \widetilde{\mathscr{E}})$ is given by

$$(41)\qquad \widetilde{\mathscr{F}_e} = \mathscr{F}^{s,k}([0,\infty)),$$

$$(42)\qquad \widetilde{\mathscr{E}}(u,v) = \mathscr{E}^{s,k}_{[0,\infty)}(u,v) + L^0(\varphi, 1-\varphi)u(0)v(0).$$

Using a similar method as in [5], we can also deduce from this the following characterization of the C_b-generator $\widetilde{\mathfrak{G}}$ of $\widetilde{X}$:

$u \in \mathscr{D}(\widetilde{\mathfrak{G}})$ if and only if

$$(43)\qquad u \in C_b([0,\infty)), \qquad \frac{dD_s u - u\,dk}{d\widetilde{m}} \in C_b([0,\infty)),$$

$$(44)\qquad D_s u(0) - u(0)\left[k(\{0\}) + L^0(\varphi, 1-\varphi)\right] = \widetilde{\mathfrak{G}}u(0)\widetilde{m}(\{0\}).$$

Itô–McKean [12, §5.3] obtained the above charcterization in the special case that X is a standard Brownian motion on $\mathbb{R}$. In this case, $k = 0$ and also $L^0(\varphi, 1-\varphi) = 0$ because $\varphi(x) = 1$ for $x < 0$. But, if $|s(-\infty)| < \infty$ and $m(-\infty, c) = \infty$ for $c < 0$, then $\varphi(x) < 1$ for $x < 0$, and accordingly $L^0(\varphi, 1-\varphi)$ does not vanish. For the higher dimensional Brownian motion, the jump term with the Feller measure U really appears in the Dirichlet form under a time change by a PCAF of non-full support (cf. [2, §5.8]).

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