# BV functions and distorted Ornstein Uhlenbeck processes over the abstract Wiener space

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**Abstract**

Let $(E, H, \mu)$ be an abstract Wiener space and $\mathbb{H}$ be the class of functions $\rho \in L^1_+(E;\mu)$ satisfying the ray Hamza condition in every direction $\ell \in E^*$. For $\rho \in \mathbb{H}$, the closure $(\mathcal{E}^\rho, \mathcal{F}^\rho)$ of the symmetric form

$$\mathcal{E}^\rho(u,v) = \int_E \langle \nabla u(z), \nabla v(z) \rangle_H \; \rho(z) \; \mu(dz), \quad u, v \in \mathcal{F}C_b^1.$$

is a quasi-regular Dirichlet form on $L^2(F, \rho d\mu)$, $(F = \mathrm{Supp}[\rho\mu])$, yielding an associated diffusion $\mathbf{M}^\rho = (X_t, P_z)$ on $F$ called a *distorted Ornstein Uhlenbeck process.* A function $\rho$ on $E$ is called *a BV function* ($\rho \in BV(E)$ in notation) if $\rho \in \cup_{p>1} L^p(E;\mu)$ and

$$V(\rho) = \sup_{G \in (\mathcal{F}C_b^1)_{E^*}, \|G\|_H(z) \le 1} \int_E \nabla^* G(z) \rho(z) \mu(dz)$$

is finite. For $\rho \in \mathbb{H} \cap BV(E)$, there exist a positive finite measure $\|D\rho\|$ on $F$ and a weakly measurable function $\sigma_\rho : F \longrightarrow H$ such that $\|\sigma_\rho(z)\|_H = 1$ $\|D\rho\|$-a.e. and

$$\int_F \nabla^* G(z) \rho(z) \mu(dz) = \int_F \langle G(z), \sigma_\rho(z) \rangle_H \|D\rho\|(dz), \quad \forall G \in (\mathcal{F}C_b^1)_{E^*}.$$

Further, the sample path of $\mathbf{M}^\rho$ admits an expression as a sum of $E$-valued CAF's:

$$X_t - X_0 = W_t - \frac{1}{2} \int_0^t X_s ds + \frac{1}{2} \int_0^t \sigma_\rho(X_s) dL_s^{\|D\rho\|}$$

where $W_t$ is an $E$-valued Brownian motion and $L_t^{\|D\rho\|}$ is a PCAF of $\mathbf{M}^\rho$ with Revuz measure $\|D\rho\|$. A measurable set $\Gamma \subset E$ is called *Caccioppoli* if $I_\Gamma \in BV(E)$. In this case, the support of the measure $\|DI_\Gamma\|$ is concentrated in $\partial\Gamma$ and the above equations reduce to the Gauss formula and the Skorohod equation for the modified reflecting Ornstein Uhlenbeck process respectively. A related coarea formula is also presented.

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# 1 Introduction

The reflecting Brownian motion for a bounded domain $D \subset R^d$ is by definition a symmetric conservative diffusion process $\mathbf{M} = (X_t, P_x)$ on a compactification $D^*$ of $D$ such that its Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(D^*) = L^2(D)$ is regular and given by

$$\mathcal{E}(u,v) = \frac{1}{2}\int_D \nabla u(x) \cdot \nabla v(x) dx \quad \mathcal{F} = H^1(D). \tag{1.1}$$

The first construction of such process goes back to [Fu 67]. By the decomposition theorem of additive functionals formulated in [Fu 80], the sample path of $\mathbf{M}$ admits an expression

$$X_t - X_0 = B_t + N_t$$

where $B_t$ is a $d$-dimensional Brownian motion and each component of $N_t$ is a CAF of zero energy.

Dealing with a reflecting Brownian motion on a Lipschitz domain $D$ (in this case $D^* = \bar{D}$), Bass and Hsu [BaHs 90] observed that a semimartingale criteria in [Fu 80] combined with the Gauss formula

$$\int_D \text{div } v \, dx = -\int_{\partial D} v \cdot \mathbf{n}(x) S(dx) \quad v \in C^1(R^d; R^d) \tag{1.2}$$

leads us to a Skorohod type expression

$$N_t = \frac{1}{2}\int_0^t \mathbf{n}(X_s) dL_s, \tag{1.3}$$

by means of a positive continuous additive functional $L_t$ of $\mathbf{M}$ with Revuz measure being the surface measure $S$. Chen, Fitzsimmons and Williams [CFW 93] then treated a general bounded domain $D$ and proved that the expression (1.3) holds if and only if $D$ is strong Caccioppoli and in this case $L$ corresponds to a surface measure $S$ on $D^* \setminus D$ appearing in the generalized Gauss formula. Here a semimartingale criteria in [Fu 80] was considerably improved in that the smoothness requirement for $S$ was removed by showing that the smoothness is rather a consequence of the validity of the Gauss formula.

In author's recent paper [Fu 99a], this sort of improvements of the semimartingale characterizations of the additive functionals are thoroughly extended to a general quasi-regular Dirichlet form setting and applied to establishing stochastic characterizations of BV functions and Caccioppoli sets on $R^d$ in terms of distorted Brownian motions and modified reflecting Brownian motions. If a non-negative function $\rho \in L^1_{loc}(R^d)$ satisfies the Hamza type condition, then the form

$$\mathcal{E}^\rho(u,v) = \frac{1}{2}\int_{R^d} \nabla u(x) \cdot \nabla v(x) \, \rho(x) \, dx \quad u, v \in C_0^1(R^d) \tag{1.4}$$

is closable on $L^2(R^d; \rho dx)$ and the closure is a regular local Dirichlet form on $L^2(F; \rho dx)$ where $F$ is the support of the measure $\rho dx$. The associated diffusion process on $F$ is called a distorted Brownian motion. The modified reflecting Brownian motion corresponds to the case where $\rho(x) = I_D(x)$. In this case, the Dirichlet space $\mathcal{F}^{I_D}$ could be a proper subspace of the Sobolev space $H^1(D)$ and hence the term 'modified' is added.

In the present paper, we shall apply the general theory in [Fu 99a] to the typical infinite dimensional situation, namely, the abstract Wiener space setting $(E, H, \mu)$. Here the counterparts of the form (1.4) have been intensively studied under the name of classical Dirichlet forms by Albeverio, Röckner, Ma and Schmuland, and their basic properties such as closability, quasi-regularity, association of diffusions etc. are well understood ([AR 90], [MR 92], [RS 92]). Furthermore we have here a counterpart of $-\mathrm{div}$ the dual $\nabla^*$ of the $H$-derivative $\nabla$ well utilized in the Malliavin calculus ([M 97], [IW 89]):

$$\int_E \nabla^* G(z)\rho(z)\mu(dz) = \int_E \langle G(z), \nabla\rho(z)\rangle_H \; \mu(dz), \quad G \in (\mathcal{F}C_b^1)_{E^*},\ \rho \in \mathcal{F}C_b^1.$$

Thus we can extend some basic notions and relations in the geometric measure theory ( [Fe 69], [G 84], [EG 92]) together with their stochastic contents to this infinite dimensional situation.

# 2 Classical Dirichlet forms and distorted Ornstein Uhlenbeck processes

Let $(E, H, \mu)$ be an abstract Wiener space. By definition, $E$ is a separable Banach space, $H$ is a separable Hilbert space densely and continuously embedded into $B$ and $\mu$ is a Gaussian measure over $E$ satisfying

$$\int_E e^{\sqrt{-1}l(z)}\mu(dz) = \exp(-\frac{1}{2}\|\ell\|_H^2), \quad \ell \in E^*. \tag{2.1}$$

By the identification $H^* = H$, $E^*$ is viewed as a dense linear subspace of $H$ so that $\ell(z) = \langle \ell, z\rangle_H$ whenever $\ell \in E^*, z \in H$, where $\langle\cdot,\cdot\rangle_H$ denotes the $H$-inner product. We let

$$\mathcal{F}C_b^1 = \{u : u(z) = f(\ell_1(z), \ell_2(z), \cdots, \ell_m(z)),\ z \in E,\ \ell_1, \ell_2, \cdots, \ell_m \in E^*,\ f \in C_b^1(\mathbb{R}^m)\}. \tag{2.2}$$

We denote by $\nabla u$ the $H$-derivative of $u \in \mathcal{F}C_b^1$, namely, it is a map from $E$ to $H$ such that

$$\langle \nabla u(z), \ell\rangle_H = \partial_\ell u(z), \quad \forall \ell \in E^*,$$

where $\partial_\ell u(z)$ is the derivative of $u$ at $z$ in direction $\ell$, so that, for $u$ expressed as in (2.2)

$$\partial_\ell u(z) = \sum_{j=1}^m \partial_j f(\ell_1(z), \cdots, \ell_m(z))\langle \ell_j, \ell\rangle_H.$$

For $p \geq 1$, $L^p(E;\mu)$ denotes the space of $\mu$-measurable real valued functions $u$ on $E$ such that $|u|^p$ is $\mu$-integrable. $L^p_+(E;\mu)$ denotes the set of all non-negative elements in $L^p(E;\mu)$. We now introduce a important subfamily of $L^1_+(E;\mu)$.

A non-negative measurable function $h(s)$ on $\mathbb{R}^1$ is said to possess *Hamza property* if $h(s) = 0$ $ds$-a.e. on the closed set $\mathbb{R}^1 - R(h)$ where

$$R(h) = \left\{ s \in \mathbb{R}^1 : \int_{x-\epsilon}^{x+\epsilon} \frac{1}{h(s)} ds < \infty \ \exists \epsilon > 0 \right\}.$$

We say that a function $\rho \in L^1_+(E;\mu)$ satisfies *ray Hamza condition in direction* $\ell \in E^*$ ($\rho \in \mathbb{H}_\ell$ in notation) if there exists a non-negative function $\tilde{\rho}$ such that

$$\tilde{\rho} = \rho \ \mu - a.e. \quad \tilde{\rho}(z + s\ell) \text{ has Hamza property in } s \in \mathbb{R}^1 \text{ for each } z \in E. \tag{2.3}$$

We set

$$\mathbb{H} = \cap_{\ell \in E^*} \mathbb{H}_\ell.$$

A function in the family $\mathbb{H}$ is simply said to satisfy *ray Hamza condition.*

The Hamza property for a function on $\mathbb{R}^1$ is quite mild; any non-negative lower semicontinuous function has this property. Thus any *ray lower semicontinuous function* $\rho \in L^1_+(E;\mu)$ defined in an analogous manner to the above belongs to the family $\mathbb{H}$. If $\rho \in L^1_+(E;\mu)$ is ray lower semicontinuous, the indicator function $I_{E_t}$ of the level set of the type

$$E_t = \{z \in E : \rho(z) > t\}$$

is also in $\mathbb{H}$ for each $t > 0$. The indicator function $I_\Gamma$ of any open set $\Gamma \subset E$ is in $\mathbb{H}$.

The notion of *ray absolute continuity* was first introduced in [K 82]. We denote by $\mathbb{D}^{r,p}(E)$, $r > 0$, $p \geq 1$ the Sobolev spaces over the abstract Wiener space $(E, H, \mu)$ ([IW 89], [W 84]). The family of non-negative functions in the respective space will be designated by adding the subscript $+$. It is known that any function in $\mathbb{D}^{1,p}(E)$, $p > 1$, is ray absolutely continuous [Su 85]. Furthermore, any function in $\mathbb{D}^{r,p}(E)$, $\frac{1}{p} < r < 1$, is ray Hölder continuous [RR]. Therefore we have the inclusion

$$\mathbb{D}^{r,p}_+(E) \subset \mathbb{H}, \quad \frac{1}{p} < r \leq 1, \ p > 1. \tag{2.4}$$

$\mathbb{H}$ also contains the indicator functions of level sets of functions in the above spaces.

For each $\rho \in \mathbb{H}$, we let

$$\mathcal{E}^\rho(u, v) = \frac{1}{2} \int_E \langle \nabla u(z), \nabla v(z) \rangle_H \ \rho(z) \ \mu(dz), \quad u, v, \mathcal{F}C^1_b. \tag{2.5}$$

Owing to the work [AR 90], we know that $\mathcal{E}^\rho$ with domain $\mathcal{F}C^1_b$ is a well defined and closable symmetric form on $L^2(E; \rho \cdot \mu)$. Its closure is denoted by $(\mathcal{E}^\rho, \mathcal{F}^\rho)$. This is a special case of the *classical Dirichlet forms* studied in [AR 90]. We let

$$F = Supp[\rho \cdot \mu], \tag{2.6}$$

namely, $F$ is the smallest closed subset of $E$ such that $\int_{E \setminus F} \rho(z) \mu(dz) = 0$.

**Theorem 2.1** *Let $\rho \in \mathbb{H}$. $(\mathcal{E}^\rho, \mathcal{F}^\rho)$ is then a quasi-regular local Dirichlet form on $L^2(F; \rho \cdot \mu)$.*

*Proof.* This has been proved in [MR 92, IV,4b] under the assumption that $F = E$. The proof works without this assumption (see [RS 92] for the proof of capacitary tightness without this assumption). Thus $(\mathcal{E}^\rho, \mathcal{F}^\rho)$ is a quasi-regular local Dirichlet form on $L^2(E; \rho \cdot \mu)$. Since $\mathcal{F}_F = \mathcal{F}$ however, $E \setminus F$ is an open $\mathcal{E}$-exceptional set according to the definition. Hence we can restrict the underlying space $E$ to $F$ without violating the quasi-regularity and the locality of $(\mathcal{E}^\rho, \mathcal{F}^\rho)$. □

By fixing a function $\rho \in \mathbb{H}$, let us state some relevant stochastic contents. By virtue of Theorem 2.1 and [MR 92] (see also [Fu 99a]), there exists a diffusion process $\mathbf{M}^\rho = (X_t, P_z)$ on $F$ associated with the Dirichlet form $(\mathcal{E}^\rho, \mathcal{F}^\rho)$. $\mathbf{M}^\rho$ will be called a *distorted Ornstein-Uhlenbeck process.* The reason of this naming will be clearer in the next section. Since constant functions are in $\mathcal{F}^\rho$ and $\mathcal{E}^\rho(1,1) = 0$, $\mathbf{M}^\rho$ is recurrent and conservative.

The totality of positive continuous additive functionals (PCAF's) of $\mathbf{M}^\rho$ is denoted by $\mathbf{A}^+$. The space of CAF's of bounded variation can be identified with the class

$$\mathbf{A} = \mathbf{A}^+ - \mathbf{A}^+. \tag{2.7}$$

For $A \in \mathbf{A}$, its total variation process is denoted by $\{A\}$, which is an element of $\mathbf{A}^+$. We will be concerned with a subclass of $\mathbf{A}$ defined by

$$\mathbf{A}_0 = \{A \in \mathbf{A} : E_{\rho\mu}(\{A\}_t) < \infty, \quad \forall t > 0\}. \tag{2.8}$$

By the Revuz correspondence, the family $\mathbf{A}^+$ is in one to one correpondence with the family $S^+$ of positive ($\mathcal{E}^\rho$-)smooth measures on $F$ (see [Fu 99a]). Accordingly $\mathbf{A}$ is in one to one correspondence with $S = S^+ - S^+$. The element of $S$ is called a *smooth signed measure* and particularly it charges no set of zero $\mathcal{E}_1^\rho$-capacity. The element of $\mathbf{A}$ corresponding to $\nu \in S$ will be denoted by $A^\nu$.

Notice that, for each $\ell \in E^*$, the function $u(z) = \ell(z)$ belongs to the Dirichlet space $\mathcal{F}^\rho$ and

$$\mathcal{E}^\rho(\ell(\cdot), v) = \frac{1}{2} \int_E \partial_\ell v(x) \rho(z) d\mu(z) \quad \forall v \in \mathcal{F}C_b^1. \tag{2.9}$$

On the other hand, the composite AF $\ell(X_t) - \ell(X_0)$ of $\mathbf{M}^\rho$ admits a decomposition into a sum of a martingale AF of finite energy and CAF of zero energy ([Fu 99a]). Let us write the decomposition as follows:

$$\ell(X_t) - \ell(X_0) = M_t^\ell + N_t^\ell. \tag{2.10}$$

Now, for $\rho \in L^1(E; \mu)$ and $\ell \in E^*$, we say that $\rho$ *is of bounded variation in direction* $\ell$ ( $\rho \in BV_\ell(E)$ in notation) if

$$\left| \int_E \partial_\ell v(z) \rho(z) d\mu(z) \right| \le C \|v\|_\infty, \quad \forall v \in \mathcal{F}C_b^1. \tag{2.11}$$

For some positive constant $C$.

On account of the above observations, we can use [Fu 99a, Th. 6.2] or its slight modification [Fu 99b, Th. Th. 2.2] in getting the following:

**Theorem 2.2** *Let $\rho \in \mathbb{H}$ and $\ell \in E^*$.*
1. *The next three conditions are equivalent each other:*
(i) $N^\ell \in \mathbf{A}_0$.
(ii) $\rho \in BV_\ell(E)$.
(iii) *There exists a finite signed measure $\nu_\ell$ on $F$ such that*

$$\mathcal{E}^\rho(\ell(\cdot), v) = -\int_F v(z) \nu_\ell(dz) \quad v \in \mathcal{F}C_b^1. \tag{2.12}$$

*In this case, $\nu_\ell$ is automatically smooth, the equation* (2.12) *extends to any $\mathcal{E}^\rho$-quasicontinuous function $v \in \mathcal{F}_b^\rho$ and*

$$N^\ell = A^{\nu_\ell}. \tag{2.13}$$

2. *$M^\ell$ is a martingale AF with the quadratic variation process*

$$\langle M^\ell \rangle_t \; = \; t \, \|\ell\|_H, \quad t \geq 0. \tag{2.14}$$

Note that, in view of the expression (2.9), the energy measure $\mu_{\langle \ell \rangle}$ of $\ell(z) \in \mathcal{F}^\rho$ equals $\|\ell\|_H \rho(z) \cdot \mu$, from which follows the second statement of the theorem ([Fu 99a]).

In the rest of this section, we shall present some explicit description of the Dirichlet form $(\mathcal{E}^\rho, \mathcal{F}^\rho)$ for $\rho \in \mathbb{H}$, which will be utilized in §4.

First of all, we fix $\ell \in E^*$ with $\|\ell\|_H = 1$ and we set

$$H_\ell = \{s\ell : s \in \mathbb{R}^1\} (\subset H), \quad E_\ell = \overline{H \ominus H_\ell},$$

where the closure is taken in the Banach space $E$. We have then the direct sum decomposition $E = H_\ell \oplus E_\ell$ given by

$$z = s\ell + x, \quad z \in E, \; s = \ell(z), \; x = z - \ell(z)\ell.$$

Let $\pi$ be the projection onto the space $E_\ell$ and $\mu_\ell$ be the image measure of $\mu$ by $\pi$: $\mu_\ell = \pi\mu$. Then we see ([Shi 80]) for any non-negative measurable function $F(z)$ that

$$\int_E F(z)\mu(dz) = \int_{E_\ell} \int_{\mathbb{R}^1} F(s\ell + x) p(s) ds \mu_\ell(dx), \tag{2.15}$$

where $p(s) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{s^2}{2})$.

By Fubini's theorem, we see that $\rho \in \mathbb{H}_\ell$ if and only if there exists a Borel set $N \subset E_\ell$ with $\mu_\ell(N) = 0$ such that

$$\rho(s\ell + x) \text{ has the Hamza property in } s \in \mathbb{R}$$

for each $x \in E_\ell \setminus N$. By redefining $\rho(s\ell + x) = 0$, $\forall (x, s) \in N \times \mathbb{R}$, we can and we shall assume that any $\rho \in \mathbb{H}_\ell$ enjoys the above property for every $x \in E_\ell$. With each $\rho \in \mathbb{H}_\ell$, we now associate a symmetric form $(\check{\mathcal{E}}^{\rho,\ell}, \check{\mathcal{F}}^{\rho,\ell})$ defined by

$$\begin{aligned} &\check{\mathcal{F}}^{\rho,\ell} \\ &= \; \{ \; u \in L^2(E; \rho d\mu) : \exists \tilde{u} = u \; \rho d\mu - a.e. \\ &\qquad \tilde{u}(s\ell + x) \text{ is absolutely continuous in } \; s \text{ on } R(\rho(\cdot \ell + x)) \; \text{ for each } x \in E_\ell \\ &\qquad \text{and} \quad \int_{E_\ell} \int_{R(\rho(\cdot \ell + x))} \left( \frac{d\tilde{u}(s\ell + x)}{ds} \right)^2 \rho(s\ell + x) p(s) ds \mu_\ell(dx) < \infty \; \}, \end{aligned} \tag{2.16}$$

$$\begin{aligned} &\check{\mathcal{E}}^{\rho,\ell}(u, v) \\ &= \; \frac{1}{2} \int_{E_\ell} \int_{R(\rho(\cdot \ell + x))} \frac{d\tilde{u}(s\ell + x)}{ds} \; \frac{d\tilde{v}(s\ell + x)}{ds} \rho(s\ell + x) p(s) ds \mu_\ell(dx), \quad u, v \in \check{\mathcal{F}}^{\rho,\ell}. \end{aligned} \tag{2.17}$$

By virtue of [AR 90, Th.3.10], we then have

**Proposition 2.1** *For $\rho \in \mathbb{H}$, the Dirichlet form $(\mathcal{E}^\rho, \mathcal{F}^\rho)$ of* Theorem 2.1 *enjoys the following properties:*

$$\mathcal{F}^\rho \subset \cap_{\ell \in E^*} \check{\mathcal{F}}^{\rho,\ell}, \tag{2.18}$$

*For each choice of $H$-c.o.n.s. $\{\ell_j\}_{j=1,2,\cdots} \subset E^*$,*

$$\mathcal{E}^\rho(u,v) = \sum_{j=1}^\infty \check{\mathcal{E}}^{\rho,\ell_j}(u,v), \quad u,v \in \mathcal{F}^\rho. \tag{2.19}$$

**Remark 2.1** [AR 90, Th.3.2] can be read in the present case as follows: For $\rho \in L^1_+(E;\mu)$, the form

$$\mathcal{E}^{\rho,\ell}(u,v) = \frac{1}{2}\int_E \partial_\ell u \, \partial_\ell v d\mu, \quad u,v \in \mathcal{F}C_b^1, \tag{2.20}$$

is a well defined and closable symmetric form on $L^2(E;\rho\mu)$ if and only if $\rho \in \mathbb{H}_\ell$. In this case, the form $(\check{\mathcal{E}}^{\rho,\ell}, \check{\mathcal{F}}^{\rho,\ell})$ defined as above is closed on $L^2(E;\rho\mu)$ and is an extension of the form (2.20).

Under the assumption that $\rho \in \mathbb{H}_\ell$, [AR 90] gave a condition for $u \in L^2(E;\rho d\mu)$ to be in the space $\check{\mathcal{F}}^{\rho,\ell}$ in an apparently weaker way than (2.16) as follows:

$$\begin{aligned}
&\text{for } \mu_\ell\text{–a.e.fixed } x \in E_\ell,\ \exists \tilde{u}(x,s) = u(x+s\ell) \ \ ds\text{–a.e.on } R(\rho(\cdot\ell + x)) \\
&\tilde{u}(x,s) \text{ is absolutely continuous in } \ s \text{ on } R(\rho(\cdot\ell+x)) \\
&\text{and} \quad \int_{R(\rho(\cdot\ell+x))} \left(\frac{d\tilde{u}(x,s)}{ds}\right)^2 \rho(s\ell+x)p(s)ds \in L^1(E_\ell;\mu_\ell).
\end{aligned} \tag{2.21}$$

Actually this condition is equivalent to the one in (2.16). Indeed, suppose $u$ satisfies condition (2.21). Take a Borel exceptional set $N \subset E_\ell$ for $u$ and let

$$\Gamma = \{(x,s) : x \in E_\ell - N,\ s \in R(\rho(\cdot\ell+x))\}, \quad v(x,s) = \tilde{u}(x,s) I_\Gamma(x,s).$$

Then $\Gamma$ is measurable set of $E_\ell \times \mathbb{R}$ and

$$v(x,s) = \lim_{k\to\infty} \frac{k}{2}\int_{s-\frac{1}{k}}^{s+\frac{1}{k}} \tilde{u}(x,t)dt \cdot I_\Gamma(x,s) = \lim_{k\to\infty}\frac{k}{2}\int_{s-\frac{1}{k}}^{s+\frac{1}{k}} u(x+t\ell)dt \cdot I_\Gamma(x,s).$$

By the last expression of the above identity, we see that $v(x,s)$ is jointly measurable in $(x,s)$. We can then readily see that the function defined by

$$\tilde{u}(z) = v(z - \ell(z)\ell, \ell(z))$$

satisfies condition (2.16).

## 3 BV functions and distorted Ornstein Uhlenbeck processes

We continue to work with the abstract Wiener space $(E, H, \mu)$. Let us introduce a family of $E^*$-valued functions on $E$ by

$$(\mathcal{F}C_b^1)_{E^*} = \{G : G(z) = \sum_{j=1}^m g_j(z)\ell_j, \quad g_j \in \mathcal{F}C_b^1,\ \ell_j \in E^*\}. \tag{3.1}$$

Denote by $\nabla^*$ the dual of the H-derivative $\nabla$ ([IW 89]): $\nabla^*$ is a linear map from $(\mathcal{F}C_b^1)_{E^*}$ to $\mathcal{F}C_b^1$ such that

$$\int_E \nabla^* G(z)\rho(z)\mu(dz) = \int_E \langle G(z), \nabla\rho(z)\rangle_H \mu(dz),\ G \in (\mathcal{F}C_b^1)_{E^*},\ \rho \in \mathcal{F}C_b^1. \tag{3.2}$$

$\nabla^*$ is an infinite dimensional variant of $-div$. The formula (3.2) is exhibited in [IW 89, (8.23)] holding for $G$ in the space of smooth functionals $\mathbf{S}$ but it can be readily seen to hold for $G \in \mathcal{F}C_b^1$.

For $\rho \in \cup_{p>1} L^p(E; \mu)$, we put

$$\sup_{G \in (\mathcal{F}C_b^1)_{E^*}, \|G\|_H(z) \le 1} \int_E \nabla^* g(z)\rho(z)\mu(dz) = V(\rho). \tag{3.3}$$

A function $\rho$ on $E$ is said to be *of bounded variation* ($\rho \in BV(E)$ in notation) if $\rho \in \cup_{p>1} L^p(E; \mu)$ and $V(\rho)$ is finite.

**Theorem 3.1** (i) $BV(E) \subset \cap_{\ell \in E^*} BV_\ell(E)$.

(ii) *Suppose $\rho \in \mathbb{H}$. If $\rho \in BV(E)$, then there exist a positive finite measure $\|D\rho\|$ on $E$ and a weakly measurable function $\sigma_\rho : E \longrightarrow H$ such that $\|\sigma_\rho(z)\|_H = 1$ $\|D\rho\|$-a.e. and the next equation holds:*

$$\int_E \nabla^* G(z)\rho(z)\mu(dz) = \int_E \langle G(z), \sigma_\rho(z)\rangle_H \|D\rho\|(dz), \quad \forall G \in (\mathcal{F}C_b^1)_{E^*}. \tag{3.4}$$

*Further, $\|D\rho\|$ is $\mathcal{E}^\rho$-smooth in the sense that it charges no set of zero $\mathcal{E}_1^\rho$-capacity. The domain of integration $E$ in the both hand sides of* (3.4) *can be replaced by $F$ the support of $\rho\mu$.*

(iii) *Conversely, if the equation* (3.4) *holds for $\rho \in \cup_{p>1} L^p(E; \mu)$ and for some positive finite measure $\|D\rho\|$ and a function $\sigma_\rho$ with the stated property, then $\rho \in BV(E)$ and $V(\rho) = \|D\rho\|(E)$.*

(iv) $\cup_{p>1} \mathbb{D}^{1,p}(E) \subset BV(E) \cap \mathbb{H}$ *and, for $\rho \in \cup_{p>1} \mathbb{D}^{1,p}(E)$,*

$$\|D\rho\| = \|\nabla\rho\|_H \cdot \mu, \quad V(\rho) = \int_E \|\nabla\rho\|_H\ \mu(dz), \quad \sigma_\rho(z) = \frac{1}{\|\nabla\rho\|_H} \nabla\rho(z)\ I_{\{\|\nabla\rho\|_H > 0\}}(z).$$

*Proof.* (i) Assume $\rho \in BV(E)$. Then $\rho \in L^p(E; \mu)$ for some $p > 1$. Take $G \in (\mathcal{F}C_b^1)_{E^*}$ of the type

$$G(z) = g(z)\ell \quad g \in \mathcal{F}C_b^1,\ \ell \in E^*,\ \|\ell\|_H = 1. \tag{3.5}$$

We have then

$$\nabla^* G(z) = -\partial_\ell g + g \cdot \ell(z)$$

by definition ([IW 89], [W 84]). Accordingly,

$$\int \partial_\ell g(z)\ \rho(z)\ \mu(dz) = -\int \nabla^* G(z)\ \rho(z)\ \mu(dz)\ +\ \int \rho(z) g(z) \ell(z) \mu(dz). \tag{3.6}$$

For any $g \in \mathcal{F}C_b^1$ satisfying $|g(z)| \leq 1$, the right hand side is not greater than

$$V(\rho) + \|\ell(\cdot)\|_{L^q} \|\rho\|_{L^p} < \infty \quad \frac{1}{p} + \frac{1}{q} = 1,$$

and hence $\sup_{g \in \mathcal{F}C_b^1, |g(z)| \leq 1} \int_E \partial_\ell g(z) \rho(z) \mu(dz)$ is dominated by the same value, namely, $\rho \in BV_\ell(E)$.

(ii) Suppose $\rho \in \mathbb{H} \cap BV(E)$. By (i) and Thoerem 2.2, there exists, for each $\ell \in E^*$, a finite signed measure $\nu_\ell$ on $E$ for which the equation (2.12) holds. We let

$$D_\ell \rho\ =\ 2\,\nu_\ell\ +\ \ell(z)\ \rho(z)\ \mu. \tag{3.7}$$

In view of (3.6), we have, for any $G$ of the type (3.5), the relation

$$\int_E \nabla^* G(z)\ \rho(z)\ \mu(dz) = \int_E g(z)\ D_\ell \rho(dz), \tag{3.8}$$

from which follows

$$V(D_\ell \rho)(E) = \sup_{|g| \leq 1} \int_E g(z)\ D_\ell \rho(dz) \leq V(\rho), \tag{3.9}$$

where $V(D_\ell \rho)$ denotes the total variation measure of the signed measure $D_\ell \rho$.

Next, choose any $H$-c.o.n.s. $\ell_1, \ell_2, \cdots, \ell_n, \cdots \in E^*$ and let

$$\gamma_\rho = \sum_{j=1}^{\infty} 2^{-j}\ V(D_{\ell_j} \rho), \qquad v_j(z) = \frac{d\ D_{\ell_j} \rho(z)}{d\ \gamma_\rho(z)},\ j = 1, 2, \cdots. \tag{3.10}$$

$\gamma_\rho$ is a positive finite measure ($\gamma_\rho(E) \leq V(\rho)$) charging no set of zero $\mathcal{E}_1^\rho$ capacity and $v_j$ can be taken to be Borel measurable. We have then, for any

$$G_n(z) = \sum_{j=1}^{n} g_j(z) \ell_j \in (\mathcal{F}C_b^1)_{E^*}, \quad n = 1, 2, \cdots, \tag{3.11}$$

the equation

$$\int_E \nabla^* G_n(z)\ \rho(z)\ \mu(dz)\ =\ \sum_{j=1}^{n} \int_E g_j(z)\ v_j(z)\ \gamma_\rho(dz). \tag{3.12}$$

Since $|v_j(z)| \leq 2^j,\ j = 1, 2, \cdots$, and $\mathcal{F}C_b^1$ is dense in $L^2(E; \gamma_\rho)$ ([MR 92, §II.3]), we can find $v_{j,m} \in \mathcal{F}C_b^1,\ j = 1, \cdots, n,\ m = 1, 2, \cdots$, such that

$$\lim_{m \to \infty} v_{j,m}(z) = v_j(z) \quad \gamma_\rho\text{–a.e.}$$

Substituting

$$g_{j,m}(z) \;=\; \frac{v_{j,m}(z)}{\sqrt{\sum_{k=1}^{n} v_{k,m}(z)^2 + \frac{1}{m}}}, \tag{3.13}$$

for $g_j(z)$ in (3.11) and (3.12), we get a bound

$$\sum_{j=1}^{n} \int_E g_{j,m}(z) v_j(z) \gamma_\rho(dz) \;\leq V(\rho),$$

because $\|G_n(z)\|_H^2 = \sum_{j=1}^{n} g_{j,m}(z)^2 \leq 1$. By letting $m \to \infty$, we arrive at a uniform bound in $n$

$$\int_E \sqrt{\sum_{j=1}^{n} v_j(z)^2}\; \gamma_\rho(dz) \;\leq\; V(\rho).$$

Now we let

$$\|D\rho\| = \sqrt{\sum_{j=1}^{\infty} v_j(z)^2}\; \gamma_\rho. \tag{3.14}$$

$$\sigma(z) = \begin{cases} \sum_{j=1}^{\infty} \frac{v_j(z)}{\sqrt{\sum_{k=1}^{\infty} v_k(z)^2}} \cdot \ell_j & \text{if } \sum_{k=1}^{\infty} v_k(z)^2 > 0 \\ 0 & \text{otherwise} \end{cases} \tag{3.15}$$

Then,

$$\|D\rho\|(E) \;\leq\; V(\rho), \qquad \|\sigma(z)\|_H \;=\; 1 \quad \|D\rho\|\text{-a.e.}, \tag{3.16}$$

$\|D\rho\|$ is $\mathcal{E}^\rho$-smooth and $\sigma$ is weakly measurable in the sense that $\langle \ell, \sigma(z)\rangle$ is measurable in $z \in E$. By rewriting the right hand side of (3.12), we further see that the desired equation (3.4) holds for $G = G_n$ expressible as (3.11) for the chosen c.o.n.s. $\{\ell_j\}$.

It remains to prove (3.4) for any $G$ of the type (3.5). In view of (3.6), the equation (3.4) then reads

$$-\int_E \partial_\ell g(z)\; \rho(z)\; \mu(dz) + \int_E g(z)\ell(z)\rho(z)\mu(dz) \;=\; \int_E g(z)\langle \ell, \sigma(z)\rangle_H \|D\rho\|(dz). \tag{3.17}$$

We put

$$k_n = \sum_{j=1}^{n} \langle \ell, \ell_j\rangle_H \ell_j, \quad G_n(z) = g(z)k_n.$$

It holds then that

$$\lim_{n\to\infty} \int_E \partial_{k_n} g\; \rho\; d\mu = \int_E \partial_\ell g\; \rho\; d\mu,$$

because

$$|\partial_{k_n} g(z) - \partial_\ell g(z)| = |\langle k_n - \ell, \nabla g(z)\rangle_H| \leq \|k_n - \ell\|_H \|\nabla g(z)\|_H,$$

and $\|\nabla g(z)\|_H$ is bounded. Further

$$|\int_E g(z)k_n(z)\rho(z)\mu(dz) - \int_E g(z)\ell(z)\rho(z)\mu(dz)| \le C_1\|\rho\|_{L^p}\|k_n(\cdot)-\ell(\cdot)\|_{L^q} = C_2\|\rho\|_{L^p}\|k_n-\ell\|_H,$$

where $C_1,\ C_2$ are positive constants and $\frac{1}{p}+\frac{1}{q}=1$ for $p>1$ with $\rho\in L^p(E;\mu)$.

Therefore, using (3.6) again, the left hand side of (3.17) is seen to coincide with

$$\lim_{n\to\infty}\int_E \nabla^* G_n(z)\ \rho\ \mu(dz).$$

Since (3.4) is already proved for $G_n$, the above expression equals

$$\lim_{n\to\infty}\int_E g(z)\ \langle k_n,\sigma(z)\rangle_H\ \|D\rho\|(dz)\ =\ \int_E g(z)\ \langle \ell,\sigma(z)\rangle_H\ \|D\rho\|(dz)$$

the right hand side of (3.17).

(iii) Suppose $\rho\in\cup_{p>1}L^p(E;\mu)$ satisfies the equation (3.4) for some positive finite measure $\|D\rho\|$ and a function $\sigma_\rho$ with the property stated in the paragraph preceding (3.4). Clearly

$$V(\rho)\le\|D\rho\|(E)$$

and $\rho\in BV(E)$. To obtain the converse inequality, choose any $H$-c.o.n.s. $\{\ell_j\}$ from $E^*$ and set

$$\sigma_j(z)=\langle\ell_j,\sigma(z)\rangle_H\quad j=1,2,\cdots.$$

Fix an arbitrary $n$. As in the proof of (ii), we can find functions

$$v_{j,m}\in\mathcal{F}C_b^1,\ m=1,2,\cdots,\ \text{with}\ \lim_{m\to\infty}v_{j,m}(z)=\sigma_j(z)\ \|D\rho\|\text{-a.e. } j=1,\cdots,n.$$

Define then $g_{j,m}(z)$ by (3.13) and substitute $G_{n,m}(z)=\sum_{j=1}^n g_{j,m}(z)\ell_j$ for $G(z)$ in (3.4) yielding

$$\sum_{j=1}^n\int_E g_{j,m}(z)\sigma_j(z)\|D\rho\|(dz)\le V(\rho).$$

By letting $m\to\infty$, we get

$$\int_E\left(\sum_{j=1}^n\sigma_j^2(z)\right)^{\frac{1}{2}}\|D\rho\|(dz)\le V(\rho).$$

We finally let $n\to\infty$ to obtain $\|D\rho\|(E)\le V(\rho)$.

(iv) Obviously the duality relation (3.2) extends to $\rho\in\cup_{p>1}\mathbb{D}^{1,p}(E)$. By defining $\|D\rho\|$ and $\sigma(z)$ in the stated way, the extended relation (3.2) is reduced to equation (3.4).

□

In the rest of this section, let us fix $\rho \in \mathbb{H} \cap BV(E)$ and consider the conservative diffusion process

$$\mathbf{M}^\rho = (\Omega, \mathcal{M}, \{\mathcal{M}_t\}, \theta_t, X_t, P_z)$$

over $F \subset E$ associated with the classical Dirichlet form $(\mathcal{E}^\rho, \mathcal{F}^\rho)$ of Theorem 2.1. $\mathbf{M}$ is called the distorted Ornstein Uhlenbeck process associated with $\rho$ and its state space $F$ is the topological support of $\rho\mu$. Without loss of generality, we may assume that the sample path

$$X_t(\omega) \longrightarrow F$$

is continuous in $t \in [0, \infty)$ for each $\omega \in \Omega$. We now present a semimartingale decomposition of $\mathbf{M}$ which legitimates the use of the term 'distorted Ornstein Uhlenbeck process'.

Recall that the notion of a (real valued) additive functional (AF in abbreviation) of $\mathbf{M}^\rho$ involves a defining set $\Lambda \in \mathcal{M}_\infty$ and an exceptional set $N \subset F$ with

$$\theta_t(\Lambda) \subset \Lambda, \quad P_z(\Lambda) = 1 \; \forall z \in F \setminus N.$$

$N$ is a properly exceptional set of $\mathbf{M}^\rho$ and for each $\omega \in \Lambda$ the AF is required to satisfy due porperties ([Fu 99a]). The notion of $E$-valued continuous additive functional can be defined in the same way.

A mapping

$$A_t(\omega) \;: [0, \infty) \times \Omega \longrightarrow E$$

is called an *$E$-valued CAF* of $\mathbf{M}^\rho$ if
$\ell(A_t(\omega))$ is $\mathcal{M}_t$-measurable for each $t \geq 0$ and each $\ell \in E^*$,
there exist a defining set $\Lambda$ and exceptional set $N$ as above and, for each $\omega \in \Lambda$,

$$A_0(\omega) = 0, \quad A_t(\omega) \text{ is continuous in } t \in [0, \infty) \quad \text{and}$$

$$A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t\omega), \quad t, s \geq 0.$$

Two $E$-valued CAF's $A^{(1)}, \; A^{(2)}$, are regarded to be equivalent if

$$\text{for each } t \geq 0, \; P_z(A_t^{(1)} = A_t^{(2)}) = 1 \quad \mathcal{E}^\rho\text{−}q.e \; z \in F.$$

In this case, we can find a common defining set $\Lambda$ and exceptional set $N$ such that $A_t^{(1)} = A_t^{(2)}$ for any $t \geq 0, \; \omega \in \Lambda$. For any $E$-valued CAF $A_t(\omega)$, $\ell(A_t(\omega))$ is obviously a real valued CAF with the same defining set and exceptional set.

Simple examples of $E$-valued CAF's with full defining set $\Omega$ and with no exceptional set are

$$A_t(\omega) = X_t(\omega) - X_0(\omega), \quad A_t(\omega) = \int_0^t X_s(\omega)ds \text{ (Riemann integral).}$$

Consider next a function

$$\tau : E \longrightarrow H$$

such that $\tau$ is $H$-bounded and weakly measurable in the sense that $\sup_{z\in E}\|\tau(z)\|_H$ is finite and $\langle \ell, \tau(z)\rangle_H$ is Borel measurable for any $\ell \in H^* = H$. Then $\tau$ is, as a mapping from $E$ into itself, also $E$-bounded and weakly measurable. Therefore the composite process $\tau(X_t(\omega))$ enjoys the same property as a mapping from $[0,\infty)$ to $E$ for each fixed $\omega \in \Omega$.

Let $L_t(\omega)$ be a real valued PCAF with defining set $\Lambda$ and exceptional set $N$. Then we see that, for each $\omega \in \Lambda$, $\tau(X_t(\omega))$ is Bochner integrable in $t$ with respect to $dL_t(\omega)$ and the Bochner integral (cf. [Y 68])

$$\int_0^t \tau(X_s(\omega))dL_s(\omega), \quad t \geq 0,\ \omega \in \Lambda$$

becomes an $E$-valued CAF with the same defining and exceptional sets as $L_t$.

An $E$-valued stochastic process $W_t(\omega)$ is called an *$\{\mathcal{M}_t\}$-Brownian motion on $E$* under a probability measure $Q$ on $(\Omega, \mathcal{M})$ if
$W_t$ is continuous in $t \geq 0$ $Q$-almost surely and,
for each $\ell \in E^*$, $\ell(W_t(\omega))$ is $\mathcal{M}_t$-measurable and further

$$E^Q\left(e^{\sqrt{-1}(\ell(W_t)-\ell(W_s))}|\mathcal{M}_s\right) \ = \ \exp(-\frac{1}{2}(t-s)\|\ell\|_H^2), \quad t > s \geq 0.$$

The second condition above is equivalent to the requirement that the real valued process $\ell(W_t)$ is a one dimensional $\{\mathcal{M}_t\}$-Brownian motion for each $\ell \in E^*$ with $\|\ell\|_H = 1$. Keeping these notions in mind, let us proceed to a decomposition theorem.

**Theorem 3.2** *Let $\rho \in \mathbb{H} \cap BV(E)$ and consider the $\mathcal{E}^\rho$-smooth measure $\|D\rho\|$ and an $H$-valued function $\sigma_\rho$ appearing in* Theorem 3.1 (ii)*. Then the sample path of the associated distorted Ornstein Uhlenbeck process $\mathbf{M}^\rho$ admits the following expression as a sum of three $E$-valued CAF's:*

$$X_t(\omega) - X_0(\omega) \ = \ W_t \ - \ \frac{1}{2}\int_0^t X_s(\omega)ds \ + \ \frac{1}{2}\int_0^t \sigma_\rho(X_s(\omega))\, dL_s^{\|D\rho\|}(\omega). \ \ t \geq 0. \qquad (3.18)$$

*Here, $L_t^{\|D\rho\|}(\omega)$ is a real valued PCAF associated with $\|D\rho\|$ by the Revuz correspondence. The $E$-valued CAF $W_t$ has the same defining set and exceptional set as $L_t^{\|D\rho\|}$.*

*Moreover, $W_t(\omega)$ is an $\{\mathcal{M}_t\}$-Brownian motion on $E$ under $P_\gamma$ for each $\mathcal{E}^\rho$-smooth probability measure $\gamma$ on $F$.*

*Proof.* Since the left hand side and the last two terms of the right hand side of equation (3.18) are $E$-valued CAF as described above, $W_t$ can be defined by this equation as an $E$-valued CAF with the same defining set and exceptional set as $L_t^{\|D\rho\|}$. From (3.18) follows a decomposition of real valued AF

$$\ell(X_t) \ - \ \ell(X_0) \ = \ \ell(W_t) \ - \ \frac{1}{2}\int_0^t \ell(X_s)ds \ + \ \frac{1}{2}\int_0^t \langle \ell, \sigma^\rho(X_s(\omega))\rangle_H \, dL_s^{\|D\rho\|}(\omega). \qquad (3.19)$$

Let us compare (3.19) with the decomposition (2.10):

$$\ell(X_t) - \ell(X_0) = M_t + N_t.$$

Since (2.9) and (3.17) lead us to the identity

$$\mathcal{E}^\rho(\ell(\cdot), g) \;=\; \frac{1}{2}\int_E g(z)\ell(z)\rho(z)\mu(dz) \;-\; \frac{1}{2}\int_E g(z)\langle \ell, \sigma(z)\rangle_H \,\|D\rho\|(dz)$$

holding for any $g \in \mathcal{F}C_b^1$, we have by Theorem 2.2 that

$$N_t^\ell = -\frac{1}{2}\int_0^t \ell(X_s)ds \;+\; \frac{1}{2}\int_E \langle \ell, \sigma(X_s)\rangle_H \, dL_s^{\|D\rho\|}. \tag{3.20}$$

Hence we get from (3.19) and (3.20) that

$$\ell(W_t) = M_t^\ell \quad P_z\text{-a.s. for } \mathcal{E}^\rho\text{-q.e. } z \in F, \tag{3.21}$$

the $\mathcal{E}^\rho$-exceptional set depending on $\ell$ in general.

By virtue of Theorem 2.2, $\ell(W_t)$ is a martingale AF with quadratic variation $t\|\ell\|_H$ $P_z$-a.s. for $\mathcal{E}^\rho$-q.e. $z \in F$. Owing to the martingale characterization of Brownian motion ([IW 89]),we see that, for any $\ell \in E^*$ with $\|\ell\|_H = 1$, the real valued process $\ell(W_t)$ is an $\{\mathcal{M}_t\}$-Brownian motion under $P_\gamma$ for each $\mathcal{E}^\rho$-smooth probability measure $\gamma$ on $F$. Hence $W_t$ is an $\mathcal{M}_t$-Brownian motion on $E$ under $P_\gamma$.

□

# 4 Caccioppoli sets and modified reflecting Ornstein Uhlenbeck processes

We still work with the abstract Wiener space $(E, H, \mu)$.

**Lemma 4.1 (lower semicontinuity)** *Let $p > 1$. If $\rho_k \in BV(E) \cap L^p(E;\mu)$ is $L^p(E;\mu)$-convergent to $\rho \in L^p(E;\mu)$ as $k \to \infty$, then*

$$V(\rho) \le \liminf_{k\to\infty} V(\rho_k).$$

*Proof.* For any $G(z) \in (\mathcal{F}C_b^1)_{E^*}$ with $\|G\|_H \le 1$,

$$\int_E \nabla^* G(z)\rho(z)\mu(dz) = \lim_{k\to\infty}\int_E \nabla^* G(z)\rho_k(z)\mu(dz) \le \liminf_{k\to\infty} V(\rho_k).$$

□

**Lemma 4.2** *Let $\{T_t, t > 0\}$ be the Ornstein Uhlenbeck semigroup.*
*Then for any $\rho \in \cup_{p>1} L^p(E;\mu)$,*

$$\int_E \nabla^* G(z) T_t\rho(z)\mu(dz) = e^{-t}\int_E \nabla^*(T_t G)(z)\rho(z)\mu(dz),\ \forall G \in (\mathcal{F}C_b^1)_{E^*}. \tag{4.1}$$

*Proof.* It suffices to prove (4.1) for $G = \sum_{j=1}^n g_j \ell_j$ with any polynomials $g_j$ and for any polynomial $\rho$. Using (3.2), symmetry of $T_t$ and the well known identity ([W 84])

$$\partial_\ell (T_t \rho) = e^{-t} T_t (\partial_\ell \rho),$$

we see that the left hand side of (4.1) equals

$$\begin{aligned} \int_E \langle G(z), \nabla(T_t\rho)(z)\rangle_H \, \mu(dz) &= \sum_{j=1}^n \int_E g_j(z) \partial_{\ell_j}(T_t\rho)(z)\mu(dz) \\ = \; e^{-t} \sum_{j=1}^n \int_E g_j(z) T_t(\partial_{\ell_j}\rho)(z)\mu(dz) &= e^{-t} \int_E \langle T_t G, \nabla \rho\rangle_H \, \mu(dz), \end{aligned}$$

which coincides with the right hand side of (4.1) by virtue of (3.2) again. □

**Proposition 4.1** *For any $\rho \in BV(E) \cap L^p(E;\mu)$ $(p > 1)$, there exists a sequence of functions $\rho_k \in \mathbb{D}^{1,p}(E)$ such that*

$$\lim_{k\to\infty} \rho_k = \rho \text{ in } L^p(E;\mu), \quad \lim_{k\to\infty} V(\rho_k) = V(\rho).$$

*Proof.* Let $\{T_t, t > 0\}$ be the Ornstein Uhlenbeck semigroup. It is known ([Su 88]) that, for any $\rho \in L^p(E;\mu)$ for $p > 1$,

$$T_t\rho \in \mathbb{D}^{1,p}, \quad T_t\rho \to \rho \text{ in } L^p(E;\mu) \quad t \downarrow 0.$$

By Lemma 4.1, we have $V(\rho) \le \liminf_{t\downarrow 0} V(T_t\rho)$. On the other hand, for any $G \in (\mathcal{F}C_b^1)_{E^*}$ with $\|G\|_H(z) \le 1$, we get from Lemma 4.2,

$$\int_E \nabla^* G(z) T_t\rho(z)\mu(dz) = e^{-t} \int_E \nabla^*(T_t G)(z)\rho(z)\mu(dz) \le e^{-t} V(\rho),$$

which implies

$$V(T_t\rho) \le e^{-t} V(\rho) \quad \text{and} \quad \limsup_{t\downarrow 0} V(T_t\rho) \le V(\rho).$$

□

For a function $\rho(z)$ on $E$, we consider its level sets defined by

$$E_t^\rho = \{z \in E : \rho(z) > t\}. \tag{4.2}$$

**Theorem 4.1 (coarea formula)** *For any non-negative $\rho \in BV(E)$,*

$$V(\rho) \;=\; \int_0^\infty V(I_{E_t^\rho})\, dt. \tag{4.3}$$

*Proof.* $V(\rho)$ admits an expression as in Theorem 3.1 (iv) when $\rho \in \mathbb{D}^{1,p}(E)$ for some $p > 1$. The identity (4.3) is first proved in this case and then extended to a general $\rho \in BV(E)$ by using the approximation in Proposition 4.1. Full proof is exactly analogous to the proof of [EG 92, §5.5, Th. 1] in the finite dimensional case. $\square$

An $\mu$-measurable subset $\Gamma$ of $E$ is said to be *Caccioppoli* if $I_\Gamma \in BV(E)$. Theorem 4.1 means that a.e. level sets of a non-negative BV function are Caccioppoli. In virtue of Thoerem 3.1 (iv), we have

**Corollary 4.1** *For any* $\rho \in \cup_{p>1}\mathbb{D}^{1,p}_+(E)$,

$$I_{E_t^\rho} \in \mathbb{H} \cap BV(E) \qquad \text{for a.e. } t \geq 0.$$

Consider now a $\mu$-measurable set $\Gamma \subset E$ satisfying condition

$$I_\Gamma \in \mathbb{H} \cap BV(E). \tag{4.4}$$

Denote the corresponding objects $\sigma_{I_\Gamma}$, $\|DI_\Gamma\|$ in Theorem 3.1 (ii) by $-\mathbf{n}_\Gamma$, $\|\partial\Gamma\|$ respectively. Then formula (3.4) reads

$$\int_\Gamma \nabla^* G(z)\mu(dz) = -\int_F \langle G(z), \mathbf{n}_\Gamma\rangle_H \ \|\partial\Gamma\|(dz), \quad \forall G \in (\mathcal{F}C_b^1)_{E^*},$$

where the domain of integration $F$ of the right hand side is the support of $I_\Gamma \cdot \mu$. $F$ is contained in $\bar{\Gamma}$ but we shall further show that the domain of integration of the right hand side can be restricted to $\partial\Gamma$. In doing so, we need to utilize the associated distorted Ornstein Uhlenbeck process $\mathbf{M}^{I_\Gamma} = (X_t, P_z)$ on $F$ , which will be called the *modified reflecting Ornstein Uhlenbeck process for* $\Gamma$.

**Theorem 4.2** *Suppose a $\mu$-measurable set $\Gamma \subset E$ satisfies condition* (4.4)*. Then the support of $\|\partial\Gamma\|$ is contained in the boundary $\partial\Gamma$ of $\Gamma$, and accordingly a generalized Gauss formula holds:*

$$\int_\Gamma \nabla^* G(z)\mu(dz) = -\int_{\partial\Gamma} \langle G(z), \mathbf{n}_\Gamma\rangle_H \ \|\partial\Gamma\|(dz), \quad \forall G \in (\mathcal{F}C_b^1)_{E^*}. \tag{4.5}$$

*Proof.* For any $G$ of the type (3.5), we have from (2.9), (3.6) and (3.8) that

$$\mathcal{E}^{I_\Gamma}(\ell(\cdot), g) - \frac{1}{2}\int_\Gamma g(z)\ell(z)\mu(dz) = -\frac{1}{2}\int_F g(z)\ D_\ell I_\Gamma(dz). \tag{4.6}$$

Since the finite signed measure $D_\ell I_\Gamma$ charges no set of zero $\mathcal{E}_1^{I_\Gamma}$-capacity, the equation (4.6) readily extends to any $\mathcal{E}^{I_\Gamma}$-quasicontinuous function $g \in \mathcal{F}_b^{I_\Gamma}$.

Denote by $\Gamma^0$ the interior of $\Gamma$. Then $\Gamma^0 \subset F \subset \bar{\Gamma}$. In view of the construction of the measure $\|DI_\Gamma\|$ in Theorem 3.1, it suffices to show that, for any fixed $\ell \in E^*$ with $\|\ell\|_H = 1$,

$$D_\ell I_\Gamma(\Gamma^0) = 0. \tag{4.7}$$

Take an arbitrary $\epsilon > 0$ and set

$$U = \{z \in E : d(z, E \setminus \Gamma^0) > \epsilon\},$$

$$V = \{z \in E : d(z, E \setminus \Gamma^0) \geq \epsilon\},$$

where $d$ is the metric distance of the space $E$. Then $\bar{U} \subset V$ and $V$ is a closed set contained in the open set $\Gamma^0$. By making use of the modified reflecting Ornstein Uhlenbeck process $\mathbf{M}^{I_\Gamma} = (X_t, P_z)$ on $F$, we define a non-negative bounded function $h$ by

$$h(z) = 1 - E_z\left(e^{-\tau_V}\right) \qquad z \in F, \tag{4.8}$$

where $\tau_V$ denotes the first exit time from the set $V$. $h$ is in the space $\mathcal{F}_b^{I_\Gamma}$ and further $\mathcal{E}^{I_\Gamma}$ quasicontinuous because it is $\mathbf{M}^{I_\Gamma}$ finely continuous. Moreover

$$h(z) > 0 \quad \forall z \in U, \qquad h(z) = 0 \quad \forall z \in F \setminus V. \tag{4.9}$$

Let

$$\nu(dz) = h(z) D_\ell I_\Gamma(dz), \tag{4.10}$$

and

$$I_g = \mathcal{E}^{I_\Gamma}(\ell(\cdot), g \cdot h) - \frac{1}{2}\int_\Gamma g(z)h(z)\ell(z)\mu(dz). \tag{4.11}$$

Equation (4.6) for $\mathcal{E}^{I_\Gamma}$-quasicontinuous function $g \cdot h \in \mathcal{F}_b^{I_\Gamma}$ then leads us to

$$I_g = -\frac{1}{2}\int_F g(z)\nu(dz), \qquad \forall g \in \mathcal{F}C_b^1.$$

In order to prove (4.7), it is enough to show that $I_g = 0$ for any function $g(z)$ of $z \in E$ of the type

$$g(z) = f(\ell(z), \ell_2(z), \cdots, \ell_m(z)), \quad \ell_2, \cdots, \ell_m \in E^*,\ f \in C_0^1(R^m), \tag{4.12}$$

because we have then $I_g = 0$ for any $g \in \mathcal{F}C_b^1$, and consequently $\nu = 0$ by virtue of the fact that $\mathcal{F}C_b^1$ is a determining class of a finite signed measure ([ST 92]).

On account of Proposition 2.1, we have the expression

$$\mathcal{E}^{I_\Gamma}(\ell(\cdot), g \cdot h) = \check{\mathcal{E}}^{I_\Gamma,\ell}(\ell(\cdot), g \cdot h) = \frac{1}{2}\int_{E_\ell}\int_{R_x} \frac{d(g\tilde{h})(s\ell + x)}{ds} p(s) ds \mu_\ell(dx), \tag{4.13}$$

where $R_x = R(I_\Gamma(\cdot\ell + x))$ and $\tilde{h}$ is a $\mu$-version of $h$ appearing in the description of (2.16). Let

$$V_x = \{s\ell : s\ell + x \in V\}, \quad \Gamma_x^0 = \{s\ell : s\ell + x \in \Gamma^0\}, \quad F_x = \{s\ell : s\ell + x \in F\}.$$

We then have the inclusion $V_x \subset \Gamma_x^0 \subset R_x \subset F_x$. By (4.9), $h(s\ell + x) = 0$ for any $x \in E_\ell$ and for any $s \in R_x \setminus V_x$. On the other hand, by selecting a Borel set $N \subset E_\ell$ with $\mu_\ell(N) = 0$, we have for each $x \in E_\ell \setminus N$,

$$\tilde{h}(s\ell + x) = h(s\ell + x) \quad ds\text{–a.e.}$$

Since $\tilde{h}(\cdot\ell + x)$ is absolutely continuous in $s$, we can conclude that

$$\tilde{h}(s\ell + x) = 0 \quad \forall x \in E_\ell \setminus N, \quad \forall s \in R_x \setminus V_x.$$

Fix $x \in E_\ell \setminus N$ and let $I$ be any connected component of the one dimensional open set $R_x$. Further, for any function $g$ of the type (4.12), we denote by $K_x$ the support of $g(\cdot\ell + x)$ and choose a finite open interval $J$ containing $K_x$. Then $I \cap V_x \cap K_x$ is a closed set contained in a finite open interval $I \cap J$ and

$$(g\tilde{h})(s\ell + x) = 0 \quad \forall s \in (I \cap J) \setminus (I \cap V_x \cap K_x).$$

Therefore an integration by part gives

$$\int_{I\cap J} \frac{d(g\tilde{h})(s\ell + x)}{ds} p(s)ds = \int_{I\cap J} (g\tilde{h})(s\ell + x)sp(s)ds.$$

Combining this with (4.11) and (4.13), we arrive at

$$I_g = \frac{1}{2}\int_{E_\ell\setminus N}\int_{R_x} (g\tilde{h})(s\ell + x)sp(s)ds\mu_\ell(dx) - \frac{1}{2}\int_E (gh)(z)\ell(z)I_\Gamma(z)\mu(dz) = 0.$$

□

We say that two $\mu$-measurable sets $\Gamma_1,\ \Gamma_2$ are equivalent if $\mu(\Gamma_1 \ominus \Gamma_2) = 0$. Neither condition (4.4) nor the topological support of $I_\Gamma \cdot \mu$ depends on the choice of a representative from the same equivalence class, while the topological boundary $\partial\Gamma$ does depend on the choice. Theorem 4.2 says that, the support of this measure sits in the intersection of $\partial\Gamma$ for every choice of the representative $\Gamma$.

Finally we state Theorem 3.2 for $\rho = I_\Gamma$.

**Theorem 4.3** *Suppose a $\mu$-measurable set $\Gamma$ satisfies condition* (4.4). *Then the sample path of the corresponding modified reflecting Ornstein Uhlenbeck process $\mathbf{M}^{I_\Gamma} = (\Omega, \{\mathcal{M}_t\}, X_t, P_z)$ for $\Gamma$ admits the following expression as a sum of three $E$-valued CAF's:*

$$X_t(\omega) - X_0(\omega) \;=\; W_t \;-\; \frac{1}{2}\int_0^t X_s(\omega)ds \;-\; \frac{1}{2}\int_0^t \mathbf{n}_\Gamma(X_s(\omega))\, dL_s^{\|\partial\Gamma\|}(\omega). \quad t \geq 0. \tag{4.14}$$

*Here, $L_t^{\|\partial\Gamma\|}(\omega)$ is a real valued PCAF associated with $\|\partial\Gamma\|$ by the Revuz correspondence and enjoys the property*

$$\int_0^t I_{\partial\Gamma}(X_s(\omega))dL_s^{\|\partial\Gamma\|}(\omega) = L_t^{\|\partial\Gamma\|}(\omega), \quad t \geq 0. \tag{4.15}$$

*The $E$-valued CAF $W_t$ has the same defining set and exceptional set as $L_t^{\|\partial\Gamma\|}$. Moreover, $W_t(\omega)$ is an $\{\mathcal{M}_t\}$-Brownian motion on $E$ under $P_\gamma$ for each $\mathcal{E}^{I_\Gamma}$-smooth probability measure $\gamma$ on $F$.*

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