On the space of BV functions and a related stochastic calculus in infinite dimensions

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Dedicated to Professor Leonard Gross on the occasion of his 70th birthday

Abstract

Functions of bounded variation (BV functions) are defined on an abstract Wiener space (E, H, μ) in a way similar to that in finite dimensions. Some characterizations are given, which justify describing a BV function as a function in $L(\log L)^{1/2}$ with the first order derivative being an H-valued measure. It is also shown that the space of BV functions is obtained by a natural extension of the Sobolev space $\mathbb{D}^{1,1}$. Moreover, some stochastic formulae related to BV functions are investigated.

Key words: BV function, abstract Wiener space, Orlicz space, surface measure, Dirichlet form, distorted Ornstein-Uhlenbeck process, generalized Itô's formula

1 Introduction

A real valued function ρ defined on an open set $U \subset \mathbb{R}^d$ is said to be of bounded variation ($\rho \in BV(U)$ in notation) if the distributional derivatives $\partial_i \rho,\ 1 \le i \le d$, are finite signed measures on U, or equivalently, if

$$V(\rho) = \sup \left\{ \int_U \rho \operatorname{div}\varphi \, dx \,\middle|\, \varphi \in C_0^1(U; \mathbb{R}^d), \|\varphi\|_\infty \le 1 \right\} < \infty. \tag{1.1}$$

Thus the space $BV(U)$ is a natural extension of the classical Sobolev space $W^{1,1}(U)$ with $V(\rho) = \|\nabla \rho\|_1$ and it has played important roles in solving diverse fine variational problems in finite dimensions ([9, 15, 8, 3]).

In a recent paper [13], the notion of a BV function on an abstract Wiener space (E, H, μ) was introduced as a function ρ on E for which a quantity analogous to (1.1) is finite. However, ρ was required to be in $\bigcup_{p>1} L^p(E; \mu)$, excluding the Malliavin Sobolev space $\mathbb{D}^{1,1}$ from the space $BV(E)$. Furthermore, in order to formulate an analogue to the Gauss formula holding for ρ and an associated measure $\|D\rho\|$ on E, the closability of a

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pre-Dirichlet form $\mathcal{E}^\rho$ related to ρ was crucially assumed. Then, under the last assumption, a semimartingale decomposition of the associated distorted Ornstein-Uhlenbeck process living on the support of $\rho\, d\mu$ was also presented in [13].

A purpose of the present paper is to extend the analytical part of [13] considerably by requiring a BV function to sit in a broader Orlicz space $L(\log L)^{1/2}$ and by establishing an associated Green formula without any closability assumption (§3). The space $BV(E)$ will now contain the Sobolev space $\mathbb{D}^{1,1}$ as a proper subspace and the structures of those spaces will be characterized in terms of the measures $\|D\rho\|$, $\rho \in BV(E)$. Moreover, we shall give in §3 a refinement of a theorem in [13] concerning the support of the measure $\|D\rho\|$ by showing that it vanishes outside a quasi support of $\mathbf{1}_{\{\rho\neq a\}} \cdot \mu$ for every $a \in \mathbb{R}$.

In §4, we shall assume that $\rho \in BV(E)$ is non-negative and $\mathcal{E}^\rho$ is closable. The martingale part of the associated distorted Ornstein-Uhlenbeck process $\mathbf{M}^\rho = (X_t, \mathcal{M}_t, P_z)$ will then be shown to be a Brownian motion on E under P_z for quasi-every starting point z. This fact was proven in [13] only under P_γ for smooth probability measures γ. By making use of an analogue to the classical Green formula obtained in §3, we shall then show in §4 a generalized Itô's formula for $\mathbf{M}^\rho$, which has been formulated in [12] in finite dimensions.

When $\rho \in BV(E)$ is an indicator function of a set A, a Green formula (Theorem 3.12) and a result on the support of $\|D\rho\|$ (Theorem 3.15) indicate that $\|D\rho\|$ and σ_ρ in the formula are regarded as a surface measure and a normal vector field of a 'boundary' of A, respectively. Such notions in infinite dimensions have been investigated in various contexts, such as in [16, 28, 17, 18, 19, 2, 4, 10]. Our approach is based on a theory of Dirichlet forms and aims at applications to stochastic analysis on sets whose boundaries do not have good smoothness.

2 Preliminaries

Let (E, H, μ) be an abstract Wiener space. Namely, E is a separable Banach space, H is a separable Hilbert space densely and continuously embedded in E, and μ is a Gaussian measure on E which satisfies that

$$\int_E \exp\left(\sqrt{-1}\ell(z)\right)\mu(dz) = \exp\left(-\|\ell\|_H^2/2\right), \quad \ell \in E^*.$$

Here, we regard the topological dual E^* of E as a subspace of H by the natural inclusion $E^* \subset H^*$ and the identification $H^* \simeq H$. The inner product and the norm of H is denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|_H$, respectively. Let for each $k \in \mathbb{N}\cup\{\infty\}$,

$$\mathcal{F}C_b^k = \left\{ u : E \to \mathbb{R} \;\middle|\; \begin{array}{l} u(z) = f(\ell_1(z),\ldots,\ell_m(z)),\ \ell_1,\ldots,\ell_m \in E^*,\ f \in C_b^k(\mathbb{R}^m) \\ \text{for some } m \in \mathbb{N} \end{array} \right\}.$$

For $u \in \mathcal{F}C_b^1$, the H-derivative of u, denoted by ∇u, is a map from E to H defined by the relation

$$\langle \nabla u(z), \ell\rangle = \partial_\ell u(z), \quad \ell \in E^* \subset H,$$

where $\partial_\ell u(z) = \lim_{\varepsilon\to 0}(u(z+\varepsilon\ell) - u(z))/\varepsilon$, $\ell \in E^* \subset H \subset E$.

For a separable Hilbert space K, a Borel measure ν on a metric space X and $p \in [1, \infty]$, $L^p(X \to K; \nu)$ denotes the usual L^p space consisting of K-valued functions on X. We shall often omit each symbol X, K and ν if $X = E$, $K = \mathbb{R}$, and $\nu = \mu$, respectively. Also, $L^p_+(\nu)$ denotes the space of all nonnegative functions belonging to $L^p(\nu)$. The norm $\|\cdot\|_p$ always means $L^p(E \to K; \mu)$-norm. For $\rho \in L^1$, we define a symmetric bilinear form $\mathcal{E}^\rho : \mathcal{F}C_b^1 \times \mathcal{F}C_b^1 \to \mathbb{R}$ by

$$\mathcal{E}^\rho(u, v) = \frac{1}{2} \int_E \langle \nabla u(z), \nabla v(z) \rangle \rho(z) \, \mu(dz), \quad u, v \in \mathcal{F}C_b^1.$$

$\mathcal{E}^\rho$ can be regarded as a bilinear form on $L^2(F^\rho; |\rho| \cdot \mu)$ where F^ρ is a support of $|\rho| \cdot \mu$, since ∇ has the following consistency property by Proposition 7.1.4 in [6, Chapter I]: if $u \in \mathcal{F}C_b^1$ and $v \in \mathcal{F}C_b^1$ coincide on a measurable set A, then $\nabla u = \nabla v$ on A μ-a.e. In the following, a measurable function on E is also regarded as a function on F^ρ by the natural restriction. The set of all functions $\rho \in L^1_+$ such that $(\mathcal{E}^\rho, \mathcal{F}C_b^1)$ is closable on $L^2(F^\rho; \rho \cdot \mu)$ will be denoted by $QR(E)$. Its closure $(\mathcal{E}^\rho, \mathcal{F}^\rho)$ is then automatically a quasi-regular local Dirichlet form by the results of [22, 25] (see also [13, Theorem 2.1]). Functions belonging to $\mathbb{H}$ defined in [13], especially positive L^1-functions bounded away from 0, are elements of $QR(E)$. We denote by $\mathcal{F}^\rho_b$ the set of all bounded functions in $\mathcal{F}^\rho$. Following [27, 14], we denote by $\mathcal{F}^\rho_e$ the extended Dirichlet space of $(\mathcal{E}^\rho, \mathcal{F}^\rho)$: $u \in \mathcal{F}^\rho_e$ if and only if $|u| < \infty$ $\rho \cdot \mu$-a.e. and there exists a sequence $\{u_n\}$ in $\mathcal{F}^\rho$ such that $\mathcal{E}^\rho(u_m - u_n, u_m - u_n) \to 0$ as $n \geq m \to \infty$ and $u_n \to u$ $\rho \cdot \mu$-a.e. as $n \to \infty$. For example, a function $\ell(\cdot) : z \in E \mapsto \ell(z) \in \mathbb{R}$ belongs to $\mathcal{F}^\rho_e$ for every $\ell \in E^*$. Indeed, when Φ_n is a smooth function on $\mathbb{R}$ such that $0 \leq \Phi_n' \leq 1$ on $\mathbb{R}$, $\Phi_n(x) = x$ on $[-n, n]$ and $|\Phi_n(x)| = n + 1$ on $\mathbb{R} \setminus [-n-2, n+2]$, $\{\Phi_n \circ \ell(\cdot)\}_{n \in \mathbb{N}}$ is the desired sequence. $\mathcal{E}^\rho$ extends to a bilinear form on $\mathcal{F}^\rho_e$ in a natural way.

For each $\rho \in QR(E)$, there exists an associated diffusion process $\mathbf{M}^\rho = (X_t, \mathcal{M}_t, P_z)$ on F^ρ with $(\mathcal{E}^\rho, \mathcal{F}^\rho)$. We denote by $\mathbf{A}^\rho_+$ the set of all positive continuous additive functionals (PCAF in abbreviation) of $\mathbf{M}^\rho$, and define $\mathbf{A}^\rho = \mathbf{A}^\rho_+ - \mathbf{A}^\rho_+$. For $A \in \mathbf{A}^\rho$, its total variation process is denoted by $\{A\}$. We also define $\mathbf{A}^\rho_0 = \{A \in \mathbf{A}^\rho \mid E_{\rho \cdot \mu}(\{A\}_t) < \infty \text{ for all } t > 0\}$. Each element in $\mathbf{A}^\rho_+$ has a corresponding positive $\mathcal{E}^\rho$-smooth measure on F^ρ by the Revuz correspondence. The totality of such measures will be denoted by S^ρ_+. Accordingly, $\mathbf{A}^\rho$ has a correspondence with $S^\rho = S^\rho_+ - S^\rho_+$, the set of $\mathcal{E}^\rho$-smooth signed measures.

For each $u \in \mathcal{F}^\rho_e$, we have the following decomposition:

$$\tilde{u}(X_t) - \tilde{u}(X_0) = M^{[u]}_t + N^{[u]}_t,$$

where $\tilde{u}$ is an $\mathcal{E}^\rho$-quasicontinuous modification of u, $M^{[u]}$ is a martingale AF of finite energy and $N^{[u]}$ is a CAF of zero energy. Also, $M^{[u]}$ and $N^{[u]}$ are uniquely determined. When $u = \ell(\cdot)$ for some $\ell \in E^*$, we also write $M^{[\ell]}$ and $N^{[\ell]}$, instead of $M^{[u]}$ and $N^{[u]}$, respectively.

Let $\rho \in L^1$ and $\ell \in E^*$. Following [13], we say that ρ is of bounded variation in direction ℓ ($\rho \in BV_\ell(E)$ in notation) if there is a positive constant C such that

$$\left| \int_E \partial_\ell v(z) \rho(z) \, \mu(dz) \right| \leq C \|v\|_\infty \quad \text{for every } v \in \mathcal{F}C_b^1.$$

Theorem 2.1 *Let $\rho \in L^1_+$ and $\ell \in E^*$.*

(i) *The following conditions are mutually equivalent.*

(a) $\rho \in BV_\ell(E)$.

(b) *There exists a finite signed measure ν_ℓ on E such that for every $v \in \mathcal{F}C_b^1$,*

$$\frac{1}{2}\int_E \partial_\ell v(z)\rho(z)\,\mu(dz) = -\int_E v(z)\,\nu_\ell(dz). \tag{2.1}$$

In this case, ν_ℓ necessarily belongs to $S^{\rho+1}$.

(ii) *Suppose further that $\rho \in QR(E)$. Then the following condition is also equivalent to the above.*

(c) $N^{[\ell]} \in \mathbf{A}_0^\rho$.

In this case, ν_ℓ in (b) *necessarily satisfies that $\nu_\ell|_{E\setminus F^\rho} = 0$, $\nu_\ell|_{F^\rho} \in S^\rho$ and $N^{[\ell]}$ is in Revuz correspondence with it. Furthermore, it holds for any $\mathcal{E}^\rho$-quasicontinuous function v in $\mathcal{F}_b^\rho$ that*

$$\mathcal{E}^\rho(v, \ell(\cdot)) = -\int_{F^\rho} v(z)\,\nu_\ell(dz).$$

Proof. We shall use Theorem 6.2 in [11] for the proof. We remark that Proposition 3.1 in [11] holds for u in the extended Dirichlet space. Then Theorems 3.2, 4.2 and 6.2 in [11] also hold for such u.

Since

$$\frac{1}{2}\int_E \partial_\ell v(z)(\rho(z)+1)\mu(dz) = \frac{1}{2}\int_E \partial_\ell v(z)\rho(z)\mu(dz) + \frac{1}{2}\int_E v(z)\ell(z)\mu(dz),$$

$\rho \in BV_\ell(E)$ if and only if $\rho+1 \in BV_\ell(E)$. Obviously, $\rho+1 \in QR(E)$ for any $\rho \in L^1_+$. The assertion (i) follows from Theorem 6.2 in [11] because $\ell(\cdot) \in \mathcal{F}_e^{\rho+1}$ and $\mathcal{E}^{\rho+1}(v, \ell(\cdot)) = \frac{1}{2}\int_E \partial_\ell v(z)(\rho(z)+1)\mu(dz)$. For the proof of the assertion (ii), consider the following condition:

(b') There exists some $\nu'_\ell \in S^\rho$ such that for every $v \in \mathcal{F}C_b^1$,

$$\mathcal{E}^\rho(v, \ell(\cdot)) = -\int_{F^\rho} v(z)\,\nu'_\ell(dz).$$

From Theorem 6.2 in [11], (a), (b'), and (c) are mutually equivalent and $N^{[\ell]}$ is in Revuz correspondence with ν'_ℓ. Suppose (b'). By considering a measure ν_ℓ on E defined by $\nu_\ell|_{F^\rho} = \nu'_\ell$ and $\nu_\ell|_{E\setminus F^\rho} = 0$, the condition (b) holds. Since ν_ℓ in (b) is uniquely determined if it exists, we get the rest of the assertions. ∎

3 Structures of BV space

First, we introduce three function spaces on E. The H-derivative ∇ defined on $\mathcal{F}C_b^1$ is closable as an operator from L^1 to $L^1(E \to H)$. The domain of its closure is denoted by $\mathbb{D}^{1,1}$, equipped with the norm $\|f\|_{1,1} = \|f\|_1 + \|\nabla f\|_1$. The closure of ∇ is denoted by the same symbol.

Let

$$A_{1/2}(x) = \int_0^x (\log(1+s))^{1/2}\, ds, \quad x \ge 0,$$

and let Ψ be its complementary function, namely,

$$\Psi(y) := \int_0^y (A'_{1/2})^{-1}(t)\, dt = \int_0^y (\exp(t^2) - 1)\, dt.$$

Then it holds for $x \ge 0$ and $y \ge 0$ that

$$xy \le A_{1/2}(x) + \Psi(y). \tag{3.1}$$

Define

$$\begin{aligned} L(\log L)^{1/2} &= \{f \mid A_{1/2}(|f|) \in L^1\}, \\ L^\Psi &= \{g \mid \Psi(c|g|) \in L^1 \text{ for some } c > 0\}. \end{aligned}$$

From the general theory of Orlicz spaces (see e.g. [24, Chapter 3]), we have the following properties.

(i) $L(\log L)^{1/2}$ and L^Ψ are Banach spaces under the norms $\|f\|_{L(\log L)^{1/2}} = \inf\{\alpha > 0 \mid \int_E A_{1/2}(|f|/\alpha)\, d\mu \le 1\}$ and $\|g\|_{L^\Psi} = \inf\{\alpha > 0 \mid \int_E \Psi(|g|/\alpha)\, d\mu \le 1\}$, respectively. (Note: we adopt a terminology different from [24]; e.g. $N_\Psi(\cdot)$ is used in [24] instead of $\|\cdot\|_{L^\Psi}$).

(ii) For $f \in L(\log L)^{1/2}$ and $g \in L^\Psi$, we have

$$\|fg\|_1 \le 2\|f\|_{L(\log L)^{1/2}} \|g\|_{L^\Psi}, \tag{3.2}$$
$$\|fg\|_1 \le (\|A_{1/2}(|f|)\|_1 + 1)\|g\|_{L^\Psi}. \tag{3.3}$$

We give only a proof of (ii) here. Taking $x = |f(z)|/\|f\|_{L(\log L)^{1/2}}$ and $y = |g(z)|/\|g\|_{L^\Psi}$ in (3.1) and integrating both sides, we get (3.2). Taking $x = |f(z)|$ and $y = |g(z)|/\|g\|_{L^\Psi}$ in (3.1) and integrating both sides, we obtain (3.3). We state a direct implication of (3.2) as a next lemma.

Lemma 3.1 *Suppose $\varphi \in L^\Psi$. Then, $\varphi f \in L^1$ for any $f \in L(\log L)^{1/2}$. If a sequence $\{f_n\}$ converges to f in $L(\log L)^{1/2}$, then $\lim_{n\to\infty} \int_E \varphi f_n\, d\mu = \int_E \varphi f\, d\mu$.*

Letting $g \equiv 1$ in (3.2), we see that $L(\log L)^{1/2}$ is continuously embedded in L^1. The following observation is also useful: as a set, $L(\log L)^{1/2} = \{f \mid |f|(\log^+|f|)^{1/2} \in L^1\}$ and $L^\Psi = \{g \mid \exp(c|g|^2) \in L^1 \text{ for some } c > 0\}$, where $\log^+ x = \max(\log x, 0)$. This is because the next estimates hold for some positive constants C_1 and C_2:

$$\begin{aligned} C_1 x(\log^+ x)^{1/2} &\le A_{1/2}(x) \le x + x(\log^+ x)^{1/2}, \quad x \ge 0, \\ \exp(y^2/2) - C_2 &\le \Psi(y) \le \exp(2y^2), \quad y \ge 0. \end{aligned}$$

Also, we have the following embedding theorem.

Proposition 3.2 *The space $\mathbb{D}^{1,1}$ is continuously embedded in $L(\log L)^{1/2}$.*

Proof. The proof is based on the argument in [21, p. 272]. Let $\Phi(r) = (2\pi)^{-1/2} \int_{-\infty}^{r} \exp(-t^2/2)\, dt$, $r \in \mathbb{R}$ and $\mathcal{U}(x) = \Phi' \circ \Phi^{-1}(x)$, $0 < x < 1$. Then $\lim_{x \downarrow 0} \frac{\mathcal{U}(x)}{x\sqrt{2\log 1/x}} = 1$ (cf. [21, p. 271]). We can take a constant $\delta > 0$ such that $\mathcal{U}(x) \ge x\sqrt{\log(1+1/x)}$ for all $x \in (0, \delta]$. We may also take $\delta \le 1/(e-1)$.

Suppose $f \in \mathcal{F}C_b^1$ and $\|f\|_{1,1} \le 1 \big/ \sqrt{\log(1+1/\delta)}$ (≤ 1). The isoperimetric inequality for Gaussian measure implies that

$$\|\nabla f\|_1 \ge \int_0^\infty \mathcal{U}(\mu(\{|f| \ge s\}))\, ds.$$

If $s \ge 1/\delta$, then $\mu(\{|f| \ge s\}) \le \|f\|_1/s \le 1/s \le \delta$ and

$$\begin{aligned} \mathcal{U}(\mu(\{|f| \ge s\})) &\ge \mu(\{|f| \ge s\})\sqrt{\log(1 + 1/\mu(\{|f| \ge s\}))} \\ &\ge \mu(\{|f| \ge s\})\sqrt{\log(1+s)}. \end{aligned}$$

Therefore,

$$\begin{aligned} 1 &\ge \sqrt{\log(1+1/\delta)}\|f\|_{1,1} \ge \sqrt{\log(1+1/\delta)}\|f\|_1 + \|\nabla f\|_1 \\ &\ge \sqrt{\log(1+1/\delta)} \int_0^\infty \mu(\{|f| \ge s\})\, ds + \int_{1/\delta}^\infty \mu(\{|f| \ge s\})\sqrt{\log(1+s)}\, ds \\ &\ge \int_0^\infty \mu(\{|f| \ge s\}) \left(\sqrt{\log(1+1/\delta)} + \sqrt{\log(1+s)} \cdot \mathbf{1}_{\{s \ge 1/\delta\}} \right) ds \\ &\ge \int_0^\infty \mu(\{|f| \ge s\})\sqrt{\log(1+s)}\, ds \\ &= \int_E d\mu \int_0^\infty ds\, \mathbf{1}_{\{|f| \ge s\}} \sqrt{\log(1+s)} \\ &= \int_E d\mu \int_0^{|f|} \sqrt{\log(1+s)}\, ds = \|A_{1/2}(|f|)\|_1. \end{aligned}$$

Therefore, $\|f\|_{L(\log L)^{1/2}} \le 1$. This concludes the claim. ∎

We denote by $(\mathcal{F}C_b^1)_{E^*}$ the set of all E^*-valued functions on E expressed as $\sum_{j=1}^m g_j(z)\ell_j$ with $g_j \in \mathcal{F}C_b^1$ and $\ell_j \in E^*$, $j = 1, \dots, m$ for some $m \in \mathbb{N}$. We also denote by ∇^* the (formal) dual operator with domain $(\mathcal{F}C_b^1)_{E^*}$ of ∇. When $G(z) = g(z)\ell$, $g \in \mathcal{F}C_b^1$, $\ell \in E^*$, we have $\nabla^* G(z) = -\partial_\ell g(z) + g(z)\ell(z)$.

For $\rho \in L(\log L)^{1/2}$, define

$$V(\rho) = \sup\left\{ \int_E (\nabla^* G)\rho\, d\mu \,\middle|\, G \in (\mathcal{F}C_b^1)_{E^*},\ \|G(z)\|_H \le 1 \text{ for every } z \in E \right\} \ (\le \infty).$$

Since $\nabla^* G \in L^\Psi$ for each $G \in (\mathcal{F}C_b^1)_{E^*}$, the integral above is well-defined from Lemma 3.1.

Definition 3.3 Let $BV(E) = \{\rho \in L(\log L)^{1/2} \mid V(\rho) < \infty\}$. We say that ρ in $BV(E)$ is of bounded variation.

By Proposition 3.2 and Lemma 3.1, we see that the next duality relation holds for any $\rho \in \mathbb{D}^{1,1}$,

$$\int_E \nabla^* G(z)\rho(z)\mu(dz) = \int_E \langle G(z), \nabla\rho(z)\rangle \mu(dz), \quad G \in (\mathcal{F}C_b^1)_{E^*},$$

and hence, we have as in [13],

Lemma 3.4 *For $\rho \in \mathbb{D}^{1,1}$, $V(\rho) = \|\nabla\rho\|_1$. In particular, $\mathbb{D}^{1,1} \subset BV(E)$.*

The Ornstein-Uhlenbeck semigroup $\{T_t\}$ is defined as usual: $T_t f(x) = \int_E f(e^{-t}x + \sqrt{1-e^{-2t}}y)\, \mu(dy)$, where f is a function on E taking values in a separable Hilbert space.

Proposition 3.5 *For every $t > 0$, the operator $\nabla T_t : \mathcal{F}C_b^1 \to L^1(E \to H)$ extends uniquely to a bounded operator from $L(\log L)^{1/2}$ to $L^1(E \to H)$.*

Proof. This result is implicitly proved in [20] , but we give a proof for readers' convenience. Let $\varphi \in \mathcal{F}C_b^1$ with $\|\varphi\|_{L(\log L)^{1/2}} \le 1$, $\ell \in E^*$ with $\|\ell\|_H = 1$, and $t > 0$. Note that $\|\ell(\cdot)\|_{L^\Psi}$ is independent of the choice of ℓ. From a direct computation (see e.g. [29]),

$$\partial_\ell T_t\varphi(x) = \frac{e^{-t}}{(1-e^{-2t})^{1/2}} \int_E \varphi(e^{-t}x + (1-e^{-2t})^{1/2}y)\ell(y)\, \mu(dy).$$

Set $\theta = \arccos(e^{-t})$ and $R_\theta(x,y) = (x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta)$. Then,

$$\begin{aligned} |\partial_\ell T_t\varphi(x)| &= \cot\theta \left| \int_E (\varphi\otimes 1)(R_\theta(x,y))\ell(y)\, \mu(dy) \right| \\ &\le c\left(\|A_{1/2}(|(\varphi\otimes 1)(R_\theta(x,\cdot))|)\|_1 + 1 \right). \end{aligned}$$

Here, (3.3) was used in the second line and c is a constant depending only on t. Then,

$$\begin{aligned} \|\nabla T_t\varphi(x)\|_H &= \sup_{\ell\in E^*,\, \|\ell\|_H=1} |\partial_\ell T_t\varphi(x)| \\ &\le c(\|A_{1/2}(|(\varphi\otimes 1)(R_\theta(x,\cdot))|)\|_1 + 1). \end{aligned}$$

From the rotational invariance of $\mu\otimes\mu$,

$$\begin{aligned} \|\nabla T_t\varphi\|_1 &\le c\left(\iint_{E\times E} A_{1/2}(|(\varphi\otimes 1)(R_\theta(x,y))|)\mu(dx)\mu(dy) + 1 \right) \\ &= c\left(\iint_{E\times E} A_{1/2}(|(\varphi\otimes 1)(x,y)|)\mu(dx)\mu(dy) + 1 \right) \\ &= c\left(\int_E A_{1/2}(|\varphi(x)|)\mu(dx) + 1 \right) \le 2c. \end{aligned}$$

Since $\mathcal{F}C_b^1$ is dense in $L(\log L)^{1/2}$, we get the conclusion. ∎

Proposition 3.6 *Take any $f \in L(\log L)^{1/2}$.*

(i) $T_t f \in \mathbb{D}^{1,1}$, $\qquad t > 0$.

(ii) *$T_t f$ converges to f in $L(\log L)^{1/2}$ as $t \downarrow 0$.*

(iii) $V(T_t f) \le e^{-t} V(f) \le \infty, \quad t > 0.$

(iv) $\lim_{t\downarrow 0} V(T_t f) = V(f)$.

Proof. (i) Take a sequence of functions $\{f_n\}$ in $\mathcal{F}C_b^1$ such that f_n converges to f in $L(\log L)^{1/2}$. From Proposition 3.5, $\{\nabla T_t f_n\}$ converges in $L^1(E \to H)$. On the other hand, $\{T_t f_n\}$ converges to $T_t f$ in L^1. By the closedness of ∇ on $\mathbb{D}^{1,1}$, we conclude that $T_t f \in \mathbb{D}^{1,1}$.

(ii) This is clear when f is bounded continuous. For a general f, we have by the Jensen inequality

$$A_{1/2}(|T_t f(x)|/\alpha) \le \int_E A_{1/2}(|f(e^{-t}x + \sqrt{1-e^{-2t}}y)|/\alpha)\,\mu(dy), \quad \alpha > 0.$$

Integrating the both hand sides by $\mu(dx)$, we get

$$\|A_{1/2}(|T_t f|/\alpha)\|_1 \le \|A_{1/2}(|f|/\alpha)\|_1, \quad \alpha > 0,$$

which means that $\|T_t f\|_{L(\log L)^{1/2}} \le \|f\|_{L(\log L)^{1/2}}$. The claim follows from a usual approximation argument.

(iii) By (i), we get the following formula in the same way as in [13]:

$$\int_E \nabla^* G(z) T_t f(z) \mu(dz) = e^{-t} \int_E \nabla^*(T_t G)(z) f(z) \mu(dz), \quad G \in (\mathcal{F}C_b^1)_{E^*}, \quad f \in L(\log L)^{1/2},$$

which immediately implies (iii).

(iv) As in Lemma 4.1 in [13], we can prove that $V(f) \le \varliminf_{t\downarrow 0} V(T_t f)$. By combining (iii), the claim follows. ∎

Now we can give a characterization of the space $BV(E)$ as follows.

Theorem 3.7 *It holds that*

$$BV(E) = \left\{ \rho \in L^1 \;\middle|\; \begin{array}{l} \text{there exists a sequence } \{\rho_n\} \subset \mathbb{D}^{1,1} \text{ such that} \\ \rho_n \to \rho \text{ in } L^1 \text{ and } \sup_n \|\nabla \rho_n\|_1 < \infty. \end{array} \right\}. \tag{3.4}$$

Moreover, if ρ_n and ρ are as in the right hand side of (3.4)*, then* $V(\rho) \le \varliminf_{n\to\infty} \|\nabla \rho_n\|_1$.

Proof. Let the right-hand side of (3.4) be denoted by BV_1. First, we prove $BV(E) \subset BV_1$. This is proved in the same way as in Proposition 4.1 in [13]. Let $\rho_n = T_{1/n}\rho$, $n \in \mathbb{N}$. Then $\rho_n \to \rho$ in L^1 as $n \to \infty$, and from Proposition 3.6 and Lemma 3.4, $\rho_n \in \mathbb{D}^{1,1}$, $\|\nabla \rho_n\|_1 = V(\rho_n) \le V(\rho)$ for every n. Therefore, $\rho \in BV_1$.

Next, we prove $BV_1 \subset BV(E)$. For $\rho \in BV_1$, let $\{\rho_n\}$ be as in the definition of BV_1. Let $M = \varliminf_{n\to\infty} \|\nabla \rho_n\|_1 < \infty$. From Proposition 3.2, $\{\rho_n\}$ is bounded in $L(\log L)^{1/2}$. By taking a subsequence, we may assume that ρ_n converges to ρ μ-a.e. From Fatou's lemma, $\rho \in L(\log L)^{1/2}$.

Let Φ_m be a smooth function on $\mathbb{R}$ such that $0 \le \Phi_m' \le 1$ on $\mathbb{R}$, $\Phi_m(x) = x$ on $[-m, m]$ and $|\Phi_m(x)| = m+1$ on $\mathbb{R} \setminus [-m-2, m+2]$. Then for $G \in (\mathcal{F}C_b^1)_{E^*}$,

$$\begin{aligned}\left|\int_E (\nabla^* G)\Phi_m \circ \rho \, d\mu\right| &= \lim_{n\to\infty}\left|\int_E (\nabla^* G)\Phi_m \circ \rho_n \, d\mu\right| = \lim_{n\to\infty}\left|\int_E \langle G, \nabla(\Phi_m \circ \rho_n)\rangle \, d\mu\right| \\ &\le \varliminf_{n\to\infty} \|G\|_\infty \|\nabla(\Phi_m \circ \rho_n)\|_1 \le \varliminf_{n\to\infty} \|G\|_\infty \|\nabla \rho_n\|_1 \le M\|G\|_\infty.\end{aligned}$$

Since $\Phi_m \circ \rho \to \rho$ as $m \to \infty$ in $L(\log L)^{1/2}$, $\int_E (\nabla^* G)\Phi_m \circ \rho \, d\mu \to \int_E (\nabla^* G)\rho \, d\mu$ by Lemma 3.1. Therefore, $|\int_E (\nabla^* G)\rho \, d\mu| \le M\|G\|_\infty$. Hence, $\rho \in BV(E)$ and $V(\rho) \le M$. ▮

Corollary 3.8 *Let Λ be a function on $\mathbb{R}$ so that $|\Lambda(x) - \Lambda(y)| \le |x - y|$ for all $x, y \in \mathbb{R}$. Then for every $\rho \in BV(E)$, $\Lambda \circ \rho$ also belongs to $BV(E)$ and $V(\Lambda \circ \rho) \le V(\rho)$. In particular, the space $BV(E)$ is a vector lattice.*

Proof. Let $\rho_n = T_{1/n}\rho$, $n \in \mathbb{N}$. Then $\{\Lambda \circ \rho_n\} \subset \mathbb{D}^{1,1}$, $\Lambda \circ \rho_n \to \Lambda \circ \rho$ in L^1 as $n \to \infty$, and

$$\sup_n \|\nabla(\Lambda \circ \rho_n)\|_1 \le \sup_n \|\nabla \rho_n\|_1 \le V(\rho)$$

by Lemma 3.4 and Proposition 3.6. This implies that $\Lambda \circ \rho \in BV(E)$ and $V(\Lambda \circ \rho) \le V(\rho)$ by virtue of Theorem 3.7. ▮

We now extend Theorem 3.1 (ii) in [13] together with a uniqueness statement.

Theorem 3.9 *For each $\rho \in BV(E)$, there exists a positive finite measure ν on E and an H-valued Borel function σ on E such that $\|\sigma\|_H = 1$ ν-a.e. and for every $G \in (\mathcal{F}C_b^1)_{E^*}$,*

$$\int_E (\nabla^* G)\rho \, d\mu = \int_E \langle G, \sigma\rangle \, d\nu. \tag{3.5}$$

The measure ν belongs to $S^{|\rho|+1}$. If moreover $\rho \in QR(E)$, then $\nu|_{E\setminus F^\rho} = 0$ and $\nu|_{F^\rho} \in S^\rho$. Also, ν and σ are uniquely determined; namely, if ν' and σ' are another pair satisfying the relation (3.5) for all $G \in (\mathcal{F}C_b^1)_{E^*}$, then $\nu = \nu'$ and $\sigma = \sigma'$ ν-a.e.

Proof. First, suppose $\rho \ge 0$ μ-a.e. By Theorem 2.1, for each $\ell \in E^*$, there exists a signed measure ν_ℓ on E which belongs to $S^{\rho+1}$ (resp. such that $\nu_\ell|_{E\setminus F^\rho} = 0$ and $\nu_\ell|_{F^\rho} \in S^\rho$ if $\rho \in BV(E) \cap QR(E)$) satisfying

$$\frac{1}{2}\int_E \partial_\ell v(z)\rho(z)\mu(dz) = -\int_E v(z)\,\nu_\ell(dz), \quad v \in \mathcal{F}C_b^1.$$

Define $D_\ell\rho = 2\nu_\ell + \ell(\cdot)\rho \cdot \mu$. Then for an H-valued function G expressed as $G(z) = g(z)\ell$ with $g \in \mathcal{F}C_b^1$, $\ell \in E^*$ and $\|\ell\|_H = 1$, we have

$$\int_E (\nabla^* G)\rho \, d\mu = \int_E (-\partial_\ell g + g\ell(\cdot))\rho \, d\mu = \int_E g(z)\, D_\ell\rho(dz).$$

Therefore, $V(D_\ell\rho)$, the total variation measure of $D_\ell\rho$, satisfies $V(D_\ell\rho)(E) \le V(\rho)$.

For a general ρ, let ρ_+ and ρ_- be the positive part and the negative part of ρ, respectively. Then from Corollary 3.8, $\rho_\pm \in BV(E)$ and $V(\rho_\pm) \le V(\rho)$. Therefore, $V(D_\ell\rho_\pm)(E) \le V(\rho)$. Define $D_\ell\rho = D_\ell\rho_+ - D_\ell\rho_-$. Then $V(D_\ell\rho)(E) \le 2V(\rho)$.

Take $\{h_j\}_{j=1}^{\infty} \subset E^*$ as a c.o.n.s. of H. Let $\gamma = \sum_{j=1}^{\infty} 2^{-j} V(D_{h_j}\rho)$ and $v_j(z) = \frac{dD_{h_j}\rho}{d\gamma}(z)$, $j \in \mathbb{N}$. Then γ is a positive finite measure, $\gamma(E) \le 2V(\rho)$, and $\gamma \in S^{|\rho|+1}$ (resp. $\gamma|_{E\setminus F^\rho} = 0$ and $\gamma|_{F^\rho} \in S^\rho$ if $\rho \in BV(E) \cap QR(E)$). We may assume that each v_j is Borel measurable. From the same argument as in the proof of Theorem 3.1 (ii) in [13], we can construct ν and σ from γ and v_j so that (3.5) holds for all G expressed as $G(z) = \sum_{j=1}^{n} g_j(z) h_j$ with $g_j \in \mathcal{F}C_b^1$, $j = 1, \dots, n$ for some $n \in \mathbb{N}$. In order to finish the proof of the first claim, it suffices to prove the validity of (3.5) for $G(z) = g(z)\ell$, where $g \in \mathcal{F}C_b^1$ and $\ell \in E^*$ with $\|\ell\|_H = 1$. The relation to prove is

$$-\int_E (\partial_\ell g)\rho\, d\mu + \int_E g\ell(\cdot)\rho\, d\mu = \int_E g\langle \ell, \sigma\rangle\, d\nu. \tag{3.6}$$

Let $\ell_n = \sum_{j=1}^{n} \langle \ell, h_j\rangle h_j$, $n \in \mathbb{N}$. Denote the linear span of $\{\ell_n, \ell\}$ in H by H_n, and its orthogonal complement by $H_n^\perp$. Take a unitary operator U_n on H satisfying that H_n is U_n-invariant, $U_n(\ell_n) = \ell$, and $U_n|_{H_n^\perp}$ is an identity mapping. U_n can be extended to a continuous operator on E, leaving μ invariant. Set $g_n(z) = g(U_n(z))$, $z \in E$. We already know that (3.6) holds if ℓ and g are replaced by ℓ_n and g_n, respectively. We shall observe that each term converges appropriately as $n \to \infty$. Since $\partial_{\ell_n} g_n(z) = (\partial_\ell g)(U_n(z))$, it holds by the dominated convergence theorem that

$$\lim_{n\to\infty} \int_E (\partial_{\ell_n} g_n)\rho\, d\mu = \int_E (\partial_\ell g)\rho\, d\mu$$

and

$$\lim_{n\to\infty} \int_E g_n \langle \ell_n, \sigma\rangle\, d\nu = \int_E g\langle \ell, \sigma\rangle\, d\nu.$$

From the U_n-invariance of μ, it holds that $\rho \circ U_n^{-1} \to \rho$ in $L(\log L)^{1/2}$ as $n \to \infty$. Indeed, this is proved by approximating ρ by bounded continuous functions and using a triangle inequality. Then from Lemma 3.1,

$$\int_E g_n \ell_n(\cdot)\rho\, d\mu = \int_E g\ell(\cdot)(\rho \circ U_n^{-1})\, d\mu \to \int_E g\ell(\cdot)\rho\, d\mu \quad \text{as } n \to \infty.$$

Therefore, (3.6) holds.

We shall proceed to the proof of uniqueness. Suppose that σ' and ν' are another pair. Then,

$$\int_E \langle G, \gamma\rangle\, d\xi = 0 \quad \text{for every } G \in (\mathcal{F}C_b^1)_{E^*},$$

where $\xi = \nu + \nu'$ and $\gamma = \sigma \dfrac{d\nu}{d\xi} - \sigma' \dfrac{d\nu'}{d\xi}$. Taking a uniformly bounded sequence $\{G_n\} \subset (\mathcal{F}C_b^1)_{E^*}$ so that $\langle G_n, \gamma\rangle \to \|\gamma\|_H$ ξ-a.e., we get $\gamma = 0$ ξ-a.e. Therefore, $\|\sigma\|_H \dfrac{d\nu}{d\xi} = \|\sigma'\|_H \dfrac{d\nu'}{d\xi}$ ξ-a.e. Since $\|\sigma\|_H = 1$ ν-a.e., $\|\sigma\|_H \dfrac{d\nu}{d\xi} = \dfrac{d\nu}{d\xi}$ ξ-a.e. Similarly, $\|\sigma'\|_H \dfrac{d\nu'}{d\xi} = \dfrac{d\nu'}{d\xi}$ ξ-a.e. Then, $\dfrac{d\nu}{d\xi} = \dfrac{d\nu'}{d\xi}$ ξ-a.e., which implies $\nu = \nu'$. Also, it follows that $\sigma = \sigma'$ ν-a.e. from $\gamma = 0$ ξ-a.e. and $\nu = \nu'$. ∎

We shall hereafter write $\|D\rho\|$ and σ_ρ in place of ν and σ, respectively.

Corollary 3.10 *It holds that*

$$BV(E) = \left\{ \rho \in L(\log L)^{1/2} \;\middle|\; \begin{array}{l} \textit{there exist a sequence } \{\rho_n\} \subset \mathbb{D}^{1,1}, \textit{ a positive finite} \\ \textit{measure } \nu \textit{ on } E \textit{ and an } H\textit{-valued Borel function } \sigma \textit{ on } E \\ \textit{such that } \rho_n \to \rho \textit{ in } L(\log L)^{1/2}, \\ \sup_n \|\nabla\rho_n\|_1 < \infty, \|\sigma\|_H = 1 \ \nu\text{-a.e., } \textit{and} \\ \lim_{n\to\infty} \int_E \langle G, \nabla\rho_n\rangle \, d\mu = \int_E \langle G, \sigma\rangle \, d\nu \textit{ for all } G \in (\mathcal{F}C_b^1)_{E^*}. \end{array} \right\}. \tag{3.7}$$

Furthermore, σ and ν in the right-hand side coincide with σ_ρ and $\|D\rho\|$, respectively.

Proof. Let the right-hand side in (3.7) be denoted by BV_2. Take $\rho \in BV(E)$, and let $\rho_n = T_{1/n}\rho$ for $n \in \mathbb{N}$. Then $\rho_n \in \mathbb{D}^{1,1}$, $\rho_n \to \rho$ in $L(\log L)^{1/2}$, and $V(\rho_n) \le V(\rho)$ by Proposition 3.6. For any $G \in (\mathcal{F}C_b^1)_{E^*}$,

$$\int_E \langle G, \nabla\rho_n\rangle \, d\mu = \int_E (\nabla^* G)\rho_n \, d\mu \to \int_E (\nabla^* G)\rho \, d\mu = \int_E \langle G(z), \sigma_\rho(z)\rangle \, \|D\rho\|(dz) \quad \text{as } n \to \infty.$$

Therefore, $\rho \in BV_2$. The inverse inclusion is trivial from Theorem 3.7. The latter assertion follows from Theorem 3.9. ▍

REMARK 3.11 We can also replace $\mathbb{D}^{1,1}$ by $\mathcal{F}C_b^1$ and two $L(\log L)^{1/2}$'s by L^1 in the right-hand side of (3.7). Compare (3.7) with characterizations of $\mathbb{D}^{1,1}$:

$$\begin{aligned} \mathbb{D}^{1,1} &= \left\{ \rho \in L^1 \;\middle|\; \begin{array}{l} \text{there exist a sequence } \{\rho_n\} \subset \mathcal{F}C_b^1 \text{ and } J \in L^1(E \to H) \text{ such} \\ \text{that } \rho_n \to \rho \text{ in } L^1 \text{ and } \nabla\rho_n \to J \text{ in } L^1(E \to H). \end{array} \right\} \\ &= \left\{ \rho \in L(\log L)^{1/2} \;\middle|\; \begin{array}{l} \text{there exist a sequence } \{\rho_n\} \subset \mathcal{F}C_b^1 \text{ and} \\ J \in L^1(E \to H) \text{ such that } \rho_n \to \rho \text{ in } L(\log L)^{1/2} \\ \text{and } \nabla\rho_n \to J \text{ in } L^1(E \to H). \end{array} \right\}. \end{aligned}$$

We can now present a formula analogous to the classical Green formula. We consider the Ornstein-Uhlenbeck operator $L = -\nabla^*\nabla$, which can be expressed as

$$Lu(z) = \sum_{i=1}^m \partial_i^2 f(\ell_1(z), \dots, \ell_m(z)) - \sum_{i=1}^m \partial_i f(\ell_1(z), \dots, \ell_m(z))\ell_i(z)$$

if $u(z) = f(\ell_1(z), \dots, \ell_m(z)) \in \mathcal{F}C_b^2$ and $\ell_1, \dots, \ell_m \in E^*$ are orthonormal in H.

Theorem 3.12 *Let $\rho \in BV(E)$. For any $u \in \mathcal{F}C_b^2$ and any $v \in \mathcal{F}C_b^1$,*

$$\mathcal{E}^\rho(u, v) = -\frac{1}{2}\int_E v(Lu)\rho \, d\mu - \frac{1}{2}\int_E v(z)\langle \nabla u(z), \sigma_\rho(z)\rangle \|D\rho\|(dz). \tag{3.8}$$

If further $\rho \in QR(E)$, then the equation obtained by replacing E by F^ρ in (3.8) holds for any $u \in \mathcal{F}C_b^2$ and any $\mathcal{E}^\rho$-quasicontinuous function $v \in \mathcal{F}_b^\rho$.

Proof. Take ρ_n as in the right hand side of (3.7). An integration by part gives

$$\int_E \langle \nabla u, \nabla v\rangle \rho_n \, d\mu = -\int_E v(Lu)\rho_n \, d\mu - \int_E v\langle \nabla u, \nabla\rho_n\rangle \, d\mu.$$

Letting n tend to infinity, we get the desired formula. ▍

The next theorem is a converse to Lemma 3.4 and characterizes the space $\mathbb{D}^{1,1}$ as a subspace of $BV(E)$.

Theorem 3.13 *Let $\rho \in BV(E)$. If $\|D\rho\| \ll \mu$, Then $\rho \in \mathbb{D}^{1,1}$ and*

$$\|D\rho\| = \|\nabla\rho\|_H \cdot \mu, \quad \sigma_\rho = \frac{\nabla\rho}{\|\nabla\rho\|_H} \cdot \mathbf{1}_{\{\nabla\rho\neq 0\}}.$$

Proof. It suffices to prove that if there exists $J \in L^1(E \to H)$ such that

$$\int_E (\nabla^* G)\rho \, d\mu = \int_E \langle G, J\rangle \, d\mu, \qquad \text{for all } G \in (\mathcal{F}C_b^1)_{E^*},$$

then $\rho \in \mathbb{D}^{1,1}$ and $\nabla\rho = J$. Since $\rho \in L(\log L)^{1/2}$, $T_t\rho$ belongs to $\mathbb{D}^{1,1}$ and for any $G \in (\mathcal{F}C_b^1)_{E^*}$,

$$\int_E \langle G, \nabla T_t\rho\rangle \, d\mu = \int_E e^{-t}(\nabla^* T_t G)\rho \, d\mu = e^{-t}\int_E \langle T_t G, J\rangle \, d\mu = e^{-t}\int_E \langle G, T_t J\rangle \, d\mu.$$

Therefore, $\nabla T_t\rho = e^{-t}T_t J$. This converges to J in L^1 as $t \to 0$, which implies that $\rho \in \mathbb{D}^{1,1}$ and $\nabla\rho = J$. ▌

Remark 3.14 We have a coarea formula

$$V(\rho) = \int_{-\infty}^{\infty} V(\mathbf{1}_{\{\rho > t\}}) \, dt, \quad \rho \in BV(E),$$

just as Theorem 4.1 in [13]. Then for every $\rho \in BV(E)$, $\mathbf{1}_{\{\rho>t\}}$ belongs to $BV(E)$ for a.e. t with respect to the Lebesgue measure. However, $\mathbf{1}_A \in \mathbb{D}^{1,1}$ if and only if $\mu(A) = 0$ or 1. Indeed, if $\mathbf{1}_A \in \mathbb{D}^{1,1}$, then $\nabla\mathbf{1}_A = \nabla(\mathbf{1}_A)^2 = 2\cdot\mathbf{1}_A\,\nabla\mathbf{1}_A$, which implies that $\nabla\mathbf{1}_A = 0$ μ-a.e. Then for all $t > 0$, $\nabla T_t\mathbf{1}_A = e^{-t}T_t\nabla\mathbf{1}_A = 0$ μ-a.e. Since $T_t\mathbf{1}_A \in \mathcal{F}^1$, it is well-known that $T_t\mathbf{1}_A =$ constant μ-a.e., therefore, $\mathbf{1}_A =$ constant μ-a.e. Hence, there are many functions which belong to $BV(E)$ but do not belong to $\mathbb{D}^{1,1}$.

Finally in this section, we study the support of $\|D\rho\|$. Let $\rho \in QR(E)$. We denote the $\mathcal{E}^\rho$-quasi support of $\nu \in S_+^\rho$ by $\mathcal{E}^\rho$-q. Supp ν. When A is a measurable subset of E, we define

$$\overline{A}^\rho = \mathcal{E}^\rho\text{-q.}\operatorname{Supp}(\mathbf{1}_A \cdot \rho \cdot \mu), \quad \partial^\rho A = \overline{A}^\rho \cap \overline{E \setminus A}^\rho.$$

Theorem 3.15 *Let $\rho \in BV(E)$. Then for every $a \in \mathbb{R}$, we have $\|D\rho\|(E \setminus \overline{\{\rho \neq a\}}^{|\rho|+1}) = 0$, namely,*

$$\mathcal{E}^{|\rho|+1}\text{-q.}\operatorname{Supp}\,\|D\rho\| \subset \overline{\{\rho \neq a\}}^{|\rho|+1} \quad \mathcal{E}^{|\rho|+1}\text{-q.e.}$$

When $\rho = \mathbf{1}_A \in BV(E)$ for a certain set A, we have

$$\mathcal{E}^1\text{-q.}\operatorname{Supp}\,\|D\rho\| \subset \partial^1 A \quad \mathcal{E}^1\text{-q.e.}$$

Proof. Let $a \in \mathbb{R}$. From the way of construction of $\|D\rho\|$ in the proof of Theorem 3.9, it is enough to prove that $D_\ell \rho(E \setminus F_a) = 0$ for each $\ell \in E^*$, where $F_a := \overline{\{\rho \neq a\}}^{|\rho|+1}$. By Lemma 4.6.1 in [14], there is a nonnegative and $\mathcal{E}^{|\rho|+1}$-quasicontinuous function u in $\mathcal{F}_b^{|\rho|+1}$ such that $F_a = \{u = 0\}$ $\mathcal{E}^{|\rho|+1}$-q.e. We can take a uniformly bounded sequence $\{u_n\} \subset \mathcal{F}C_b^1$ such that $u_n \to u$ in $\mathcal{F}^{|\rho|+1}$ and $\mathcal{E}^{|\rho|+1}$-q.e. as $n \to \infty$. For any $g \in \mathcal{F}C_b^1$, we have

$$\begin{aligned}\int_E g(z)u_n(z)\, D_\ell\rho(dz) &= \int_E g(z)u_n(z)\, D_\ell(\rho - a)(dz) \\ &= -\int_E \partial_\ell(gu_n)(\rho - a)\, d\mu + \int_E gu_n\ell(\cdot)(\rho - a)\, d\mu \\ &= -\int_E \{(\partial_\ell g)u_n + g\langle \nabla u_n, \ell\rangle\}(\rho - a)\, d\mu + \int_E gu_n\ell(\cdot)(\rho - a)\, d\mu.\end{aligned}$$

Keeping $\mathcal{E}^{|\rho|+1}$-smoothness of $D_\ell\rho$ in mind, we obtain by letting $n \to \infty$ that

$$\int_E g(z)u(z)\, D_\ell\rho(dz) = -\int_E \{(\partial_\ell g)u + g\langle \nabla u, \ell\rangle\}(\rho - a)\, d\mu + \int_E gu\ell(\cdot)(\rho - a)\, d\mu. \quad (3.9)$$

Since $u = 0$ on F_a μ-a.e., $\nabla u = 0$ on F_a μ-a.e. by Theorem 7.1.1 in [6, Chapter I]. Therefore, the right-hand side of (3.9) vanishes. This means that $D_\ell\rho(E \setminus F_a) = 0$, which finishes the proof of the first part.

When $\rho = \mathbf{1}_A \in BV(E)$, by applying the first claim with $a = 0$ and $a = 1$, we get

$$\mathcal{E}^1\text{-q.}\,\mathrm{Supp}\,\|D\rho\| \subset \overline{A}^1 \cap \overline{E \setminus A}^1 = \partial^1 A \quad \mathcal{E}^1\text{-q.e.} \qquad \blacksquare$$

4 Distorted Ornstein-Uhlenbeck process and Itô's formula

Since E is separable, E^* is also separable in the weak*-topology (see e.g. [26, p.90]). Let $\{\ell_n\}$ be a countable dense subset of E^*.

Lemma 4.1 *Let $\{B_t\}$ be an E-valued continuous process starting at 0. If $\{\ell_n(B_t)\}$ is an $\{\mathcal{M}_t\}$-martingale with quadratic variation $t\|\ell_n\|_H^2$ for every n, then $\{B_t\}$ is an $\{\mathcal{M}_t\}$-Brownian motion on E.*

Proof. By the martingale representation theorem, each $\{\ell_n(B_t)\}$ is a 1-dimensional $\{\mathcal{M}_t\}$-Brownian motion with a constant time change. Take any $\ell \in E^*$ with $\|\ell\|_H = 1$. There exists a subsequence $\{\ell_{n_k}\}$ of $\{\ell_n\}$ converging to ℓ in the weak* sense. Then,

$$\begin{aligned}\lim_{k\to\infty} \exp\left(-\|\ell_{n_k}\|_H^2/2\right) &= \lim_{k\to\infty} \int_E \exp\left(\sqrt{-1}\ell_{n_k}(z)\right) \mu(dz) \\ &= \int_E \exp\left(\sqrt{-1}\ell(z)\right) \mu(dz) = \exp(-1/2),\end{aligned}$$

therefore $\lim_{k\to\infty} \|\ell_{n_k}\|_H = 1$. For $\xi \in \mathbb{R}$, $t > s > 0$ and an $\mathcal{M}_s$-measurable bounded function f,

$$\begin{aligned}\mathbb{E}\left[\exp\left(\sqrt{-1}\xi(\ell(B_t) - \ell(B_s))\right) f\right] &= \lim_{k\to\infty} \mathbb{E}\left[\exp\left(\sqrt{-1}\xi(\ell_{n_k}(B_t) - \ell_{n_k}(B_s))\right) f\right] \\ &= \lim_{k\to\infty} \exp\left(-(t-s)\xi^2\|\ell_{n_k}\|_H^2/2\right) \mathbb{E}[f] \\ &= \exp\left(-(t-s)\xi^2/2\right) \mathbb{E}[f].\end{aligned}$$

Namely, $\{\ell(B_t)\}$ is a 1-dimensional $\{\mathcal{M}_t\}$-Brownian motion. Therefore, $\{B_t\}$ is an $\{\mathcal{M}_t\}$-Brownian motion on E. ∎

By using this lemma, Theorem 3.2 in [13] is improved as follows.

Theorem 4.2 *Let $\rho \in BV(E) \cap QR(E)$. Then the sample path of the distorted Ornstein-Uhlenbeck process $\mathbf{M}^\rho = (X_t, \mathcal{M}_t, P_z)$ associated with $(\mathcal{E}^\rho, \mathcal{F}^\rho)$ admits the following expression as a sum of three E-valued CAF's*:

$$X_t(\omega) - X_0(\omega) = W_t(\omega) - \frac{1}{2}\int_0^t X_s(\omega)\,ds + \frac{1}{2}\int_0^t \sigma_\rho(X_s(\omega))\,dA_s^{\|D\rho\|}(\omega), \quad t \geq 0. \quad (4.1)$$

Here, the $\mathcal{E}^\rho$-smooth measure $\|D\rho\|$ and the H-valued function σ_ρ are defined as in Theorem 3.9; $A^{\|D\rho\|}$ is a real valued PCAF associated with $\|D\rho\|$ via the Revuz correspondence. Moreover, for $\mathcal{E}^\rho$-q.e. $z \in F^\rho$, $\{W_t(\omega)\}$ is an $\{\mathcal{M}_t\}$-Brownian motion on E under P_z.

Proof. We can define an E-valued CAF W_t by the equation (4.1). As in the same way of the proof of Theorem 3.2 in [13], for every $\ell \in E^*$, for $\mathcal{E}^\rho$-q.e. z, $\{\ell(W_t)\}$ is a martingale under P^z with quadratic variation $t\|\ell\|_H^2$. Since a countable union of exceptional sets is also exceptional, Lemma 4.1 completes the proof. ∎

We now turn to a generalized Itô's formula which has been formulated in [12] for the additive functionals of the distorted Brownian motion on $\mathbb{R}^d$. For this purpose, we prepare a lemma for quasi-sure analysis on Hilbert space valued functions. Though it is quite standard and we need it only for the Ornstein-Uhlenbeck semigroup, we shall formulate it under a general framework and give a proof for completeness. Let $(\mathcal{E}, \mathcal{F})$ be a quasi-regular symmetric Dirichlet form on a state space (Ω, m), where Ω is a Hausdorff topological space with a countable base and m is a σ-finite Borel measure on Ω. Let $\{P_t\}$ and (ω_t, Q_z) be a Markovian semigroup on $L^2(\Omega; m)$ and a Markov process on Ω associated with $(\mathcal{E}, \mathcal{F})$, respectively. The expectation with respect to Q_z is denoted by E^{Q_z}. Let K be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|_K$. For a K-valued step function G on Ω expressed as $G = \sum_{i=1}^n \mathbf{1}_{A_i} k_i$ with $k_i \in K$, $m(A_i) < \infty$, and $A_i \cap A_j = \emptyset$ if $i \neq j$, define $P_t G = \sum_{i=1}^n (P_t \mathbf{1}_{A_i}) k_i$, $t > 0$. Then P_t extends uniquely to a bounded operator on $L^2(\Omega \to K; m)$ and satisfies that $\|P_t G\|_K \leq P_t(\|G\|_K)$ m-a.e. for all $G \in L^2(\Omega \to K; m)$. In the following lemma, $\tilde{f}$ means an $\mathcal{E}$-quasicontinuous modification of a function f on Ω if it exists. Note that an $\mathcal{E}$-quasicontinuous modification of a K-valued function is uniquely determined up to $\mathcal{E}$-exceptional set like real-valued functions, because of the separability of K.

Lemma 4.3 (i) *Let $\{G_n\}$ be a sequence of $\mathcal{E}$-quasicontinuous functions in $L^2(\Omega \to K; m)$ and $G \in L^2(\Omega \to K; m)$. If there exists a sequence $\{w_n\} \subset \mathcal{F}$ such that*

$$\|G_n - G\|_K \leq w_n \ m\text{-a.e. } \textit{for all } n \textit{ and } w_n \to 0 \textit{ in } \mathcal{F} \textit{ as } n \to \infty,$$

then G has an $\mathcal{E}$-quasicontinuous modification $\tilde{G}$, and $G_{n_k} \to \tilde{G}$ $\mathcal{E}$-q.e. for some subsequence $\{G_{n_k}\}$.

(ii) *Let G be a Borel measurable function in $L^2(\Omega \to K; m)$ and $t > 0$. Then $E^{Q_\cdot}[G(\omega_t)]$ is an $\mathcal{E}$-quasicontinuous modification of $P_t G$.*

(iii) *Let G_n, $G \in L^2(\Omega \to K; m)$ and $G_n \to G$ in $L^2(\Omega \to K; m)$ as $n \to \infty$. Then for each $t > 0$, there is a subsequence $\{G_{n_k}\}$ such that $\widetilde{P_t G_{n_k}} \to \widetilde{P_t G}$ $\mathcal{E}$-q.e.*

Proof. Let Cap denote the capacity associated with $(\mathcal{E}, \mathcal{F})$.

(i): Let $\varepsilon > 0$. Since $w_n \to 0$ in $\mathcal{F}$, there exists a sequence $\{n_l\} \uparrow \infty$ and an open set U_1 such that $\mathrm{Cap}(U_1) < \varepsilon/2$ and $\widetilde{w_{n_l}}$ converges to 0 uniformly on $E \setminus U_1$. Since $\|G_m - G_n\|_K \le |w_m| + |w_n|$ m-a.e., it holds that $\|G_m - G_n\|_K \le |\widetilde{w_m}| + |\widetilde{w_n}|$ $\mathcal{E}$-q.e. There is an open set U_2 such that $\mathrm{Cap}(U_2) < \varepsilon/2$, $G_n|_{E\setminus U_2}$ is continuous for every n, and the inequality above holds on $E \setminus U_2$ for every m and n. Let $U = U_1 \cup U_2$. Then $\mathrm{Cap}(U) < \varepsilon$ and $\{G_{n_l}\}$ converges uniformly on $E \setminus U$. By diagonalization argument, we can take a subsequence of $\{G_n\}$ which converges to some $\mathcal{E}^1$-quasicontinuous function $\tilde{G}$ $\mathcal{E}$-q.e., and clearly $G = \tilde{G}$ m-a.e.

(ii): Take K-valued step functions $\{G_n\}$ such that $G_n \to G$ in $L^2(\Omega \to K; m)$ as $n \to \infty$. Each $P_t G_n$ has an $\mathcal{E}$-quasicontinuous modification in view of the result for scalar valued functions. It holds that

$$\|\widetilde{P_t G_n} - P_t G\|_K \le P_t(\|G_n - G\|_K) \quad m\text{-a.e.},$$

and the right-hand side of the inequality above converges to 0 in $\mathcal{F}$ as $n \to \infty$. By (i), $P_t G$ has an $\mathcal{E}^1$-quasicontinuous modification. On the other hand, $E^{Q_z}[G(\omega_t)]$ exists for $\mathcal{E}$-q.e. z since $E^{Q_z}[\|G(\omega_t)\|_K] < \infty$ $\mathcal{E}$-q.e.

For each $k \in K$, we have

$$\langle \widetilde{P_t G}, k\rangle = P_t(\langle G, k\rangle) = E^{Q_\cdot}[\langle G, k\rangle(\omega_t)] \quad m\text{-a.e.}$$

Therefore, $\langle \widetilde{P_t G}, k\rangle = E^{Q_\cdot}[\langle G, k\rangle(\omega_t)]$ $\mathcal{E}$-q.e. since both are $\mathcal{E}$-quasicontinuous. This implies that $\widetilde{P_t G}(z) = E^{Q_z}[G(\omega_t)]$ for $\mathcal{E}$-q.e. z.

(iii): From (ii), $P_t G_n$ has an $\mathcal{E}$-quasicontinuous modification for every n. Since

$$\|\widetilde{P_t G_n} - P_t G\|_K \le P_t(\|G_n - G\|_K) \quad m\text{-a.e.}$$

and the right-hand side converges to 0 in $\mathcal{F}$ as $n \to \infty$, the assertion follows from (i). ∎

Recall Theorem 3.12 where the Ornstein-Uhlenbeck operator $L = -\nabla^*\nabla$ appears. Let $\{T_t\}$ be its associated Ornstein-Uhlenbeck semigroup as before. Its corresponding Dirichlet form is nothing but $(2\mathcal{E}^1, \mathcal{F}^1)$. The following theorem is a counterpart of Theorem 3.3 in [12]. Below, all functions are regarded as Borel measurable.

Theorem 4.4 *Suppose either of the following.*

(a) *$\rho \in BV(E) \cap QR(E)$, $u(z) = f(\ell_1(z), \dots, \ell_m(z)) \in \mathcal{F}C_b^1$, and $u^\varepsilon(z) = f^\varepsilon(\ell_1(z), \dots, \ell_m(z))$, where f^ε is an ordinary mollification of f on $\mathbb{R}^m$.*

(b) *$\rho \in BV(E) \cap QR(E) \cap L^\infty$, $u = w|_{F^\rho}$ for some $w \in \mathcal{F}_b^1$ such that $\|\nabla w\|_H$ is μ-essentially bounded and ∇w has an $\mathcal{E}^1$-quasicontinuous modification, and $u^\varepsilon = T_\varepsilon w$. In this case, ∇u denotes $\nabla w|_{F^\rho}$.*

Then, the next conditions are equivalent.

(i) $N^{[u]} \in \mathbf{A}_0^\rho$.

(ii) *There exists a finite signed measure* $\nu_{u,\rho}$ *on* F^ρ *such that*

$$\lim_{\varepsilon\to 0}\int_{F^\rho} v(Lu^\varepsilon)\rho\,d\mu = \int_{F^\rho} v\,d\nu_{u,\rho} \quad \textit{for every } v \in \mathcal{F}C_b^1.$$

In this case, it holds that for any $v \in \mathcal{F}C_b^1$,

$$\mathcal{E}^\rho(u,v) = -\frac{1}{2}\int_{F^\rho} v\,d\nu_{u,\rho} - \frac{1}{2}\int_{F^\rho} v(z)\langle \widetilde{\nabla u}(z), \sigma_\rho(z)\rangle \,\|D\rho\|(dz),$$

where $\widetilde{\nabla u}$ *is an* $\mathcal{E}^\rho$*-quasicontinuous modification of* ∇u. *Moreover,* $\nu_{u,\rho} \in S^\rho$. *Let* A^{Lu} *and* $A^{\|D\rho\|}$ *denote the CAFs associated with* $\nu_{u,\rho}$ *and* $\|D\rho\|$, *respectively. Then* P_z-a.e. *for* $\mathcal{E}^\rho$*-q.e.* $z \in F^\rho$, *the equation*

$$u(X_t) - u(X_0) = M_t^{[u]} + \frac{1}{2}A_t^{Lu} + \frac{1}{2}\int_0^t \langle \widetilde{\nabla u}, \sigma_\rho\rangle(X_s)\,dA_t^{\|D\rho\|}$$

holds. Here $M^{[u]}$ *is a continuous martingale AF with quadratic variation*

$$\langle M^{[u]}\rangle_t = \int_0^t \|\nabla u\|_H^2(X_s)\,ds.$$

Further, for some sequence $\{\varepsilon_n\} \downarrow 0$,

$$\lim_{n\to\infty}\int_0^t (Lu^{\varepsilon_n})(X_s)\,ds = A_t^{Lu} \quad \textit{locally uniformly in } t.$$

Proof. We shall give a proof only in the case (b). The case (a) is similarly (and more easily) proved. First we remark that $\mathcal{E}^1$-exceptional sets are $\mathcal{E}^\rho$-exceptional sets and $\mathcal{E}^1$-convergence implies $\mathcal{E}^\rho$-convergence because $\rho \in L^\infty$. From the theorem of [7] and an argument in the proof of [1, Theorem 2.4], we can take a sequence $\{v_n\} \subset \mathcal{F}C_b^\infty$ such that $v_n \to w$ μ-a.e. and both $\{\|v_n\|_\infty\}$ and $\{\|\nabla v_n\|_\infty\}$ are uniformly bounded. By the Banach-Saks theorem, a sequence of the Cesàro mean $\{u_n\}$ of some subsequence of $\{v_n\}$ satisfies that $u_n \in \mathcal{F}C_b^\infty$, $u_n \to w$ μ-a.e. and in $\mathcal{F}^1$, and both $\{\|u_n\|_\infty\}$ and $\{\|\nabla u_n\|_\infty\}$ are uniformly bounded.

For each $\varepsilon > 0$, $T_\varepsilon u_n \to u^\varepsilon$ in $\mathcal{F}^1$ and $LT_\varepsilon u_n \to Lu^\varepsilon$ in $L^2(\mu)$ as $n \to \infty$. Also, $\nabla T_\varepsilon u_n = e^{-\varepsilon}T_\varepsilon \nabla u_n$, $\nabla u^\varepsilon = e^{-\varepsilon}T_\varepsilon\nabla w$ and $\nabla u_n \to \nabla w$ in $L^2(E \to H;\mu)$ as $n \to \infty$. By Lemma 4.3, ∇u^ε has an $\mathcal{E}^1$-quasicontinuous modification $\widetilde{\nabla u^\varepsilon}$ and, by taking a subsequence if necessary,

$$\nabla T_\varepsilon u_n \to \widetilde{\nabla u^\varepsilon} \quad \mathcal{E}^1\text{-q.e.} \quad \text{as } n \to \infty.$$

Then, letting $n \to \infty$ in our version of the Green formula (Theorem 3.12) for $T_\varepsilon u_n$ in place of u, we have

$$\mathcal{E}^\rho(u^\varepsilon, v) = -\frac{1}{2}\int_{F^\rho} v(Lu^\varepsilon)\rho\,d\mu - \frac{1}{2}\int_{F^\rho} v(z)\langle\widetilde{\nabla u^\varepsilon}(z), \sigma_\rho(z)\rangle\|D\rho\|(dz), \quad v \in \mathcal{F}C_b^1,$$

and accordingly

$$N_t^{[u^\varepsilon]} = \frac{1}{2}\int_0^t (Lu^\varepsilon)(X_s)ds + \frac{1}{2}\int_0^t \langle \widetilde{\nabla u^\varepsilon}, \sigma_\rho\rangle(X_s)\, dA_s^{\|D\rho\|}.$$

Let $\widetilde{\nabla w}$ be an $\mathcal{E}^1$-quasicontinuous modification of ∇w. We may assume that $\widetilde{\nabla w}$ is Borel measurable. Then $\widetilde{\nabla u} := \widetilde{\nabla w}|_{F^\rho}$ is an $\mathcal{E}^\rho$-quasicontinuous modification of ∇u and Borel measurable. Also, by Lemma 4.3 (ii),

$$\widetilde{\nabla u^\varepsilon}(z) = e^{-\varepsilon}\widetilde{T_\varepsilon \nabla w}(z) = e^{-\varepsilon}E_z^{O-U}[\widetilde{\nabla w}(X_\varepsilon^{O-U})] \quad \mathcal{E}^1\text{-q.e. } z,$$

where (X_t^{O-U}) is the Ornstein-Uhlenbeck process on E and E_z^{O-U} represents an expectation with respect to the distribution of the process starting at z. Since $\widetilde{\nabla w}$ is finely continuous $\mathcal{E}^1$-q.e., which is proved in the same way as in Theorem 4.2.2. in [14], we get

$$e^{-\varepsilon}E_z^{O-U}[\widetilde{\nabla w}(X_\varepsilon^{O-U})] \to \widetilde{\nabla w}(z) \quad \mathcal{E}^1\text{-q.e. } z \text{ as } \varepsilon \downarrow 0.$$

Therefore, $\widetilde{\nabla u^{\varepsilon_n}} \to \widetilde{\nabla u}$ $\mathcal{E}^\rho$-q.e. as $n \to \infty$ for an arbitrary sequence $\{\varepsilon_n\}$ decreasing to 0. Keeping the fact that $u^\varepsilon \to u$ in $\mathcal{F}^\rho$ as $\varepsilon \downarrow 0$ in mind, we have

$$\mathcal{E}^\rho(u,v) = -\lim_{\varepsilon\downarrow 0}\frac{1}{2}\int_{F^\rho} v(Lu^\varepsilon)\rho\, d\mu - \frac{1}{2}\int_{F^\rho} v(z)\langle\widetilde{\nabla u}(z), \sigma_\rho(z)\rangle\, \|D\rho\|(dz), \quad v \in \mathcal{F}C_b^1. \tag{4.2}$$

By Theorem 2.2 in [12], the equivalence of (i) and (ii) and all other assertions hold except for the last one, which in turn follows from Corollary 5.2.1 in [14]. ▮

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References

[1] S. Aida, T. Masuda, and I. Shigekawa: Logarithmic Sobolev inequalities and exponential integrability, J. Funct. Anal. **126** (1994), 83–101.

[2] H. Airault and P. Malliavin: Intégration géométrique sur l'espace de Wiener, Bull. Sci. Math. (2) **112** (1988), 3–52.

[3] L. Ambrosio, N. Fusco and D. Pallara, Functions of bounded variation and free discontinuity problems, Oxford University Press, 2000.

[4] V. I. Bogachev: Smooth measures, the Malliavin calculus and approximations in infinite-dimensional spaces, 18th Winter School on Abstract Analysis (Srní, 1990). Acta Univ. Carolin.—Math. Phys. **31** (1990), 9–23.

[5] V. I. Bogachev: Gaussian Measures, American Mathematical Society, 1998.

[6] N. Bouleau and F. Hirsch: Dirichlet forms and analysis on Wiener space, de Gruyter Studies in Mathematics **14**, Walter de Gruyter, Berlin, 1991.

[7] O. Enchev, and D. W. Stroock: Rademacher's theorem for Wiener functionals, Ann. Probab. **21** (1993), 25–33.

[8] L. C. Evans and R. F. Gariepy: Measure theory and fine properties of functions, CRC Press, Boca Raton, New York, Tokyo, 1992.

[9] H. Federer: Geometric measure theory, Springer, Berlin-Heidelberg-New York, 1969.

[10] D. Feyel and A. de La Pradelle: Hausdorff measures on the Wiener space, Potential Anal. **1** (1992), 177–189.

[11] M. Fukushima: On semimartingale characterizations of functionals of symmetric Markov processes, Electronic Journal of Probability **4** (1999), paper no.18, pp. 1–32.

[12] M. Fukushima: On Ito's formulae for additive functionals of symmetric diffusion processes, Stochastic Processes, Physics and Geometry: New Interplays, A volume in Honor of Sergio Albeverio, CMS Conference Proc., to appear.

[13] M. Fukushima: BV functions and distorted Ornstein Uhlenbeck processes over the abstract Wiener space, J. Funct. Anal. **174** (2000), 227–249.

[14] M. Fukushima, Y. Oshima and M. Takeda: Dirichlet forms and symmetric Markov processes, de Gruyter Studies in Mathematics **19**, Walter de Gruyter, Berlin, 1994.

[15] E. Giusti: Minimal surfaces and functions of bounded variation, Monographs in Mathematics **80**, Birkhäuser, 1984.

[16] V. Goodman: A divergence theorem for Hilbert space, Trans. Amer. Math. Soc. **164** (1972), 411–426.

[17] A. Hertle: Gaussian surface measures and the Radon transform on separable Banach spaces, Measure theory, Oberwolfach 1979 (Proc. Conf., Oberwolfach, 1979), pp. 513–531, Lecture Notes in Math. **794**, Springer, Berlin, 1980.

[18] A. Hertle: Gaussian plane and spherical means in separable Hilbert spaces, Measure theory, Oberwolfach 1981 (Oberwolfach, 1981), pp. 314–335, Lecture Notes in Math. **945**, Springer, Berlin-New York, 1982.

[19] A. Hertle: On the asymptotic behaviour of Gaussian spherical integrals, Probability in Banach spaces, IV (Oberwolfach, 1982), 221–234, Lecture Notes in Math. **990**, Springer, Berlin-New York, 1983.

[20] Y. Inahama: Meyer's inequality of type $L(\log L)^\alpha$ on abstract Wiener spaces, J. Math. Kyoto Univ. **39**(1999), 87–113.

[21] M. Ledoux: Isoperimetry and Gaussian analysis, Lectures on Probability Theory and Statistics (Saint-Flour, 1994), 165–294, Lecture Notes in Math. **1648**, Springer, 1996.

[22] Z. M. Ma and M. Röckner: Introduction to the theory of (nonsymmetric) Dirichlet forms, Universitext, Springer-Verlag, Berlin, 1992.

[23] G. Pisier: Riesz transform; A simple analytic proof of P. A. Meyer's inequality, Lecture Notes in Math. **1321**, Springer, 1988, 485–501.

[24] M. M. Rao and Z. D. Ren: Theory of Orlicz spaces, Monographs and Textbooks in Pure and Applied Mathematics **146**, Marcel Dekker, Inc., New York, 1991.

[25] M. Röckner and B. Schmuland: Tightness of general $C_{1,p}$ capacities on Banach space, J. Funct. Anal. **108** (1992), 1–12.

[26] W. Rudin: Functional Analysis, second ed., McGraw-Hill.

[27] M. L. Silverstein: Symmetric Markov processes, Lecture Notes in Math. **426**, Springer-Verlag, Berlin-New York, 1974.

[28] A. V. Skorohod: Integration in Hilbert space, Translated from the Russian by Kenneth Wickwire, Ergebnisse der Mathematik und ihrer Grenzgebiete **79**, Springer-Verlag, New York-Heidelberg, 1974.

[29] H. Sugita: Positive generalized Wiener functions and potential theory over abstract Wiener spaces, Osaka J. Math. **25** (1988), 665–696.