TIME CHANGES OF SYMMETRIC DIFFUSIONS AND FELLER MEASURES

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1. Introduction

In the present work, we will be concerned with a formula going back to Douglas ([6]):

$$\frac{1}{2}\int_D |\nabla Hf(x)|^2 \, dx = \frac{1}{2}\int_{\partial D\times\partial D\setminus d} (f(\xi)-f(\eta))^2 \, U(\xi,\eta)d\xi d\eta, \tag{1.1}$$

where Hf denotes the harmonic function on the planer unit disk D with boundary value f and $U(\xi,\eta)=\frac{1}{4\pi(1-\cos(\xi-\eta))}$.

In 1962, J.L. Doob [4] extended formula (1.1) to the case where D is a general Green space and ∂D is its Martin boundary by adopting as U the Naim kernel. The first author identified the Naim kernel with the Feller kernel soon after in [8] and then utilized the resulting formula (1.1) as a basis to describe all possible symmetric Markovian extensions of the absorbing Brownian motion on a bounded Euclidean domain in [9]. The Feller kernel had been introduced by W. Feller [7] for the minimal Markov process on a countable state space for the purpose of describing all possible boundary conditions on some ideal boundaries. A common feature of the mentioned approaches was in that we are only given a minimal process on D a priori and we try to capture its Markovian extensions including the construction of intrinsic boundaries.

Since that decade, the investigations of Markov processes and associated Dirichlet forms have been developed considerably and we can now take the following different but much more stochastic view on the formula (1.1). What is given in advance is the reflecting Brownian motion X on $\overline{D}$ and we consider its time changed process Y on ∂D with respect to a local time on ∂D. The left hand side of (1.1) is the Dirichlet form for Y (the trace of the Dirichlet form for X), while the right hand side is its specific Beurling-Deny representation. (1.1) tells us that Y is of pure jump and that its jumping mechanism, namely, the Lévy system is governed by the Feller kernel U which can be easily and intrinsically defined depending only on the absorbing Brownian motion X_D on D.

This view point allows us to extend the formula (1.1) with a great generality. Indeed, we will consider in this paper a general symmetric diffusion process X with a general state space E and its time changed process Y on an arbitrary closed subset F of E. We will show in §5 that the jumping measure and the killing measure for Y can be identified with the Feller measure U and the supplementary Feller measure V respectively introduced in §2 depending only on the absorbed process X_G on $G=E\setminus F$.

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The organization of the present paper is as follows. In §2, we consider a conservative Borel right process X which is symmetric with respect to a σ-finite measure m on a general state space E. Let F be a closed subset of E and X_G be the process on $G = E \setminus F$ obtained from X by killing upon its hitting time of F. We shall introduce a notion of the energy functional L_G for each pair of X_G-almost excessive functions on G, a variant of the concept due to P.A. Meyer (cf. [13]). By means of L_G, we can readily define the Feller measure $U(d\xi, d\eta)$ (a bi-measure on $F \times F$), the supplementary Feller measure $V(d\xi)$ (a measure on F informally called an escape measure) and also the Feller kernel $U(\xi, \eta)$ when the Poisson kernel exists. In Example 2.1, we exhibit explicit expressions of those quantities for the case that X is the d-dimensional Brownian motion ($d \geq 3$) and F is the $(d-1)$-dimensional compact smooth hypersurface.

From §3 on, we assume that X is a diffusion, namely, its sample paths are continuous. In §3, we will focus our attention on the excursions of the sample paths of X away from the closed set F and we shall identify the Feller measure and supplementary Feller measure with the expectations of certain homogeneous random measures generated by the end points of excursions. Here we shall make use of a description of the joint distribution of end points of excursions previously studied by Hsu [14] for the reflecting Brownian motion on a smooth domain.

From §4 on, we further assume that X is associated with a regular Dirichlet space $(E, m, \mathcal{F}, \mathcal{E})$ (without loss of generality owing to the transfer method). In §4, we shall first prove that F always admits an admissible measure μ in the sense that μ charges no set of zero capacity and possesses full quasi-support F. We then show, by applying a general reduction theorem formulated in the final section §8, that the time changed process Y of X with respect to the positive continuous additive functional with Revuz measure μ can be restricted outside some X-and Y-polar set to be a Hunt process. This reduction will enable us to use a general theorem in [11] directly to express the jumping measure and the killing measure in the Beuring-Deny representation of the Dirichlet form for Y by means of the Lévy system of Y.

By making use of the results in §3 and §4, we shall prove in §5 the stated main assertion (Theorem 5.1) that the jumping measure and killing measure for the time changed process Y are identical with the Feller measure and the supplementary Feller measure respectively. In particular, we shall see that the Lévy system of Y admits a simple expression in terms of the Feller kernel when the Poisson kernel exists.

Theorem 5.1 tells us that the trace Dirichlet form $\mathcal{E}(Hf, Hf)$ always dominates the generalized Douglas integral with the Feller measure. By assuming that $m(G)$ is finite, we shall prove in §6 that they are identical under the condition that the energy measures $\mu_{\langle u \rangle}$ do not charge the set F for any $u \in \mathcal{F}$. This condition is satisfied when the energy measures are absolutely continuous with respect to m (the densities are so called square field operators $\Gamma(u)$) and $m(F) = 0$. We shall also characterize this condition in terms of the notion of the reflected Dirichlet space of the part of $\mathcal{E}$ on the set G formulated by M.L. Silverstein [19],[20] and Z.Q. Chen [3]. In the course of the proof, we shall make a full use of several results in [11] to recover and extend the method in [4] and [8] of computing the Dirichlet norm of the classical harmonic function.

In §7, we shall apply the obtained results to the reflecting Brownian motion on the closure of a bounded Lipschitz domain $D \subset \mathbb{R}^d$ associated with the Dirichlet space $H^1(D)$. In this case, the relative boundary ∂D is known to be identical with the Martin boundary of D so that Doob's representation of (1.1) shall be recovered by the present approach.

In §8, we shall formulate a general theorem of reduction of a right process to a Hunt process properly associated with a regular Dirichlet form.

2. Feller measure U, supplementary Feller measure V and Feller kernel

Let E be a Lusin topological space and m be a σ-finite positive Borel measure on E. Let $X = (X_t, P^x)$ be a conservative Borel right Markov process on E which is *m-symmetric* in the sense that its transition function p_t satisfies

$$\int_E p_t f(x)\ g(x) m(dx) = \int_E f(x)\ p_t g(x) m(dx), \qquad \forall f, g \in \mathcal{B}^+.$$

Fix a closed set F and put $G := F^c$. Denote by T the hitting time of F. Let

$$p_t^0(x, A) := P^x(X_t \in A, t < T),\ x \in G,\ A \subset G$$

be the transition function of X_G, the absorbed process of X on G, which is obtained by killing X at leaving G. Then X_G is symmetric with respect to the measure $m_G = 1_G \cdot m$ ([11]). The resolvent of X_G will be denoted by R^0_α.

A measurable function u on G is said to be α-excessive for X_G if for every $x \in G$

$$u(x) \geq 0, \qquad e^{-\alpha t} p_t^0 u(x) \uparrow u(x), \quad t \downarrow 0.$$

If the above properties holds for m_G-a.e. $x \in G$, then u is said to be α-almost excessive. A 0-excessive (resp. 0-almost excessive) function is simply called excessive (resp. almost excessive). Let us denote by $\mathcal{S}_G$ the totality of X_G-almost excessive functions on G finite m_G-a.e.
$\langle u, v\rangle_{m_G}$ will denote the integral of uv with the measure m_G.

Lemma 2.1. *For any $u, v \in \mathcal{S}_G$,*

$$\frac{1}{t}\langle u - p_t^0 u, v\rangle_{m_G} \tag{2.1}$$

is non-decreasing as $t \downarrow 0$. If moreover v is p_t^0-invariant in the sense that $p_t^0 v = v,\ t > 0$, then (2.1) *is independent of $t > 0$.*

Proof. We set

$$e(t) = \langle u - p_t^0 u, v\rangle_{m_G}.$$

Then, for $t, s \geq 0$,

$$e(t+s) = e(t) + \langle p_t^0 u - p_{t+s}^0 u, v\rangle_{m_G} = e(t) + \langle u - p_s^0 u, p_t^0 v\rangle_{m_G} \leq e(t) + e(s),$$

the last inequality being replaced by equality if v is p_t^0-invariant. □

Let us define the *energy functional* of $u, v \in \mathcal{S}_G$ by

$$L_G(u, v) = \lim_{t \downarrow 0} \frac{1}{t}\langle u - p_t^0 u, v\rangle_{m_G}. \tag{2.2}$$

We note that $L_G(u, v)$ is nothing but the value of the energy functional of the excessive measure $u \cdot m_G$ and the excessive function v for X_G in the sense of Dellacherie-Meyer and Getoor when X_G is transient and $u \cdot m_G$ is purely excessive([13, Prop.3.6]). We also have the formula

$$L_G(u, v) = \lim_{\alpha \to \infty} \alpha \langle u - \alpha R^0_\alpha u, v\rangle_{m_G}, \tag{2.3}$$

as an increasing limit because, by the Fubini theorem,

$$\alpha\langle u - \alpha R^0_\alpha u, v\rangle_{m_G} = \int_0^\infty e^{-t}(t/\alpha)^{-1}\langle u - p^0_{t/\alpha} u, v\rangle_{m_G} t dt. \tag{2.4}$$

For $\alpha \geq 0$ let H^α be the α-order hitting measure for F, i.e.,

$$H^\alpha(x, B) := E^x(e^{-\alpha T} 1_B(X_T); T < \infty),\ x \in G, B \in \mathcal{B}(E).$$

H^0 will be denoted by H. $H^\alpha(x,\cdot)$ is carried by F, since F is closed. It is easy to see that, for any $f \in \mathcal{B}(F)^+$, $H^\alpha f$ is α-excessive for X_G.

We also consider the function on G defined by

$$q(x) = P^x(T = \infty)(= 1 - H1(x)), \qquad x \in D. \tag{2.5}$$

Then q is not only excessive for X^G but also p_t^0-invariant.

We now let, for $f, g \in b\mathcal{B}(F)^+$,

$$U(f \otimes g) = L_G(Hf, Hg), \qquad V(f) = L_G(Hf, q). \tag{2.6}$$

We call U the *Feller measure* for F with respect to m because it is a bi-measure in the sense that $U(I_B \otimes I_C)$ is a (possibly infinite) measure in $B \in \mathcal{B}(F)$ (resp. $C \in \mathcal{B}(F)$) for each fixed C (resp. B). V is a (possibly infinite) measure on F and will be called the *supplementary Feller measure* or more informally the *escape measure* for F. We will see in §4 that U is a σ-finite measure on $F \times F$ off the diagonal and V is a σ-finite measure on F.

For $\alpha > 0$, we also define the α-order Feller measure U_α for F by

$$U_\alpha(f \otimes g) = \alpha\langle H^\alpha f, Hg\rangle_{m_G} \quad f, g \in b\mathcal{B}(F)^+. \tag{2.7}$$

Lemma 2.2. *We have the following formulae for $f, g \in b\mathcal{B}(F)^+$:*

$$U(f \otimes g) = \lim_{t \to 0} \frac{1}{t} E^{Hg \cdot m_G}(T \le t, f(X_T)), \tag{2.8}$$

$$U(f \otimes g) = \lim_{\alpha \to \infty} U_\alpha(f \otimes g). \tag{2.9}$$

Proof. The first formula follows from

$$P^x(T \le t, f(X_T)) = Hf(x) - p_t^0 Hf(x), \qquad x \in G.$$

The second one is a consequence of (2.3) and $H^\alpha f = Hf - \alpha R_\alpha^0 Hf$. $\square$

The notion U goes back to W. Feller[7] where a version of U was introduced by (2.9) and utilized to describe possible boundary conditions for a minimal Markov process on a countable state space.

The supplementary Feller measure V has more specific properties:

Lemma 2.3. (i) *For any $t > 0$, $\alpha > 0$,*

$$V(f) = \frac{1}{t} E^{q \cdot m_G}(T \le t, f(X_T)), \qquad f \in b\mathcal{B}(F)^+. \tag{2.10}$$

$$V(f) = \alpha\langle H^\alpha f, q\rangle_{m_G}, \qquad f \in b\mathcal{B}(F)^+. \tag{2.11}$$

(ii) *If $m(G) < \infty$, then $V = 0$.*
(iii) *If $m(G) < \infty$ and $P^x(T < \infty) > 0$ for m-a.e. $x \in G$, then $P^x(T < \infty) = 1$ for q.e. $x \in G$.*

Proof. (i) follows from p_t^0-invariance of q, Lemma 2.1 and (2.4). If $m(G)$ is finite, then the right hand side of (2.10) tends to zero as $t \to \infty$ and we get (ii). (iii) follows from (i) and (ii). □

When the hitting measure $H(x, \cdot)$ has a suitable density with respect to a certain measure μ on F, then the Feller measure U has also a density with respect to $\mu \times \mu$. In the rest of this section, we assume that there exists a σ-finite measure μ on F and a finite-valued function $K(x,\xi),\ x \in G,\ \xi \in F$, strictly positive $m_G \times \mu$-a.e. such that

$$H(x,B) = \int_B K(x,\xi)\mu(d\xi) \qquad \forall B \in \mathcal{B}(F), \quad \text{for } m_G - \text{a.e. } x \in G, \tag{2.12}$$

and $K(\cdot,\xi)$ is X_G-almost excessive for every $\xi \in F$. The function $K^\xi(x) = K(x,\xi)$ will be called a *Poisson kernel* with respect to μ.

We put

$$U(\xi,\eta) = L_G(K^\xi, K^\eta), \qquad \xi,\ \eta \in F, \tag{2.13}$$

which will be called a *Feller kernel* with respect to μ.

In fact, if we define the α-order Poisson kernel by

$$K_\alpha(x,\xi) = K(x,\xi) - \alpha R_\alpha^0 K^\xi(x), \qquad x \in G, \quad \xi \in F, \tag{2.14}$$

and the α-order Feller kernel by

$$U_\alpha(\xi,\eta) = \alpha \langle K_\alpha^\xi, K^\eta \rangle_{m_G} \qquad \xi,\ \eta \in F, \tag{2.15}$$

then, by (2.3),

$$U(\xi,\eta) = \lim_{\alpha\to\infty} U_\alpha(\xi,\eta), \qquad \xi,\ \eta \in F \tag{2.16}$$

and we get from (2.9) that

$$U(d\xi,d\eta) = U(\xi,\eta)\mu(d\xi)\mu(d\eta).$$

Example 2.1 (Brownian motion and a compact hypersurface)**.** Let X be the standard Brownian motion on $\mathbb{R}^d$ with $d \geq 3$. Let S be a C^3 compact hypersurface so that $G = \mathbb{R}^d \setminus S$ is the union of the interior domain D_i and exterior domain D_e. The absorbed Brownian motion X_G has the transition density

$$p_t^0(x,y) = n(t,x-y) - E^x[n(t-T,X_T,y);T<t], \quad n(t,x) = \frac{1}{(2\pi t)^{d/2}} \exp(-\frac{|x|^2}{2t}), \tag{2.17}$$

where T is the hitting time of S by X. $p_t^0(x,y),\ x,y \in D_i$ (resp. $x,y \in D_e$) is the fundamental solution of the heat equation

$$\frac{\partial u(t,x)}{\partial t} = \frac{1}{2}\Delta_x u(t,x), \quad ,t>0,\ x \in D_i,\ (\text{resp. } x \in D_e) \tag{2.18}$$

with the Dirichlet boundary condition

$$u(t,x) = 0, \qquad x \in S.$$

Denote by σ the surface measure on S. Then we can get the expressions

$$P^x(T \in ds, X_T \in d\xi) = g(s,x,\xi)ds\sigma(d\xi)$$

with

$$g(s,x,\xi) = \frac{1}{2}\frac{\partial p_s^0(x,\xi)}{\partial n_\xi^i}, \quad x \in D_i,\ \xi \in S, \tag{2.19}$$

$$g(s,x,\xi)=\frac{1}{2}\frac{\partial p_s^0(x,\xi)}{\partial n_\xi^e},\quad x\in D_e,\ \xi\in S, \tag{2.20}$$

where n_ξ^i and n_ξ^e denote the inward normal and outward normal at $\xi\in S$ respectively. A proof of (2.19) was given in [1, pp262]. We give a similar proof of (2.20) for completeness.

We extend a smooth function h on S to D_e by

$$h(y)=E^y(h(X_T);T<\infty),\qquad y\in D_e.$$

It can be seen that h is a harmonic function on D_e vanishing at ∞ and hence the first dervatives of h are bounded on D_e (see the paragraph below (2.25)). On the other hand, we can see from (2.17) that, for each $T>0$ and $a>0$, there are positive constants C_1,C_2 such that

$$\left|\frac{\partial p_t^0(x,y)}{\partial y_k}\right|\le C_1\ \exp(-C_2|x-y|^2),\quad 0<t<T,\ 1\le k\le d,\ |x-y|>a. \tag{2.21}$$

For large $R>0$, we put $D_e^R=\{x\in D_e:|x|<R\}$ and denote its outer boundary by Σ_R. For a fixed $x\in D_e^R$, we have by Green's formula

$$\begin{aligned}\frac{1}{2}\int_S\frac{\partial p_s^0(x,y)}{\partial n_y^e}h(y)d\sigma(y)\quad &=\quad \int_{D_e^R}\left(-\frac{1}{2}\Delta_y p_s^0(x,y)\right)h(y)dy\\ -\frac{1}{2}\int_{\Sigma_R}\frac{\partial p_s^0(x,y)}{\partial n_y}h(y)d\sigma(y)\quad &+\quad \frac{1}{2}\int_{\Sigma_R}p_s^0(x,y)\frac{\partial h(y)}{\partial n_y}d\sigma(y).\end{aligned}$$

By the above observations, the last two integrals vanish as $R\to\infty$. Substituting (2.18) into the resulting equality and integrating in s, we arrive at

$$\int_0^t ds\int\frac{1}{2}\frac{\partial p_s^0(x,y)}{\partial n_y^e}h(y)d\sigma(y)=h(x)-p_s^0h(x)=E^x(h(X_t);T\le t)$$

proving (2.20).

Accordingly, the Poisson kernel and the α-order Poisson kernel with respect to σ admit the expressions

$$K(x,\xi)=\int_0^\infty g(s,x,\xi)ds,\quad K_\alpha(x,\xi)=\int_0^\infty e^{-\alpha s}g(s,x.\xi)ds,\quad x\in G,\ \xi\in S. \tag{2.22}$$

The α-order Feller kernel $U_\alpha(\xi,\eta)$ is the sum of $U_\alpha^i(\xi,\eta)$ and $U_\alpha^e(\xi,\eta)$ where

$$U_\alpha^i(\xi,\eta)=\alpha\int_{D^i}K_\alpha^\xi(x)K^\eta(x)dx,\quad U_\alpha^e(\xi,\eta)=\alpha\int_{D^e}K_\alpha^\xi(x)K^\eta(x)dx,\quad \xi,\eta\in S.$$

From (2.19),(2.20) and (2.22), we can get, for $\xi,\eta\in\partial D,\ \xi\neq\eta$,

$$U_\alpha^i(\xi,\eta)=\frac{1}{4}\int_0^\infty(1-e^{-\alpha t})\frac{\partial^2p_t^0(\xi,\eta)}{\partial n_\xi^i\partial n_\eta^i}dt,\ U_\alpha^e(\xi,\eta)=\frac{1}{4}\int_0^\infty(1-e^{-\alpha t})\frac{\partial^2p_t^0(\xi,\eta)}{\partial n_\xi^e\partial n_\eta^e}dt.$$

By letting $\alpha\to\infty$, we are led to the following expressions of the Feller kernel:

$$U(\xi,\eta)=\frac{1}{4}\int_0^\infty\frac{\partial^2p_t^0(\xi,\eta)}{\partial n_\xi^i\partial n_\eta^i}\ dt+\frac{1}{4}\int_0^\infty\frac{\partial^2p_t^0(\xi,\eta)}{\partial n_\xi^e\partial n_\eta^e}\ dt\qquad \xi\neq\eta, \tag{2.23}$$

$$U(\xi,\eta) = \frac{1}{2}\,\frac{\partial K(\xi,\eta)}{\partial n^i_\xi} + \frac{1}{2}\,\frac{\partial K(\xi,\eta)}{\partial n^e_\xi}, \qquad \xi \neq \eta. \tag{2.24}$$

We consider the special case that $S = \Sigma_R$ the sphere of radius R centered at the origin. The Poisson kernel with respect to the surface measure σ are then expressed as

$$\begin{aligned} K(x,\eta) &= \frac{1}{\Omega_d R}\cdot\frac{R^2-|x|^2}{|x-\eta|^d}, \quad |x|<R,\ \eta\in\Sigma_R, \\ &= \frac{1}{\Omega_d R}\cdot\frac{|x|^2-R^2}{|x-\eta|^d}, \quad |x|>R,\ \eta\in\Sigma_R, \end{aligned} \tag{2.25}$$

where Ω_d denotes the area of the unit sphere in $\mathbb{R}^d$.
Note that, for $D_e = \{|x|>R\}$ and a continuous function f on Σ_R,

$$(Hf)(x) = \int_{\Sigma_R} K(x,\eta)f(\eta)d\sigma(\eta), \qquad x\in D_e,$$

is the unique harmonic function on D_e taking value f on Σ_R and vanishing at ∞.

By (2.24), we obtain an explicit expression of the Feller kernel:

$$U(\xi,\eta) = \frac{2}{\Omega_d}\ |\xi-\eta|^{-d}, \qquad \xi,\eta\in\Sigma_R,\ \xi\neq\eta. \tag{2.26}$$

We can also obtain an explicit expression of the supplementary Feller measure V. By virtue of the above observation, $H1(x) = \dfrac{R^{d-2}}{|x|^{d-2}}$, $x\in D_e$, and consequently, we get from (2.10) and (2.20) that

$$V(d\xi) = v(\xi)\sigma(d\xi)$$

with

$$v(\xi) = \frac{1}{2t}\int_0^t ds \int_{\{|x|>R\}} \left(1-\frac{R^{d-2}}{|x|^{d-2}}\right)\ \frac{\partial p^0_s(x,\xi)}{\partial n^e_\xi}\ dx, \quad \xi\in\Sigma_R.$$

The integral on the right hand side converges in view of (2.21). This expression shows that $v(\xi)$ is actually a positive constant, say, v_0 independent of ξ so that

$$V(d\xi) = v_0\ \sigma(d\xi), \qquad d\xi\in\mathcal{B}(\Sigma_R). \tag{2.27}$$

The value of the *escape constant* v_0 will be computed at the end of §5.

Some computations similar to the above have been carried out in [17] for a certain Markov process and also in [8] and [14] for diffusions on an interior Euclidean domain.

3. End points of excursions and U and V

In the sequel, we further assume that X is a diffusion, namely, all of its sample paths are continuous on $[0,\infty)$. For any $\omega\in\Omega$, we define

$$J(\omega) = \{t\in[0,\infty):\ X_t(\omega)\in G\},$$

which is open and consists of all of excursions away from F of the sample path of ω.

We set, for $t\geq 0$,

$$R(t) = \inf(t,\infty)\cap J^c = \inf\{s>t: X_s\in F\}, \quad (\inf\emptyset=\infty,)$$

and, for $t>0$

$$L(t) = \sup[0,t)\cap J^c = \sup\{0<s<t: X_s\in F\},\ (\sup\emptyset = 0.)$$

Clearly $R(t) = T \circ \theta_t + t$ and for any $s, t \geq 0$, $R(t) \circ \theta_s + s = R(t+s)$. By continuity of paths, $X_{R(t)} \in F$ if $R(t) < \infty$ and $X_{L(t)} \in F$ on $T < t$. The process $X_{L(t)}$ stays on F until X hits F again and is adapted, but $X_{R(t)}$ is not adapted in general.

For $t > 0$, we introduce the time reversal operator at t by

$$r_t\omega(s) = \omega(t-s),$$

so that $X_s \circ r_t = X_{t-s}$, $s \in [0,t]$.

Since X is m-symmetric and conservative, we have

$$E^m(Y \circ r_t) = E^m(Y), \tag{3.1}$$

for any $\mathcal{F}_t$-measurable random variable Y (cf. [11, Lemma 4.1.2].)

We can see that

$$L(t) \circ r_t = t - T, \quad X_{L(t)} \circ r_t = X_{t-L(t)\circ r_t} = X_T,$$

on $T < t$.

Lemma 3.1. *For $t > 0$, let $I_1 \subset [0,t]$, $I_2 \subset [t,\infty)$ be non-empty intervals, and $A, B \in \mathcal{B}(F)$. Then*

$$\begin{aligned} &P^m(L(t) \in I_1, X_{L(t)} \in A, X_t \in G, R(t) \in I_2, X_{R(t)} \in B) \\ &= \int_G P^x(T \in t - I_1, X_T \in A))P^x(T \in I_2 - t, X_T \in B)m(dx). \end{aligned} \tag{3.2}$$

In particular,

$$\begin{aligned} &P^m(L(t) \in I_1, X_{L(t)} \in A, X_t \in G, X_{R(t)} \in B, R(t) < \infty) \\ &= \int_G P^x(T \in t - I_1, X_T \in A)H(x,B)m(dx). \end{aligned} \tag{3.3}$$

Furthermore,

$$\begin{aligned} &P^m(L(t) \in I_1, X_{L(t)} \in A, X_t \in G, R(t) = \infty) \\ &= \int_G P^x(T \in t - I_1, X_T \in A)q(x)m(dx). \end{aligned} \tag{3.4}$$

Proof. Clearly $\{L(t) \in I_1, X_{L(t)} \in A\} \in \mathcal{F}_t$. Since $R(t) = T \circ \theta_t + t$, by the Markov property and (3.1), we have

$$\begin{aligned} &P^m(L(t) \in I_1, X_{L(t)} \in A, X_t \in G, R(t) \in I_2, X_{R(t)} \in B) \\ &= E^m(L(t) \in I_1, X_{L(t)} \in A, P^{X_t}(X_0 \in G, X_T \in B, T \in I_2 - t)) \\ &= E^m[(1_{\{L(t)\in I_1, X_{L(t)}\in A\}}\phi(X_t)) \circ r_t] \\ &= E^m[\phi(X_0); X_T \in A, T \in t - I_1] \end{aligned}$$

where $\phi(x) = 1_G(x)P^x(X_T \in B, T \in I_2 - t)$. This completes the proof of (3.2) and (3.3). (3.4) follows from (3.3).

□

Denote by I the set of all left end points of open (excursion) intervals in J. We note for $s > 0$ that $s \in I$ if and only if $R(s-) < R(s)$ and that, in this case, $R(s-) = s$. It is convenient to add an extra point Δ to E and let

$$X_\infty = \Delta.$$

For any subset S of E, we write S_Δ for $S \cup \Delta$.

For any non-negative measurable function Ψ on $F_\Delta \times F_\Delta$, let us consider a random measure $\kappa(\Psi, \cdot)$ defined by

$$\kappa(\Psi, dt) = \sum_{0<s:\ R(s-)<R(s)} \Psi(X_{R(s-)}, X_{R(s)})\epsilon_s(dt), \tag{3.5}$$

where ϵ_s is the point mass at s. By the above note, the random measure κ may also be written as

$$\kappa(\Psi, dt) = \sum_{0<s:\ s\in I} \Psi(X_s, X_{R(s)})\epsilon_s(dt).$$

Any function f on F will be extended to F_Δ by setting $f(\Delta) = 0$. By this convention, $f \otimes g$ will denote the function on $F_\Delta \times F_\Delta$ defined by $\Psi(x, y) = f(x)g(y)$ for $f, g \in \mathcal{B}_+(F)$. We further let $(f \otimes I_\Delta)(x, y) = f(x)I_\Delta(y)$. Obviously, we have, for $f, g \in \mathcal{B}_+(F)$,

$$\kappa(f \otimes g, dt) = \sum_{0<s:\ R(s-)<R(s)<\infty} f(X_{R(s-)})g(X_{R(s)})\epsilon_s(dt), \tag{3.6}$$

$$\kappa(f \otimes I_\Delta, dt) = \sum_{0<s:R(s-)<\infty,R(s)=\infty} f(X_{R(s-)})\epsilon_s(dt). \tag{3.7}$$

For later reference, we introduce the last exit time from F defined by

$$S_F = \sup\{t > 0 : X_t \in F\}, \quad (\sup \emptyset = 0). \tag{3.8}$$

Then, $s = S_F > 0$ if and only if $R(s-) < \infty,\ R(s) = \infty$, and accordingly

$$\kappa(f \otimes I_\Delta, dt) = f(X_{S_F-})\epsilon_{S_F}(dt).$$

Lemma 3.2. *The random measure $\kappa(\Psi, \cdot)$ is homogeneous for any $\Psi \in \mathcal{B}_+(F_\Delta \times F_\Delta)$.*

Proof. Since $R(s) \circ \theta_u + u = R(u + s)$, we have $X_{R(s)} \circ \theta_u = X_{R(u+s)}$ and

$$\begin{aligned}\kappa(\Psi, dt) \circ \theta_u &= \sum_{u<s+u:\ R(u+s-)<R(u+s)} F(X_{R(u+s-)}, X_{R(u+s)})\epsilon_s(dt)\\ &= \sum_{u<s:\ R(s-)<R(s)} F(X_{R(s-)}, X_{R(s)})\epsilon_s(dt + u)\\ &= \kappa(\Psi, dt + u)\end{aligned}$$

□

Theorem 3.1. *Let $f, g \in \mathcal{B}_+(F)$. Then*

$$E^m\kappa(f \otimes g, (0, t]) = tU(f \otimes g), \qquad E^m\kappa(f \otimes I_\Delta, (0, t]) = tV(f), \quad t > 0.$$

Proof. For $n \geq 1$, let $D_n := \{t_{n,k-1} = \frac{k-1}{2^n} : k \geq 1\}$ and $I_{n,k} = [t_{n,k-1}, t_{n,k})$.

For $0 < s$, we observe that $R(s-) < R(s)$ and $(R(s-), R(s)) \cap D_n \neq \emptyset$ if and only if $R(s-) = L(t_{n,k}) \in I_{n,k},\ X_{t_{n,k}} \in G$ and $R(s) = R(t_{n,k})$ for a unique k depending on n.

Therefore, by the monotone convergence theorem and using (3.3) and (2.8), we get

$$\begin{aligned}
E^m\kappa(f\otimes g,(0,t]) &= E^m \sum_{0<s\le t:\, R(s-)<R(s)<+\infty} f(X_{R(s-)})g(X_{R(s)}) \\
&= \lim_n E^m \sum_{k:\ t_{n,k}\le t} f(X_{L(t_{n,k})})g(X_{R(t_{n,k})})1_{\{L(t_{n,k})\in I_{n,k}, X_{t_{n,k}}\in G, R(t_{n,k})<\infty\}} \\
&= \lim_n \sum_{k:\ t_{n,k}\le t} E^m f(X_{L(t_{n,k})})g(X_{R(t_{n,k})})1_{\{L(t_{n,k})\in I_{n,k}, X_{t_{n,k}}\in G, R(t_{n,k})<\infty\}} \\
&= \lim_n \sum_{k:\ t_{n,k}\le t} \int_G E^x(T\in(0,2^{-n}], f(X_T))Hg(x)m_G(dx) \\
&= \lim_n [2^n t]\int_G E^x(T\in(0,2^{-n}], f(X_T))Hg(x)m_G(dx) \\
&= tU(f\otimes g),
\end{aligned}$$

where $[2^n t]$ is the biggest integer dominated by $2^n t$.

In the same way, we get from (3.4) and (2.10),

$$\begin{aligned}
E^m\kappa(f\otimes I_\Delta,(0,t]) &= E^m \sum_{0<s\le t, R(s-)<\infty, R(s)=\infty} f(X_{L(s)}) \\
&= \lim_n \sum_{k:\ t_{n,k}\le t} E^m f(X_{L(t_{n,k})})1_{\{L(t_{n,k})\in I_{n,k}, X_{t_{n,k}}\in G, R(t_{n,k})=\infty\}} \\
&= \lim_n \sum_{k:\ t_{n,k}\le t} \int_G E^x(T\in(0,2^{-n}], f(X_T))q(x)m_G(dx) \\
&= \lim_n [2^n t]\int_G E^x(T\in(0,2^{-n}], f(X_T))q(x)m_G(dx) \\
&= tV(f).
\end{aligned}$$

□

4. Admissible measure and time changed process Y

We still work with an m-symmetric conservative diffusion process X on E. Let $(\mathcal{E},\mathcal{F})$ be the associated Dirichlet form on $L^2(E;m)$.

By virtue of the transfer method (cf.[16],[10]), we can and shall assume without loss of generality that the Dirichlet space $(E,m,\mathcal{F},\mathcal{E})$ is regular and X is an associated strong Markov process on E with continuous sample paths with infinite life time. By the regularity, we mean that, E is a locally compact separable metric space, m is a positive Radon measure on E with full support and that $\mathcal{F}\cap C_0(E)$ is dense in $\mathcal{F}$ and in $C_0(E)$. Here $C_0(E)$ denotes the space of continuous functions on E with compact support. The capacity associated with this Dirichlet form will be denoted by Cap. A set N with Cap(N) is called an $\mathcal{E}$-polar set. The phrase '$\mathcal{E}$-q.e.' will mean 'except for an $\mathcal{E}$-polar set'.

A quasi-support of a Borel measure is a smallest quasi-closed set outside of which the measure vanishes. It is unique up to the $\mathcal{E}$-q.e. equivalence.

Lemma 4.1. *For a closed set $F \subset E$ with $\mathrm{Cap}(F) > 0$, there exists a non-trivial positive Radon measure μ on E such that μ charges no $\mathcal{E}$-polar set, $\mu(E \setminus F) = 0$ and the quasi-support of μ coincides with F, $\mathcal{E}$-q.e.*

Proof. As in the preceding sections, we denote by T the hitting time of F. Take an m-integrable strictly positive function g on E and set

$$\mu(B) = P^{g \cdot m}(X_T \in B, T < \infty), \quad B \in \mathcal{B}(E).$$

Clearly $\mu(E \backslash F) = 0$ and μ is a non-trivial positive Radon measure charging no set of zero capacity. If a quasi-continuous function $f \in \mathcal{F}$ vanishes μ-a.e., then $E^{g \cdot m}(e^{-T} f(X_T)) = 0$, which implies that the quasi-continuous function $E^{\cdot}(e^{-T} f(X_T))$ vanishes m-a.e. Hence $f = 0$ $\mathcal{E}$-q.e. on F, since $\mathcal{E}$-q.e. point of F is regular for F, and we can conclude on account of [11, Th.4.6.2] that F is a quasi-support of μ. □

We call a measure μ *admissible* for the closed set F if it possesses the properties stated in the above lemma and its topological support $\mathrm{Supp}[\mu]$ equals F. The following sufficient condition for a measure μ to be admissible for F can be shown in the same way as the above proof of Lemma 4.1 (see also [11, Problem 4.6.1].)

Lemma 4.2. Let F be a closed set with $\mathrm{Cap}(F) > 0$. If there exists a σ-finite measure μ with $\mathrm{Supp}[\mu] = F$ such that F admits a Poisson kernel with respect to μ in the sense of §2, then μ is admissible for F.

From now on, we consider a closed set F with $\mathrm{Cap}(F) > 0$. We fix an admissible measure μ for F. Then μ is a smooth measure. Let $\phi(t)$ be the PCAF (positive continuous additive functional) with Revuz measure μ and $\tilde{F}$ be its support, namely,

$$\tilde{F} = \{x \in E : P^x(R_\phi = 0) = 1\}$$

where

$$R_\phi = \inf\{t > 0 : \phi(t) > 0\}.$$

Then $\tilde{F}$ is a quasi-support of μ (cf. [11, Th.5.1.5]), and hence, by choosing the exceptional set for ϕ appropriately, we may assume that

$$\tilde{F} \subset F, \qquad \mathrm{Cap}(F \setminus \tilde{F}) = 0. \tag{4.1}$$

Let $\tau = (\tau_t)$ be the right continuous inverse of ϕ:

$$\tau_t = \inf\{s : \phi(s) > t\}, \quad (\inf \emptyset = \infty). \tag{4.2}$$

We set

$$Y_t = X_{\tau_t}, t < \check{\zeta}, \quad \text{where } \check{\zeta} = \phi(\infty). \tag{4.3}$$

Then $Y = (Y_t, \check{\zeta}, P^x)_{x \in \tilde{F}}$ is a right process on the state space $\tilde{F}$ with life time $\check{\zeta}$, which is called a time change of X or the time changed process (cf. [18]). We add a cemetery Δ to $\tilde{F}$ and define

$$Y_t = \Delta, t \geq \check{\zeta},$$

so that the time changed process Y is a right process on $\tilde{F}_\Delta = \tilde{F} \cup \Delta$. We also note that

$$Y_{t-} = X_{\tau_{t-}} \in F, \ t \leq \check{\zeta}, \tag{4.4}$$

owing to the continuity of the sample path of X.

In general, the process $Y = (Y_t, \check{\zeta}, P^x)_{x \in \tilde{F}}$ is not a Hunt process. It could happen that $Y_{t-} \in F \setminus \tilde{F}$ and Y may be neither quasi-left continuous. By making use of a general reduction theorem formulated in §8 however, we can show that the restriction of Y to the outside of a suitable exceptional set is actually a Hunt process.

To this end, we recall some basic facts about the time changed process Y on $\tilde{F}$ shown in [11, Th. 6.2.1]. Y is μ-symmetric and the associated Dirichlet form (denoted by $(\check{\mathcal{E}}, \check{\mathcal{F}})$) on $L^2(F, \mu)$ is regular. Further Y is properly associated with $(\check{\mathcal{E}}, \check{\mathcal{F}})$ in the sense that $\check{p}_t u$ is a $\check{\mathcal{E}}$-quasi-continuous version of $\check{T}_t u$ for any $u \in L^2(F; \mu)$, where $\check{p}_t$ (resp. $\check{T}_t$) denotes the transition function of Y (resp. the L^2-semigroup associated with $(\check{\mathcal{E}}, \check{\mathcal{F}})$).

It is also clear from the preceding definition of the path Y_t that the left limit Y_{t-} exists in F_Δ for all $t > 0$. Hence all the conditions in Theorem 8.1 are satisfied by the time changed process Y and we are led to the next theorem for Y. The capacity on F associated with $(\check{\mathcal{E}}, \check{\mathcal{F}})$ is denoted by $\check{\mathrm{Cap}}$. A set $N \subset F$ with $\check{\mathrm{Cap}}(N) = 0$ is called an $\check{\mathcal{E}}$-polar set.

Theorem 4.1. *There exists a Borel subset $\check{F}$ of $\tilde{F}$ such that*

$$F \setminus \check{F} \text{ is } \check{\mathcal{E}}\text{−polar and } \mathcal{E}\text{−polar,} \tag{4.5}$$

$\check{F}$ is Y-invariant and the restriction $Y|_{\check{F}}$ of the time changed process Y to $\check{F}$ is a Hunt process properly associated with $\check{\mathcal{E}}$.

By the general theorem Theorem 8.1, we only know that the set $F \setminus \check{F}$ is $\check{\mathcal{E}}$-polar. But then $\tilde{F} \setminus \check{F}$ is $\mathcal{E}$-polar by virtue of [11, Lemma 6.2.5]. Hence

$$F \setminus \check{F} = (F \setminus \tilde{F}) + (\tilde{F} \setminus \check{F})$$

is $\mathcal{E}$-polar as well in view of (4.1).

Finally we notice that the Dirichlet form $(\check{\mathcal{E}}, \check{\mathcal{F}})$ admits the following description. Denote by $\mathcal{F}_e$ the extended Dirichlet space of $\mathcal{F}$ and any $u \in \mathcal{F}_e$ will be taken to be $\mathcal{E}$-quasi-continuous. Then, on account of (4.1) and [11, Th.6.2.1],

$$\check{\mathcal{F}} = \{f \in L^2(F; \mu) : f = u \ \mu\text{-a.e. on } F \text{ for some } u \in \mathcal{F}_e\}$$

$$\check{\mathcal{E}}(f, f) = \mathcal{E}(Hu, Hu), \ f \in \check{\mathcal{F}}, \ f = u \ \mu\text{-a.e. on } F, \ u \in \mathcal{F}_e, \tag{4.6}$$

where Hu is defined by

$$Hu(x) = E^x(u(X_T); T < \infty) \qquad x \in E.$$

5. Identification of jumping and killing measures of Y with U and V

For simplicity, the restriction of the time changed process Y to the set $\check{F}$ of Theorem 4.1 will be denoted by Y again. Then Y is a Hunt process on $\check{F} \cup \Delta$ properly associated with the regular Dirichlet form $(\check{\mathcal{E}}, \check{\mathcal{F}})$ on $L^2(F; \mu)$ and $F \setminus \check{F}$ is not only $\check{\mathcal{E}}$-polar but also $\mathcal{E}$-polar.

Since the Dirichlet form $(\check{\mathcal{E}}, \check{\mathcal{F}})$ on $L^2(F; \mu)$ is regular, it admits the Beurling-Deny decomposition; for any $\check{\mathcal{E}}$-quasi-continuous functions $f, g \in \check{\mathcal{F}}$,

$$\begin{aligned} \check{\mathcal{E}}(f, g) &= \check{\mathcal{E}}^{(c)}(f, g) \\ &\quad + \int_{F \times F \setminus d} (f(x) - f(y))(g(x) - g(y)) J(dx, dy) + \int_F f(x) g(x) k(dx), \end{aligned} \tag{5.1}$$

where, $\check{\mathcal{E}}^{(c)}$ is a symmetric form with a strong local property, J is a symmetric positive Radon measure on $F \times F$ off the diagonal d and k is a positive Radon measure on F. J and k are called the *jumping measure* and the *killing measure* for the Dirichlet form $(\check{\mathcal{E}}, \check{\mathcal{F}})$ respectively.

Since Y is a Hunt process properly associated with $(\check{\mathcal{E}}, \check{\mathcal{F}})$, we can use directly the general result of [11, §5.3] to describe J and k in terms of the Lévy system of Y. Let

$(N(x,dy),\psi)$ be a Lévy system of Y. More precisely $N(x,dy)$ is a kernel on $(\check{F}_\Delta, \mathcal{B}(\check{F}_\Delta))$ with $N(x,\{x\}) = 0$, $x \in \check{F}$, and $\psi = \psi(t)$ is a PCAF of Y such that, for any $\Psi \in \mathcal{B}^+(\check{F}_\Delta \times \check{F}_\Delta)$ vanishing on the diagonal,

$$E^x\left(\sum_{s\le t}\Psi(Y_{s-},Y_s)\right) = E^x\left(\int_0^t\int_{\check{F}_\Delta} N(Y_s,dy)\Psi(Y_s,y)d\psi(s)\right), \quad x \in \check{F}. \tag{5.2}$$

Let ν be the Revuz measure of ψ with respect to Y. Then, by [11, Th. 5.3.1],

$$J(dx,dy) = \frac{1}{2}N(x,dy)\nu(dx), \qquad k(dx) = N(x,\Delta)\nu(dx). \tag{5.3}$$

On account of the Revuz correspondence, we have, for any $\Psi \in \mathcal{B}^+(\check{F}_\Delta \times F_\Delta)$ vanishing on the diagonal,

$$\begin{aligned}\int_{\check{F}\times\check{F}\setminus d}\Psi(x,y)J(dx,dy) &= \lim_{t\downarrow 0}\tfrac{1}{2t}E^\mu \textstyle\sum_{0<s\le t}\Psi(Y_{s-},Y_s)I_{\check{F}}(Y_s)\\ &= \lim_{\alpha\to\infty}\tfrac{\alpha}{2}E^\mu \textstyle\sum_{0<t<\infty}e^{-\alpha t}\Psi(Y_{t-},Y_t)I_{\check{F}}(Y_t),\end{aligned} \tag{5.4}$$

$$\begin{aligned}\int_{\check{F}}\Psi(x,\Delta)k(dx) &= \lim_{t\downarrow 0}\tfrac{1}{t}E^\mu \textstyle\sum_{0<s\le t}\Psi(Y_{s-},\Delta)I_\Delta(Y_s)\\ &= \lim_{\alpha\to\infty}\alpha E^\mu \textstyle\sum_{0<t<\infty}e^{-\alpha t}\Psi(Y_{t-},\Delta)I_\Delta(Y_t),\end{aligned} \tag{5.5}$$

Theorem 5.1. *We have the following identities:*

$$J = \frac{1}{2}U \quad \text{on } \check{F}\times\check{F}\setminus d, \qquad U = 0 \quad \text{on } (F\times F\setminus d)\setminus(\check{F}\times\check{F}), \tag{5.6}$$

$$k = V \quad \text{on } \check{F}, \qquad V = 0 \quad \text{on } F\setminus\check{F}. \tag{5.7}$$

Proof. It is known that $R_\phi = \tilde{T}$ where $\tilde{T}$ is the hitting time of the support $\tilde{F}$ of ϕ. Hence

$$\tau_{\phi(t)} = \inf\{s:\ \phi(s)>\phi(t)\} = \inf\{s>t:\ \phi(s-t)\circ\theta_t>0\} = \tilde{T}\circ\theta_t + t.$$

Since $F\setminus\tilde{F}$ is $\mathcal{E}$-polar, we have

$$P^x(\tilde{T}=T) = 1, \quad \mathcal{E}\text{-q.e. } x\in E,$$

and hence

$$\tau_{\phi(t)} = R(t) \quad \forall t>0, \quad P^x\text{-a.e. for } \mathcal{E}\text{-q.e. } x\in E. \tag{5.8}$$

For any $\Psi \in \mathcal{B}^+(F\times F)$ vanishing on the diagonal, we have from (5.4),(4.4) and (5.8)

$$\begin{aligned}2\int_{\check{F}\times\check{F}\setminus d}\Psi(x,y)J(dx,dy) &= \lim_{\alpha\to\infty}\alpha E^\mu\sum_{0<t<\infty}e^{-\alpha t}\Psi(Y_{t-},Y_t)I_{\check{F}}(Y_t)\\ &= \lim_{\alpha\to\infty}\alpha E^\mu\sum_{0<t<\infty}e^{-\alpha t}\Psi(X_{\tau_{t-}},X_{\tau_t})I_{\check{F}}(X_{\tau_t})\\ &= \lim_{\alpha\to\infty}\alpha E^\mu(\Sigma_\alpha),\end{aligned}$$

where

$$\Sigma_\alpha = \sum_{0<t<\infty,R(t)<\infty} e^{-\alpha\phi(t)}\Psi(X_{R(t-)},X_{R(t)}).$$

Since μ is the Revuz measure of ϕ with respect to the conservative m-symmetric process X, we have from [11, Th. 5.1.3] and [18, (32.6)] that

$$\begin{aligned}
\alpha E^\mu(\Sigma_\alpha) &= \alpha \frac{1}{s} E^m \left(\int_0^s E^{X_u}(\Sigma_\alpha) d\phi(u) \right) = \alpha \frac{1}{s} E^m \int_0^s \Sigma_\alpha \circ \theta_u \, d\phi(u) \\
&= \alpha \frac{1}{s} E^m \int_0^s \sum_{0<t<\infty, R(t+u)<\infty} e^{-\alpha(\phi(t+u)-\phi(u))} \Psi(X_{R(t+u-)}, X_{R(t+u)}) d\phi(u) \\
&= \alpha \frac{1}{s} E^m \int_0^s e^{\alpha\phi(u)} d\phi(u) \sum_{u<t<\infty, R(t)<\infty} e^{-\alpha\phi(t)} \Psi(X_{R(t-)}, X_{R(t)}) \\
&= \frac{1}{s} E^m \left(\sum_{R(t)<\infty} e^{-\alpha\phi(t)} \Psi(X_{R(t-)}, X_{R(t)}) \int_0^s I_{\{t>u\}}(u) de^{\alpha\phi(u)} \right), \\
&= \frac{1}{s} E^m \left(\Sigma_\alpha \cdot (e^{\alpha\phi(s\wedge t)} - 1) \right).
\end{aligned}$$

Choose Ψ for which the integral $\int_{\check{F}\times\check{F}\setminus d} \Psi(x,y) J(dx,dy)$ is finite. Then $E^m(\Sigma_\alpha) < \infty$ and

$$\begin{aligned}
\alpha E^\mu(\Sigma_\alpha) = \frac{1}{s} E^m & \sum_{0<t\le s, R(t)<\infty} \Psi(X_{R(t-)}, X_{R(t)}) \\
& + \frac{1}{s} E^m \sum_{0<s<t, R(t)<\infty} e^{-(\phi(t)-\phi(s))} \Psi(X_{R(t-)}, X_{R(t)}) - \frac{1}{s} E^m(\Sigma_\alpha).
\end{aligned}$$

Notice that, for the last exit time S_F from F, we have

$$\begin{aligned}
& \frac{1}{s} E^m \sum_{0<s<t, R(t)<\infty} e^{-(\phi(t)-\phi(s))} \Psi(X_{R(t-)}, X_{R(t)}) \\
&= \frac{1}{s} E^m \left[\sum_{0<s<t, R(t)<\infty} e^{-(\phi(t)-\phi(s))} \Psi(X_{R(t-)}, X_{R(t)}); s < S_F \right] \\
&= \frac{1}{s} E^m(\Sigma_\alpha \circ \theta_s; s < S_F) \le \frac{1}{s} E^m(\Sigma_\alpha),
\end{aligned}$$

because the time set $\{t : s < t, \ R(t) < \infty\}$ is non-empty if and only if $s < S_F$. Since

$$\lim_{\alpha\to\infty} E^m(\Sigma_\alpha) = 0,$$

we arrive at the equality

$$2 \int_{\check{F}\times\check{F}\setminus d} \Psi(x,y) J(dx,dy) = \frac{1}{s} E^m \sum_{0<t\le s, R(t-)<R(t)<\infty} \Psi(X_{R(t-)}, X_{R(t)}).$$

By substituting $\Psi = f \otimes g$ for any $f, g \in C_0(F)$ with disjoint support in the above equality, we get the desired identity (5.6) by virtue of Theorem 3.1.

Exactly the same computation works to get from (5.5) the following formula holding for any $f \in \mathcal{B}_+(F)$:

$$\begin{aligned}
\int_{\check{F}} f(x) k(dx) &= \lim_{\alpha\to\infty} \alpha E^\mu \textstyle\sum_{0<t<\infty, R(t-)<\infty, R(t)=\infty} e^{-\phi(t)} f(X_{R(t-)}) \\
&= \tfrac{1}{s} E^m \textstyle\sum_{0<t\le s, R(t-)<\infty, R(t)=\infty} f(X_{R(t-)}),
\end{aligned}$$

which, combined with Theorem 3.1 leads us to (5.7). □

Corollary 5.1. *Suppose that the hitting measure has the Poisson kernel $K(x,\xi), x \in G, \xi \in F$ with respect to a σ-finite measure μ with Supp$[\mu] = F$. Then μ is admissible and the associated time changed process Y (with a possible q.e. modification of its state space $\check{F}$) has as its Lévy system*

$$(U(\xi,\eta)\mu(d\eta), t), \tag{5.9}$$

where U is the Feller kernel defined by (2.13) in terms of K and t denotes the non-random PCAF $\psi(t) = t$ of Y.

Proof. μ is admissible by Lemma 4.2. By Theorem 5.1,

$$J(d\xi, d\eta) = \frac{1}{2}U(\xi,\eta)\mu(d\xi)\mu(d\eta) \quad \text{on} \quad \check{F} \times \check{F} \setminus d,$$

and μ is the Revuz measure of the PCAF t of Y. Hence, it suffices to show that the value of the right hand side of (5.2) depends only on the function Ψ and the jumping measure J for q.e. x and that it does not depend on the special choice of N and ν expressing J as in (5.3). But this can been readily seen from known formulae [11, (5.1.12),(5.1.14)] on the Revuz correspondence of the PCAF and the smooth measure. □

Example 5.1 (escape measure for Brownian motion and sphere)**.** Just as in the last part of Example 2.1, we let X be the standard Brownian motion on $\mathbb{R}^d$ with $d \geq 3$, F be the sphere Σ_R centered at the origin with radius $R > 0$ and σ be the surface measure on Σ_R. We have seen in (2.27) that the escape measure V on Σ_R for X is a constant times of σ:

$$V(d\xi) = v_0\ \sigma(d\xi), \qquad d\xi \in \mathcal{B}(\Sigma_R).$$

By making use of Theorem 5.1, we will show that

$$v_0 = \frac{d-2}{2R}. \tag{5.10}$$

Since Σ_R admits the Poisson kernel (2.25) with respect to σ, σ is an admissible measure for Σ_R by Lemma 4.2. Let Y be the time changed process of X with respect to the PCAF with Revuz measure σ and $(\check{\mathcal{E}}, \check{\mathcal{F}})$ be the Dirichlet form of Y on $L^2(\Sigma_R, \sigma)$. By virtue of Theorem 5.1 and (2.26), we have, for any $f \in \check{\mathcal{F}}$,

(5.11)

$$\check{\mathcal{E}}(f,f) = \check{\mathcal{E}}^{(c)}(f,f) + \frac{1}{\Omega_d}\int_{\Sigma_R\times\Sigma_R\setminus d}(f(\xi)-f(\eta))^2\frac{1}{|\xi-\eta|^d}\sigma(d\xi)\sigma(d\eta) + v_0\int_{\Sigma_R} f(\xi)^2\sigma(d\xi),$$

where $\check{\mathcal{E}}^{(c)}$ is a (possibly vanishing) strongly local form.

The Dirichlet form of the Brownian motion X on $L^2(\mathbb{R}^d)$ is given by

$$\mathcal{F} = H^1(\mathbb{R}^d), \qquad \mathcal{E}(u,u) = \frac{1}{2}\int_{\mathbb{R}^d}|\nabla u(x)|^2 dx,$$

and the left hand side of (5.11) is equal to $\mathcal{E}(Hf, Hf)$ in view of (4.5). Since Σ_R is compact and $1 \in \check{\mathcal{F}}$, we get from (5.11) that

$$v_0\ \sigma(\Sigma_R) = \frac{1}{2}\int_{\mathbb{R}^d}|\nabla H1(x)|^2\ dx. \tag{5.12}$$

As was observed in Example 2.1, $H1(x) = 1$ for $|x| < R$, while $H1(x) = R^{d-2}/|x|^{d-2}$ for $|x| > R$. Hence the right hand side equals

$$\frac{(d-2)\Omega_d}{2}R^{d-2}$$

and we arrive at (5.10).

6. Trace Dirichlet form and Douglas integral with Feller measure

In the preceding two sections, we have proved the following: let $(E, m, \mathcal{F}, \mathcal{E})$ be a regular Dirichlet space and X be an associated conservative diffusion process on E. Any function in the extended Dirichlet space $\mathcal{F}_e$ is taken to be $\mathcal{E}$-quasi-continuous. Let F be a closed subset of X with $\mathrm{Cap}(F) > 0$, μ be an admissible measure for F with $\mathrm{Supp}[\mu] = F$, ϕ be a PCAF of X with Revuz measure μ and Y be a time changed process of X by means of ϕ. Y is μ-symmetric and its Dirichlet form on $L^2(F;\mu)$ is denoted by $(\check{\mathcal{E}}, \check{\mathcal{F}})$, which is also called the *trace Dirichlet form* of $\mathcal{E}$ on F. In fact, in view of (4.6) and [11, Lemma 6.2.5], we have

$$\begin{aligned} \check{\mathcal{F}} &= \mathcal{F}_e|_F \cap L^2(F;\mu) \\ \check{\mathcal{E}}(f,f) &= \mathcal{E}(Hu, Hu), \qquad f = u|_F, \quad u \in \mathcal{F}_e. \end{aligned} \tag{6.1}$$

Furthermore we have obtained in Theorem 5.1 that ,for any $f, g \in \check{\mathcal{F}}$,

$$\begin{aligned} \check{\mathcal{E}}(f,g) &= \check{\mathcal{E}}^{(c)}(f,g) \\ &\quad + \frac{1}{2}\int_{F\times F\setminus d} (f(x)-f(y))(g(x)-g(y))U(dx,dy) + \int_F f(x)g(x)V(dx), \end{aligned} \tag{6.2}$$

the representation of the trace Dirichlet form $\check{\mathcal{E}}$ in terms of the Feller measure U and the supplementary Feller measure V introduced in §2. In particular, the first integral on the right hand side is called the *Douglas integral* with the Feller measure U.

The Feller measure U and the supplementary Feller measure V are completely determined by the absorbed (minimal) process X_G of X on the set $G = X \setminus F$, while the local term $\check{\mathcal{E}}^{(c)}$ in the above decomposition is determined by the behavior of X on the set F. On the other hand, the value of the Dirichlet form $\mathcal{E}(u,u)$ for $u \in \mathcal{F}_e$ is known to be equal to the half of the total mass of the energy measure $\mu_{\langle u\rangle}$ of u. Therefore we may expect that

$$\check{\mathcal{E}}^{(c)}(f,f) = \frac{1}{2}\,\mu_{\langle Hu\rangle}(F), \qquad f = u|_F, \quad u \in \mathcal{F}_e. \tag{6.3}$$

We will not prove this. But more specifically, we show in this section that, if $\mu_{\langle u\rangle}(F)$ vanishes for any $u \in \mathcal{F}_e$, then the trace Dirichlet form equals the Douglas integral with the Feller measure under the assumption that $m(G)$ is finite.

To this end, we first show the domination of the trace Dirichlet form by the Douglas integral under the setting that $(E, m, \mathcal{F}, \mathcal{E})$ is a regular irreducible Dirichlet space and X is an associated Hunt process on E which is assumed to be conservative. We do not assume that X is of continuous sample paths.

But we make an additional assumption that

$$m(G) < \infty, \qquad \mathrm{Cap}(F) > 0. \tag{6.4}$$

We note that (6.4) and the irreducibility of $\mathcal{E}$ implies that

$$P^x(T < \infty) = 1 \qquad \text{q.e. } x \in G, \tag{6.5}$$

because then $P^x(T < \infty) > 0$ for q.e $x \in E$ by [11, Th. 4.6.6] and Lemma 2.3 applies.

For any $u \in \mathcal{F}_{e,b} = \mathcal{F}_e \cap L^\infty(E;m)$, its energy measure $\mu_{\langle u\rangle}$ is defined by

$$\int_E f(x)\mu_{\langle u\rangle}(dx) = 2\mathcal{E}(uf, u) - \mathcal{E}(u^2, f), \qquad f \in \mathcal{F} \cap C_0(E). \tag{6.6}$$

The energy measure uniquely extends to any $u \in \mathcal{F}_e$ and it holds that

$$\mathcal{E}(u,u) = \frac{1}{2}\mu_{\langle u\rangle}(E) \qquad u \in \mathcal{F}_e. \tag{6.7}$$

Let

$$\mathcal{F}^0 = \{u \in \mathcal{F} : u = 0 \quad \text{q.e. on } F\}.$$

Then $(\mathcal{F}^0, \mathcal{E})$ is a regular Dirichlet space on $L^2(G;m)$ which is associated with the absorbed process X_G ([11, Th. 4.4.3]). Recall that R^0_α denotes the resolvent operator for X_G. Since

$$R^0_1 1(x) = 1 - E^x(e^{-T}) < 1, \qquad \text{q.e. } x \in G,$$

by (6.5), we see that $(\mathcal{F}^0, \mathcal{E})$ is transient by virtue of [11, Lemma 1.6.5] and moreover the extended Dirichlet space $\mathcal{F}^0_e$ of $\mathcal{F}^0$ admits the expression

$$\mathcal{F}^0_e = \{u \in \mathcal{F}_e : u = 0 \quad \text{q.e. on } F\},$$

on account of [11, Th. 4.4.4].

Denote by $S^{(0)}_0(G)$ the space of positive Radon measures of finite 0-order energy integral with respect to $(\mathcal{F}^0_e, \mathcal{E})$. If $\nu \in S^{(0)}_0(G)$, then there exists an unique $R^0\nu \in \mathcal{F}^0_e$ called the 0-order potential of ν such that

$$\mathcal{E}(R^0\nu, v) = \int_G v d\nu \quad v \in \mathcal{F} \cap C_0(G). \tag{6.8}$$

(6.7) extends to any quasi-continuous function $v \in \mathcal{F}^0_e$.

We write $(f,g)_G = \int_G fg dm$.

We know from [11, Th. 1.5.4] that, if a non-negative measurable function f on G satisfies that $(f, R^0_{0+}f)_G < \infty$, then $R^0_{0+}f \in \mathcal{F}^0_e$ and

$$\mathcal{E}(R^0_{0+}f, v) = (f,v)_G \qquad v \in \mathcal{F}^0_e. \tag{6.9}$$

We further know from [11, Th.4.6.5] that $Hu \in \mathcal{F}_e$ for any $u \in \mathcal{F}_e$ and

$$\mathcal{E}(Hu, v) = 0 \qquad \forall v \in \mathcal{F}^0_e. \tag{6.10}$$

We prepare a lemma which generalizes the methods of computing the Dirichlet norms of classical harmonic functions employed in [4], [8].

Lemma 6.1. *For any $u \in \mathcal{F}_{e,b}$, let*

$$w = H(u^2) - (Hu)^2 \ (\in \mathcal{F}_{e,b}).$$

Then,

$$w \in \mathcal{F}^0_{e,b} \quad \text{and} \quad w = R^0\nu \quad \text{for} \quad \nu = \mu_{\langle Hu\rangle}|_G. \tag{6.11}$$

Further

$$\mu_{\langle Hu\rangle}(G) = \lim_{\alpha\to\infty} \alpha(H^\alpha 1, w)_G. \tag{6.12}$$

Proof. Since $\mathcal{F}_{e,b}$ is an algebra and $w = 0$ q.e. on F, we have that $w \in \mathcal{F}^0_{e,b}$. From (6.6) and (6.10), we have, for any $f \in \mathcal{F} \cap C_0(D) (\subset \mathcal{F}^0)$

$$\begin{aligned}\mathcal{E}(w, f) &= -\mathcal{E}((Hu)^2, f) \\ &= 2\mathcal{E}(Hu \cdot f, Hu) - \mathcal{E}((Hu)^2, f) = \int_G f d\mu_{\langle Hu \rangle},\end{aligned}$$

arriving at (6.11).

Since $H1 = 1$ q.e. on G by (6.5), we have

$$1 - H^\alpha 1 = \alpha R^0_{0+} H^\alpha 1$$

and hence $(H^\alpha 1, R^0_{0+} H^\alpha 1)_G < \infty$. Accordingly, for any $\nu \in S^{(0)}_0(G)$, we get from (6.8) and (6.9)

$$\alpha (H^\alpha 1, R^0 \nu)_G = \mathcal{E}(\alpha R^0_{0+} H^\alpha 1, R^0 \nu) = \langle \alpha R^0_{0+} H^\alpha 1, \nu \rangle = \langle 1 - H^\alpha 1, \nu \rangle \uparrow \nu(G).$$

Hence (6.12) follows from (6.11). □

Theorem 6.1. *For any* $u \in \mathcal{F}_e$,

$$\mu_{\langle Hu \rangle}(G) \le \int_{F \times F} (u(\xi) - u(\eta))^2 U(d\xi, d\eta). \tag{6.13}$$

Proof. This follows from (6.12) and the following identity in [8, (15)]:

$$\begin{aligned}&\alpha (H^\alpha 1, w)_G + \alpha \int_{F \times G} (Hu(x) - u(\xi))^2 H^\alpha (x, d\xi) m(dx) \\ &= \int_{F \times F} (u(\xi) - u(\eta))^2 U_\alpha (d\xi, d\eta), \qquad (6.14)\end{aligned}$$

which can be easily verified. □

Theorem 6.1 combined with (6.7) leads us to

Corollary 6.1. *Suppose that*

$$\mu_{\langle u \rangle}(F) = 0, \qquad \forall u \in \mathcal{F}_e. \tag{6.15}$$

Then, for any $u \in \mathcal{F}_e$,

$$\mathcal{E}(Hu, Hu) \le \frac{1}{2} \int_{F \times F} (u(\xi) - u(\eta))^2 U(d\xi, d\eta). \tag{6.16}$$

We emphasize that the condition (6.15) is satisfied if the energy measure of u is absolutely continuous with respect to m, i.e., carré de champ operator $\Gamma(u, u)$ exists for any $u \in \mathcal{F}$ and $m(F) = 0$.

We can now state the main theorem of this section.

Theorem 6.2. *Let* $(E, m, \mathcal{F}, \mathcal{E})$ *be a regular irreducible Dirichlet space whose associated Markov process on* E *is a conservative diffusion. For a closed set* $F \subset E$ *and its complement* G*, we assume the condition* (6.4). *We further assume condition* (6.15) *for the energy measures associated with* $\mathcal{E}$*. Then, for any* $u \in L^2(F, \mu) \cap \mathcal{F}_e$,

$$\mathcal{E}(Hu, Hu) = \frac{1}{2} \int_{F \times F} (u(\xi) - u(\eta))^2 U(d\xi, d\eta). \tag{6.17}$$

Proof. By (6.1) and (6.2), we have already the converse inequality to (6.16). □

We may view Theorem 6.2 from a quite different angle. The Dirichlet form $(\mathcal{F}, \mathcal{E})$ on $L^2(E; m)$ is in a sense an extension of the absorbed Dirichlet space $(\mathcal{F}^0, \mathcal{E})$ on $L^2(G; m)$. What kind of extension are we dealing with under condition (6.15) ? This question can be answered in terms of the notion of the reflected Dirichlet space initially formulated by M.L. Silverstein [19],[20] and finally by Z.Q. Chen [3].

We continue to consider a regular irreducible Dirichlet space $(E, m, \mathcal{F}, \mathcal{E})$ associated with a conservative diffusion X on E and we assume condition (6.4) for a closed set $F \subset E$ and its complement G.

Let $(\mathcal{F}_a^{ref}, \mathcal{E}^{ref})$ be the L^2 *reflected Dirichlet space* (in the sense of [3]) relative to the regular Dirichlet space $(\mathcal{F}^0, \mathcal{E})$ on $L^2(G; m)$ associated with the absorbed process X_G.

Theorem 6.3. *The condition* (6.15) *is equivalent to the following one:*

$$\mathcal{F}|_G \subset \mathcal{F}_a^{ref}, \qquad \mathcal{E}(u, v) = \mathcal{E}^{ref}(u|_G, v|_G) \quad u, v \in \mathcal{F}. \tag{6.18}$$

Proof. By (6.10) and the preceding description of the space $\mathcal{F}_e^0$, we have, for any $u \in \mathcal{F}_e$,

$$u_0 = u - Hu \in \mathcal{F}_e^0, \qquad \mathcal{E}(u, u) = \mathcal{E}(u_0, u_0) + \mathcal{E}(Hu, Hu). \tag{6.19}$$

We further know from (6.7) that condition (6.15) is equivalent to

$$\mathcal{E}(Hu, Hu) = \frac{1}{2}\, \mu_{\langle Hu \rangle}(G) \qquad \forall u \in \mathcal{F}_e. \tag{6.20}$$

Let G_k be relatively compact open sets increasing to G and L_k be the equilibrium measures of the 0-order equilibrium potentials e_k for the sets G_k relative to the extended Dirichlet space $(\mathcal{F}_e^0, \mathcal{E})$:

$$e_k \in \mathcal{F}_e^0,\ 0 \le e_k \le 1,\ e_k = 1 \text{ on } G_k, \quad \mathcal{E}(e_k, v) = \langle v, L_k \rangle_G,\ v \in \mathcal{F}_e^0.$$

We then have

$$\mu_{\langle Hu \rangle}(G) = \lim_{k \to \infty} \langle H(u^2) - (Hu)^2, L_k \rangle_G, \quad u \in \mathcal{F}_{e,b}. \tag{6.21}$$

In fact, using the notation in Lemma 6.1, we see that

$$\langle w, L_k \rangle_G = \mathcal{E}(w, e_k) = \langle e_k, \nu \rangle_G,$$

which tends as $k \to \infty$ to $\nu(G) = \mu_{\langle Hu \rangle}(G)$. By comparing the combination of (6.19), (6.20) and (6.21) with Definition 3.1 in [3] of the L^2 reflected Dirichlet space, we get the equivalence of (6.15) and (6.18). □

It has been shown by M. Takeda [22, Th.3.3] that the L^2 reflected Dirichlet space is the maximum Silverstein extension of $(\mathcal{F}^0, \mathcal{E})$ in a specific semi-order. When $(\mathcal{F}^0, \mathcal{E})$ is the Dirichlet space of the absorbing Brownian motion on an arbitrary bounded domain D, $\mathcal{F}_a^{ref}$ equals $H^1(D)$, which had been described by the first author [9] in terms of the Feller kernel on the Martin boundary (see also the next section). In view of Theorem 6.3, we thus see that the Dirichlet space $(\mathcal{F}, \mathcal{E})$ satisfying condition (6.15) corresponds to a member of the class $\mathbf{G}_1$ of [9, §8] in this special case.

7. Application to reflecting Brownian motion on a Lipschitz domain

Let D be a bounded Lipschitz domain of $\mathbb{R}^d$ with $d \geq 2$ and $\overline{D} = D \cup \partial D$ be its closure. As is well known ([15].[2]), ∂D (resp. $\overline{D}$) can then be identified with the Martin boundary M of D (resp. the Martin space $D \cup M$) and M consists only of the minimal boundary points. In what follows, we regard the relative boundary ∂D also as the Martin boundary of D under this identification.

Denote by $K(x, \xi),\ x \in D, \xi \in \partial D$, a Martin kernel. By the Martin representation theorem ([5]), any positive harmonic function h on D can be expressed as the integral of the Martin kernel against a unique positive Radon measure on ∂D called the *Martin representing measure of h corresponding to K.* We let μ be the Martin representing measure of the constant harmonic function 1 corresponding to K:

$$1 = \int_{\partial D} K(x, \xi)\mu(d\xi) \qquad x \in D. \tag{7.1}$$

We now consider the space

$$\mathcal{F} = H^1(D), \qquad \mathcal{E}(u, v) = \frac{1}{2}\int_D \nabla u \cdot \nabla v \ dx, \quad u, v \in H^1(D), \tag{7.2}$$

which can be regarded as a strongly local regular Dirichlet space on $L^2(\overline{D})$ (rather than $L^2(D)$) and hence there exists an associated conservative diffusion process $X = (X_t, P^x)_{x \in \overline{D}}$ on $\overline{D}$ uniquely up to the q.e. equivalence([11, Example 4.5.3]). We fix such a process X and call it a *reflecting Brownian motion* on $\overline{D}$.

Let T be the hitting time of ∂D of X and $H(x, \cdot)$ be the hitting distribution of X on ∂D:

$$H(x, B) = P^x(X_T \in B,\ T < \infty) \qquad x \in D, \quad B \in \mathcal{B}(\partial D).$$

Lemma 7.1. *The hitting distribution $H(x, \cdot)$ of X and the measure μ in* (7.1) *are related by*

$$H(x, B) = \int_B K(x, \xi)\mu(d\xi) \qquad \forall B \in \mathcal{B}(\partial D), \quad \text{for } q.e.\ x \in D.$$

Proof. Let X_D be the absorbed process of X obtained by killing X at time T. Thus, $X_D = (X_t, P^x, \zeta^D)$ with life time ζ^D given by

$$\zeta^D = T. \tag{7.3}$$

By virtue of [11, Th. 4.4.3], X_D is associated with the part of the Dirichlet form (7.2) on the open set D, namely,

$$\mathcal{F}_D = H_0^1(D), \qquad \mathcal{E}_D(u, v) = \frac{1}{2}\int_D \nabla u \cdot \nabla v dx, \quad u, v \in H_0^1(D). \tag{7.4}$$

Since the absorbing Brownian motion on D (the standard Brownian motion on $\mathbb{R}^d$ killed upon leaving the set D) is also associated with the Dirichlet form (7.4) ([11, Example 4.4.1]), we see that $X_D = (X_t, P^x, \zeta^D)$ coincides in law with the absorbing Brownian motion on D for q.e. starting point $x \in D$.

According to Doob's description of the structure of Brownian motion on the Martin space [5, pp727], we therefore have that

$$P^x(X_{\zeta^D -} \in B) = \int_B K(x, \xi)\mu(d\xi), \qquad \forall B \in \mathcal{B}(\partial D), \quad \text{for q.e. } x \in D. \tag{7.5}$$

The lemma follows from (7.3) and (7.5). □

Since the Martin kernel $K(x,\xi)$ is harmonic in $x \in D$, it is excessive with respect to the absorbing Brownian motion on D and consequently almost excessive with respect to X_D for each $\xi \in \partial D$. Therefore Lemma 7.1 means that the Martin kernel $K(x,\xi)$ is a Poisson kernel with respect to μ in the sense of §2. Hence, by defining the Feller kernel as (2.16) in terms of the present Martin kernel, we have the expression of the Feller measure

$$U(d\xi, d\eta) = U(\xi,\eta)\sigma(d\xi)\sigma(d\eta). \tag{7.6}$$

We also see by Lemma 4.2 that μ is an admissible measure for ∂D in the sense of §4.

On the other hand, we can see from (7.2) and (6.6) that the energy measure $\mu_{\langle u\rangle}$ of $u \in \mathcal{F}_e$ admits the expression

$$\mu_{\langle u\rangle}(dx) = |\nabla u|^2(x)dx \tag{7.7}$$

which does not charge the boundary ∂D. Hence all the conditions of Theorem 6.2 are satisfied for $F = \partial D$.

Theorem 7.1. (i) *The measure μ on ∂D defined by* (7.1) *is admissible with respect to the form* (7.2) *in the sense of* §4.
(ii) *For any $\mathcal{E}$-quasicontinuous $u \in L^2(\partial D;\mu) \cap \mathcal{F}_e$,*

$$\mathcal{E}(Hu, Hu) = \frac{1}{2}\int_{\partial D\times\partial D}(u(\xi)-u(\eta))^2 U(\xi,\eta)\mu(d\xi)\mu(d\eta), \tag{7.8}$$

where $Hu(x) = E^x(u(X_T); T<\infty), x \in \overline{D}$, and $U(\xi,\eta)$ is the Feller kernel defined in terms of the Martin kernel K.
(iii) *Let Y be the time changed process of X by means of PCAF with Revuz measure μ. Y is then recurrent and of pure jump. Y admits as its Lévy system*

$$(U(\xi,\eta)\mu(d\eta), t), \tag{7.9}$$

where t denotes the non-random PCAF $\phi(t) = t$ of Y.
(iv) *Let $(\check{\mathcal{F}}, \check{\mathcal{E}})$ be the Dirichlet space on $L^2(\partial D,\mu)$ of the time changed process Y. Then*

$$\check{\mathcal{F}} = \{f \in L^2(\partial D;\mu) : \int_{\partial D\times\partial D}(f(\xi)-f(\eta))^2 U(\xi,\eta)\mu(d\xi)\mu(d\eta) < \infty\}, \tag{7.10}$$

$$\check{\mathcal{E}}(f,f) = \frac{1}{2}\int_{\partial D\times\partial D}(f(\xi)-f(\eta))^2 U(\xi,\eta)\mu(d\xi)\mu(d\eta), \quad f \in \check{\mathcal{F}}. \tag{7.11}$$

Proof. (ii) follows from Theorem 6.2 and (7.6). (iii) follows from (ii) and Corollary 5.1. As for (iv), the inclusion $\subset$ in (7.10) and identity (7.11) are clear from (ii) and (6.1). Suppose that a function f belongs to the space appearing in the right hand side of (7.10). By virtue of [4, Th. 3.1], we then have the following expression of the function $w(x) = Hf^2(x) - (Hf(x))^2$:

$$w(x) = R^0_{0-}|\nabla(Hf)|^2(x) \qquad x \in D,$$

where R^0_α denotes the resolvent operator of the absorbing Brownian motion on D. Hence by setting $H^\alpha 1(x) = \int_{\partial D} K_\alpha(x,\xi)\mu(d\xi)$ by the kernel defined in (2.13), we easily see that

$$\int_D |\nabla(Hf)(x)|^2 dx = \lim_{\alpha\to\infty}\alpha(H^\alpha 1, w)_D.$$

On account of the identity (6.14), we see that the right hand side of the above equality is dominated by

$$\int_{\partial D\times\partial D} (f(\xi)-f(\eta))^2\, U(\xi,\eta)\mu(d\xi)\mu(d\eta)<\infty,$$

proving that $Hf\in H_e^1(D)$ (see [11, Example 1.6.1]) and consequently $f\in\check{\mathcal{F}}$. □

(7.8) and (7.10) recover the Douglas integral description of the space of harmonic functions with finite Dirichlet integrals in [4] (but with the Feller kernel instead of the Naim kernel) for the present specific Martin space (cf. [8]).

8. Reduction to Hunt processes

This section is devoted to the proof of the following general reduction theorem especially applicable to the time changed process Y in §4.

Theorem 8.1. *Let $(E,m,\mathcal{F},\mathcal{E})$ be a regular Dirichlet space and $X=(X_t,P^x)$ be a right process over a subset $E_1\subset E$ with $\mathrm{Cap}(E\setminus E_1)=0$. We assume that X is properly associated with $\mathcal{E}$ in the sense that p_tu is an $\mathcal{E}$-quasicontinuous verson of T_tu for any $u\in L^2(X;m)$, where p_t (resp. T_t) is the transition function of X (resp. the L^2-semigroup associated with $(\mathcal{E},\mathcal{F})$). We further assume that the left limit X_{t-} exists in E_Δ for every $t>0$.*
Then, there exists a Borel set $E_2\subset E_1$ such that $\mathrm{Cap}(E\setminus E_2)=0$, E_2 is X-invariant and the restriction $X|_{E_2}$ of X to E_2 is a Hunt process properly associated with $\mathcal{E}$.

We prepare two lemmas.

Lemma 8.1. (i) *For an open set $A\subset E$ of finite capacity, the function*

$$p_A^1(x)=E^x(e^{-\sigma_A})\qquad x\in E_1$$

is an $\mathcal{E}$-quasi continuous version of the 1-equilibrium potential $e_A\in\mathcal{F}$ of A. Here σ_A denotes the hitting time of the process X for the set A.
(ii) *If $\{A_n\}$ is a decreasing sequence of open subsets of E with $\lim_n\mathrm{Cap}(A_n)=0$, then*

$$\lim_{n\to\infty}p^1_{A_n}(x)=0\qquad \text{for } \mathcal{E}\text{−q.e. } x\in E_1.$$

Proof. (i) It is known that p_A^1 is a version of e_A (cf. [11, Lemma 4.2.1]). Since $p_tp_A^1$ is an $\mathcal{E}$-quasicontinuous version of T_te_A, we get the result by letting $t\downarrow 0$.
(ii) Since $\mathcal{E}(e_{A_n},e_{A_n})\downarrow 0$ as $n\to\infty$, (ii) follows from (i). □

Lemma 8.2. *For any set $N\subset E_1$ with $\mathrm{Cap}(N)=0$, there exists a Borel set $E'\subset E_1\setminus N$ such that $\mathrm{Cap}(E\setminus E')=0$ and E' is X-invariant:*

$$P^x(X_t\in E'_\Delta \text{ for all } t\ge 0,\ X_{t-}\in E'_\Delta \text{ for all } t>0)=1,$$

for all $x\in E'$.

Proof. There is a decreasing sequence of open sets A_n including the set $(E\setminus E_1)\cup N$ such that $\lim_{n\to\infty}\mathrm{Cap}(A_n)=0$. Lemma 8.1 then implies that

$$P^x(X_t \text{ or } X_{t-}\in B_0 \text{ for some } t\ge 0)=0,\ \forall x\in E_1\setminus N_1,$$

where $B_0=\cap_nA_n(\supset(E\setminus E_1)\cup N)$ and N_1 is some subset of E_1 with $\mathrm{Cap}(N_1)=0$.

We next find a decreasing sequence of open sets $A'_n\supset B_0\cup N_1$ with $\lim_{n\to\infty}\check{\mathrm{C}}\mathrm{ap}(A'_n)=0$ and let $B_1=\cap_nA'_n$. Repeating the same argument, we can find an increasing sequence of Borel subsets $\{B_n\}$ of zero $\mathcal{E}$-capacity containing $(E\setminus E_1)\cup N$ such that

$$P^x(X_t \text{ or } X_{t-}\in B_n \text{ for some } t\ge 0)=0,\ \text{for all } x\in E\setminus B_{n+1}.$$

Put $B = \cup_n B_n$. $E' = E \setminus B$ satisfies the desired properties . $\square$

Proof of Theorem 8.1. From Lemma 8.1, we can see as in the proof of [11, Lemma 4.2.2] that, for any $\mathcal{E}$-quasi-continuous function u on E,

$$P^x(\lim_{t'\uparrow t} u(X_{t'}) = u(X_{t-})\ \forall t > 0) = 1,\ \mathcal{E}\text{-q.e. } x \in E_1.$$

Choose a countable subfamily C_1 of $\mathcal{F} \cap C_0(E)$ which is dense in $C_0(E)$ and denote by Q^+ the set of all positive rational numbers. Since the functions $p_s f$ for $s \in Q^+,\ f \in C_1$ are $\mathcal{E}$-quasi-continuous, we can find a set N with $\mathrm{Cap}(N) = 0$ such that the above identity holds for each $u = p_t f,\ s \in Q^+,\ f \in C_1$, and for all $x \in E_1 \setminus N$. We then use Lemma 8.2 to get a Borel set $E_2 \subset E_1 \setminus N$ such that E_2 is X-invariant and $\mathrm{Cap}(E \setminus E_2) = 0$. Since $X|_{E_2}$ is a right process on E_2 and

$$P^x(\lim_{t'\uparrow t} p_s f(X_{t'}) = p_s f(X_{t-})\ \forall t > 0) = 1,$$

for all $x \in E_2$ and for any $s \in Q^+,\ f \in C_1$, we can also prove that $X|_{E_2}$ is quasi-left continuous on $[0, \infty)$ in exactly same manner as in the proof of [11, Lemma 7.2.5], completing the proof that $X|_{E_2}$ is a Hunt process on E_2.

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